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LETTER TO THE EDITOR

Self similarity and correlations in percolation

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Abstract. The infinite cluster above the percolation threshold is shown by a scaling theory and Monte Carlo simulations to be homogeneous on large length scales (compared with the correlation length). On shorter length scales this cluster is self similar, and its measured fractal dimensionality agrees excellently with the scaling law $D = d - \beta/\nu$. The exponents β and ν are also measured, both from the crossover between the two length scale regions and from correlations near the boundaries.

Much of the current interest in the properties of dilute systems concentrates on their geometrical structure in the vicinity of the percolation threshold, p_c (Stauffer 1979, Kirkpatrick 1979, Gefen *et al* 1981, Kapitulnik and Deutscher 1982). Studying the geometry is clearly the first step towards understanding physical properties such as long-range magnetic order or conductivity. Particular interest has been centred on the resistivity, superconducting transition temperature, critical magnetic fields and critical currents of granular materials (Deutscher 1981, Kapitulnik and Deutscher 1982). All of these properties depend on the structure of the infinite cluster, which exists for concentrations $p > p_c$. The probability to belong to this cluster is written as $P_\infty(p) \propto (p - p_c)^\beta$, and the correlation length is $\xi \propto |p - p_c|^{-\nu}$ (Stauffer 1979). On length scales which are large compared with ξ, the infinite cluster is believed to fill up the space homogeneously. On smaller scales, the correlations create fluctuations in the density. Early descriptions of the backbone of the infinite cluster used the 'links and nodes' model (Skal and Shklovskii 1974, de Gennes 1976), in which quasilinear long links connect nodes which are separated by a distance ξ. However, this description does not agree with simultaneous resistivity and superconducting critical current measurements (Deutscher 1981) nor with Monte Carlo simulations (Kirkpatrick 1979) or with electron microscope pictures of granular superconductors (Kapitulnik and Deutscher 1982), both of which exhibit *self similar* features on length scales L in the range $a \ll L \ll \xi$ (a is the microscopic lattice distance). The real pictures contain nodes on *all* of these length scales§.

Self similarity is intimately associated with the notion of fractal dimensionality (Mandelbrot 1977). There have been various attempts to identify the fractal dimensionality for the percolation problem by looking at various measures of the size of the largest *finite* cluster at and below p_c within a finite size volume (Leath and

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§ More advanced models replace the linear links by complex fractal structures, which make them self similar on length scales $a \ll L \ll \xi$ (Coniglio 1981).

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Reich 1978, Forrest and Witten 1979, Stauffer 1980). It is not clear if any of these relates to *the geometry of the infinite cluster.* Moreover, there exists no explicit quantitative verification of the fractal nature of this cluster. The aim of the present letter is to report on such a quantitative study.

In contrast to the various finite-size scaling arguments, we follow the geometrical procedure suggested by Mandelbrot (1977). Given a point on the infinite cluster, consider the number $M(L)$ of points on the same cluster within a volume L^d (of linear size L) centred at that point. Self similarity implies that this number scales as

$$M(L) = L^D, \qquad a \ll L \ll \xi, \tag{1}$$

where D is the fractal dimensionality (Mandelbrot 1977, 1982). In what follows we use both scaling theory and Monte Carlo simulations to confirm that (Kirkpatrick 1979) for $d < 6$†

$$D = d - \beta/\nu. \tag{2}$$

Another serious drawback of the earlier finite-size Monte Carlo calculations‡ has to do with boundary effects: at p_c these effects propagate throughout the sample, and cannot be separated from the self-similar region which one wants to study. In this letter we report on new Monte Carlo simulations at $p > p_c$. Since $P_\infty(p)$ grows very quickly with p, it is clear that the largest cluster within any finite sample is a part of the true infinite cluster. Moreover, since ξ is finite, we were able to separate the self-similar behaviour, occurring for $a \ll L \ll \xi$, from the effects of the boundaries. The latter were then used to obtain further information on the correlations. Both the measurement of $M(L)$ and that of the boundary effects are based on the notion that a local perturbation creates *correlations* which decay as power laws up to a distance of order ξ. Our methods provide independent measurements of D, ν and β (on the *same* family of samples), and thus a direct confirmation of (2).

We studied two-dimensional ($d = 2$) Monte Carlo simulations of site percolation, with $p > p_c = 0.5927$. For each desired value of p we generated 5000 samples, of size 187×187, and kept only those (typically more than 300) whose actual concentration was within 0.05% of p. (Without this selection, the variation is of order 0.6%, and the crossover effects described below become smeared.) We next identified the largest cluster on each of the remaining samples, picked only those samples in which the central site belonged to this cluster, and counted the sites connected to it within squares of size L around it, $M(L)$. Averaging over these samples (whose number was of order 100–200), we found the average density $\bar{\rho}(L) = M(L)/L^2$. Typical plots of $\ln \bar{\rho}(L)$ against $\ln L$, at $p - p_c = 0.035$ and 0.022, are shown in figure 1. Similar data were also found at other concentrations.

All the figures exhibit three distinct regions: the data in region I indicate a power law behaviour, $\bar{\rho}(L) \propto L^{2-D}$, with $D = 1.900 \pm 0.009$. (The error reflects the range of slopes on the log–log curve. Note that the actual slope, $d - D$, is small.) The density in region II is practically independent of L, which we identify below as $P_\infty(p)$. This function, shown in figure 2, is excellently described by $(p - p_c)^\beta$, with $\beta = 0.140 \pm 0.007$ (the errors in P_∞ reflect the range of plateau levels). Extrapolation of the lines $\bar{\rho}(L) \propto L^{2-D}$ and $\bar{\rho}(L) \propto P_\infty(p)$ yields the crossover point, which we identify as $L = \xi$.

† Above six dimensions this expression breaks (Kapitulnik *et al* 1983).

‡ Many of the results are summarised by Stauffer (1980). See also Leath and Reich (1978) and Pike and Stanley (1981).

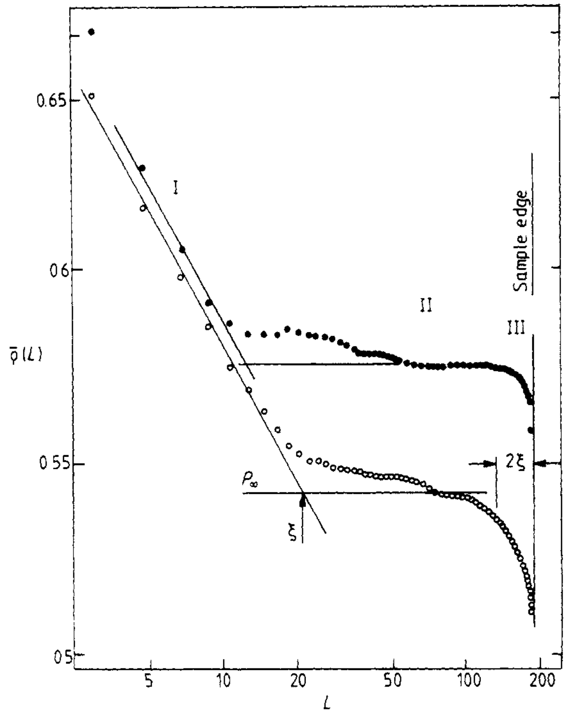

Figure 1. Average density $\bar{\rho}(L)$ against L. ●, $p-p_c=0.035$; ○, $p-p_c=0.022$.

From figure 2 we see that $\xi \propto (p-p_c)^{-\nu}$, with $\nu=1.33\pm0.08$. These values of β and ν are in excellent agreement with earlier ones. Our measured values of D, β and ν clearly satisfy (2). Region III represents the boundary effects, to be discussed below.

In order to interpret figure 1, consider the conditional probability $\rho(r)$ that a point at a distance r from the origin (which belongs to the infinite cluster) will also belong to the infinite cluster. If scaling holds, then (for $r \gg a$) the only relevant length is ξ, and we expect the scaling form (Stauffer 1978)

$$\rho(r)=P_\infty(p)f(r/\xi). \tag{3}$$

The prefactor, $P_\infty(p)$, represents the expectation that the two sites (at r and at the origin) are uncorrelated for $r \gg \xi$, when we expect that $\rho(r)\to P_\infty(p)$, i.e. that $f(x)$ approaches a constant as $x\to\infty$. For $r \ll \xi$ we expect $\rho(r)$ to be independent of ξ. This can be achieved only if $f(x)\sim x^{-\beta/\nu}$ for $x \ll 1$. We thus predict that $\rho(r)\propto r^{-\beta/\nu}$ for $r \ll \xi$. The 'mass' $M(L)$ is found via $M(L)=\int_0^L d^d r\,\rho(r)$, and one easily checks that $M(L)\propto L^{d-\beta/\nu}$ for $L<\xi$, yielding (1) and (2). For larger scales, $L>\xi$, we find $M(L)\propto (L/\xi)^d \xi^{d-\beta/\nu}$. The average density $\bar{\rho}(L)$ is thus found to behave as $L^{-\beta/\nu}$ for $L<\xi$ and as $P_\infty(p)\propto\xi^{-\beta/\nu}$ for $L>\xi$. Note that for $L<\xi$ one has $\bar{\rho}(L)\propto\rho(L)$. This can hold only if the infinite cluster is highly correlated, containing 'holes' at *all* length scales, in contrast to the naive models of Skal and Shklovskii (1974) and de Gennes (1976).

Except for the boundary effects, this behaviour is exactly the one observed in figure 1: the slope in region I should be $-\beta/\nu$, the crossover point between I and II should be at $L\sim\xi$, and the constant density in region II should be $P_\infty(p)$. The fact

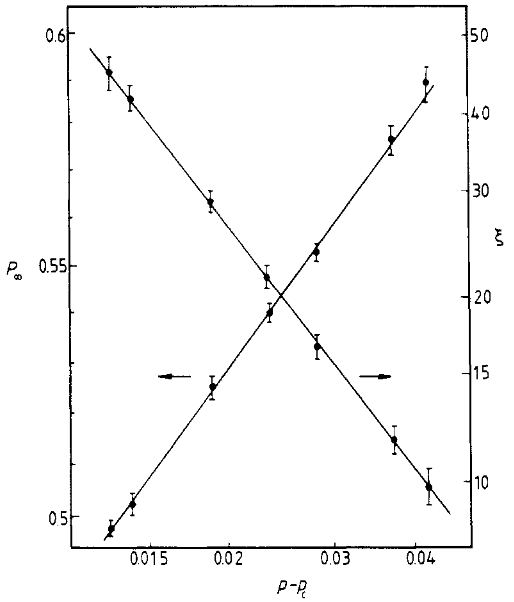

Figure 2. Concentration dependence of P_∞ and ξ, extracted from the $\bar{\rho}(L)$ plots.

that region II is reached *before the boundary effects are felt* ensures that the power law behaviour $\bar{\rho}(L) \propto L^{-\beta/\nu}$ in region I is indeed *an intrinsic property of the infinite cluster* for $L < \xi$.

It is interesting to note that all our $\bar{\rho}(L)$ plots exhibit 'oscillations' about the straight $L^{-\beta/\nu}$ line (figure 1). We argue that they reflect the fluctuations in the size distribution of the 'holes', i.e. the *lacunarity* (Mandelbrot 1977) of the largest clusters in our specific finite simulations. A measurement in the vicinity of a big 'hole' will clearly yield a reduced value of $\bar{\rho}(L)$ for L smaller than the size of the hole. This effect is clearly exhibited in figure 3, where we present the result of our measurements on the Sierpinski gasket (shown in the insert). The random version of this fractal was suggested as a model for the backbone of the infinite cluster (Kirkpatrick 1979, Gefen *et al* 1981). One clearly observes 'jumps' in $\bar{\rho}(L)$ at the sizes of the 'holes', $L = 8$, 16, 32, 64, 128. We expect such oscillations to appear whenever the average is taken over a finite number of samples, and to decay as this number is increased.

Finally, we discuss region III. The decrease in $\bar{\rho}(L)$ as L approaches the size of the sample, R, results from the fact that our analysis throws away 'finite' clusters near the boundary, which might be connected to our infinite cluster via bonds which are outside our sample (we use 'free' boundary conditions). The effect of these is expected to propagate into the sample, down to $(R-L)$ of order ξ.

If we fix some L_0, with $R-L_0 = O(a)$, we expect the function $g(L_0-L) = [P_\infty - \bar{\rho}(L)]/[P_\infty - \bar{\rho}(L_0)]$ to decay to zero for $(L_0 - L) \gg \xi$. This decay is expected to behave as $g(x) = e^{-x/\xi}$ for finite ξ, and as $g(x) \propto x^{-\beta/\nu}$ for $\xi \to \infty$ (see equation (3)). We plotted $g(r)$ against r at $p-p_c = 0.005$ (see figure 4) and $p-p_c = 0.01$, and found a very clear power law behaviour, with $\beta/\nu = 0.105 \pm 0.006$, in agreement with our other results. At higher concentrations we fitted $g(x)$ to $e^{-x/\xi}$, extracting values for both ξ and P_∞ at each concentration. Logarithmic plots of ξ and P_∞ against $(p-p_c)$ now yielded $\xi = (0.85 \pm 0.4)(p-p_c)^{1.3\pm0.1}$ and $\beta = 0.145 \pm 0.01$ consistent with our

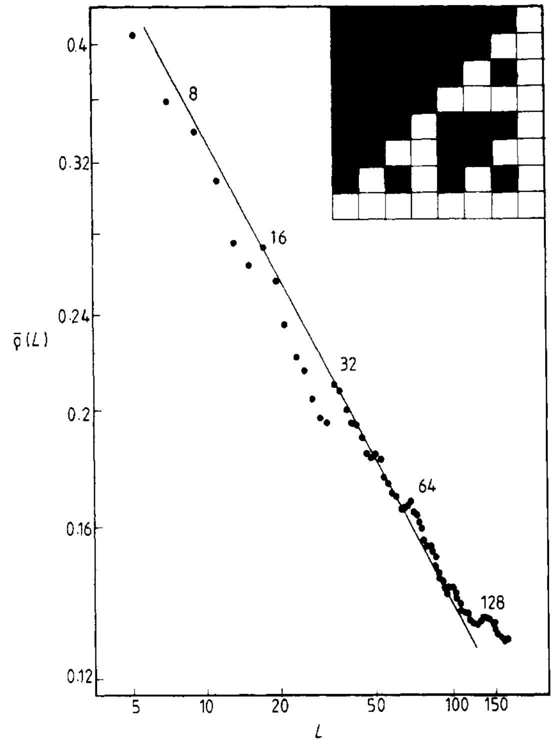

Figure 3. Density $\bar{\rho}(L)$ for the Sierpinski gasket (shown in the insert). $D = \log(3)/\log(2)$, shown by the straight line.

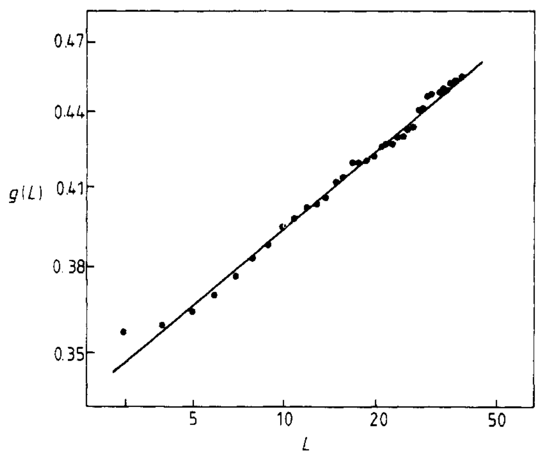

Figure 4. Dependence of density on distance from boundary, for $p - p_c = 0.005$; $L_0 = 0$.

other measurements. The boundary effects may thus be used to obtain additional independent determinations of β/ν, β and ν.

In conclusion, we have given direct proof that the infinite cluster is self similar for scales $a \ll L \ll \xi$, confirmed that its fractal dimensionality is given by (2), and shown that it is homogeneous on scales $L \gg \xi$.

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