NUMERICAL TABULATION OF THE DISTRIBUTION OF KOLMOGOROV'S STATISTIC FOR FINITE SAMPLE SIZE

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1. *Introduction*

Let X be a random variable with the continuous probability distribution function

$$F(x) = \text{Prob}\,\{X \leqq x\},$$

and let $X_1, X_2, \cdots, X_N$ be a sample of size N for X, ordered so that $X_1 \leqq X_2 \leqq \cdots \leqq X_N$. We define the empirical distribution function $F_N(x)$ by

$$F_N(x) = \begin{cases} 0 & \text{for } x < X_1 \\ \dfrac{j}{N} & \text{for } X_j \leqq x < X_{j+1}, \qquad j = 1, 2, \cdots, N-1 \\ 1 & \text{for } X_N \leqq x. \end{cases}$$

The empirical distribution function is a step-function with N jumps, each of height $1/N$, occurring at the points of the sample.

One would expect that, for N large, $F_N(x)$ will very likely be close to $F(x)$. In 1933, Kolmogorov [1] introduced the statistic

$$D_N = \text{least upper bound of } |F(x) - F_N(x)|$$

which measures the greatest absolute discrepancy between $F(x)$ and $F_N(x)$, and showed that it has the following properties which make it particularly useful for judging how "close" $F_N(x)$ is to $F(x)$:

1) the probability distribution of D_N depends on N but is independent of $F(x)$ (D_N is a "distribution-free" statistic)

2) for N large, the probability distribution of D_N is given by the relationship

$$\lim_{n\to\infty} \text{Prob}\left\{D_N < \frac{z}{N}\right\} = 1 - 2\sum_{j=1}^{\infty} (-1)^{j-1} e^{-2j^2z^2} = L(z). \tag{1.1}$$

The function $L(z)$ has been tabulated by Smirnov [2].[1] A new proof

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[1] The expression for $L(z)$ in [2] contains a misprint: $e^{-j^2z^2}$ instead of $e^{-2j^2z^2}$.

of (1.1) has been given recently by Feller [3] and a heuristic outline of a proof by Doob [4].

The asymptotic distribution (1.1) makes it possible to use the statistic D_N for testing the hypothesis that a large sample was obtained from a random variable X with a distribution function $F(x)$ which is explicitly given; it also may be used for constructing a "confidence-band" about the empirical distribution function $F_N(x)$ so that it can be asserted on a preassigned probability level that the unknown "true" distribution function $F(x)$ is entirely contained in that band. In either type of application a difficulty arises due to the fact that the known proofs of (1.1) give no indication how large N must be to make this approximation sufficiently close for practical use. An obvious way to overcome this difficulty is to compute numerically and tabulate the probability distribution of D_N for finite N up to values for which a good agreement is reached with the asymptotic formula (1.1). An adaptation of Feller's argument for such a computation was proposed in [5].

Kolmogorov, in his original paper [1], derived a system of recursion formulas which make it possible to compute for any finite N the probabilities

$$\text{Prob}\left\{D_N < \frac{c}{N}\right\} \qquad \text{for } c = 1, 2, \cdots, N.$$

These formulas were used to compute Table 1 of the present paper. They are reproduced as (A 1.1)–(A 1.4) in the Appendix where the theory of the computations is presented.

Massey [6] obtained a system of recursive formulas, equivalent with (A 1.1)–(A 1.4), as well as a procedure for replacing them by a system of difference equations. He tabulated Prob $\{D_N < c/N\}$ for $N = 5$ (5) 80 and selected values of $c \leqq 9$; there is, however, no estimate given of the error resulting from the large number of computations needed to obtain every result in this tabulation. A table of 100 $\alpha\%$ percentage points was also given by Massey [7], for $\alpha = .20$, .15, .10, .05, .01 and $N = 1$ (1) 35, to two significant digits.

Table 1 of the present paper contains values of Prob $\{D_N < c/N\}$, computed to five decimals, for $N = 1$ (1) 100 and $c = 1$ (1) 15. The method of computation used involves a "truncation" of Kolmogorov's recursion formulas (A 1.1)–(A 1.4), and has made it possible to reduce the number of computations needed and to obtain estimates of the errors due to the truncation and to the accumulated effect of round-offs on a digital computing machine.

Table 2 contains the 95% points of the distribution of D_N for $N=2$ (1) 5 (5) 30 (10) 100, and the 99% points for $N=2$ (1) 5 (5) 30 (10) 80, as well as a comparison with the corresponding values obtained from the asymptotic formula (1.1).

A comparison of Table 1 with the values tabulated by Massey in [6] shows agreement except for a few entries, particularly that for $N=5$, $c=2$. Similarly a comparison of Table 2 with Massey's table in [7] discloses only minor discrepancies, the largest being those at the 95% point for $N=25$ and at the 99% point for $N=10$, 20.

2. *Tabulation of Prob* $\{D_N<c/N\}$

Table 1 below was computed on the U. S. Bureau of Standards Western Automatic Computer (SWAC), at the Institute for Numerical Analysis.[2] The computation was programmed according to formulas (A 3.1), (A 3.2), (A 3.3) of the Appendix, modified for a binary computer; the truncation was performed at $r=12$, and the rounding off was carried out at $t'=35$ binary digits, which corresponds to about $t=10.53$ for decimal digits. This should assure everywhere an error less than $5\cdot 10^{-6}$. The final results were rounded off to 5 decimals. An alternative set of formulas was used for a check.

3. *Table of 95% and 99% points*

By $\epsilon_{N,\,.95}$ and $\epsilon_{N,\,.99}$ we denote the solutions of the equations

$$P(D_N < \epsilon_{N,\,.95}) = .95$$
$$P(D_N < \epsilon_{N,\,.99}) = .99.$$

Table 2 contains in columns (2) and (3) values of $\epsilon_{N,\,.95}$ and $\epsilon_{N,\,.99}$, to 4 decimals. Columns (4) and (5) contain the values

$$\bar{\epsilon}_{N,\,.95} = 1.3581\cdot N^{-1/2} \quad \text{and} \quad \bar{\epsilon}_{N,\,.99} = 1.6276\cdot N^{-1/2},$$

which are the asymptotic 95%- and 99%-points computed according go (1.1). The quotients $\bar{\epsilon}_{N,\,.95}/\epsilon_{N,\,.95}$ and $\bar{\epsilon}_{N,\,.99}/\epsilon_{N,\,.99}$ tabulated in columns (6) and (7) indicate the manner in which these asymptotic values approach the exact values with increasing N. It appears, in particular, that the asymptotic values are always greater than the exact ones and that for $N\geqq 80$ the approximation by (1.1) is already quite good.

[2] The writer takes this occasion to acknowledge the assistance given him by the Institute for Numerical Analysis, and to express his gratitude in particular to Dr. F. S. Acton, Dr. Gertrude Blanch, and Mrs. Roselyn S. Lipkis for their help and advice.

TABLE 1

Prob $\{D_N < c/N\}$

N / c	1	2	3	4	5	6	7	8	9	10
1	1.00000	.50000	.22222	.09375	.03840	.01543	.00612	.00240	.00094	.00036
2		1.00000	.92593	.81250	.69120	.57656	.47446	.38659	.31261	.25128
3			1.00000	.99219	.96992	.93441	.88937	.83842	.78442	.72946
4				1.00000	.99936	.99623	.98911	.97741	.96121	.94101
5					1.00000	.99996	.99960	.99849	.99615	.99222
6						1.00000	1.00000	.99996	.99982	.99943
7								1.00000	1.00000	.99998
8										1.00000

N / c	11	12	13	14	15	16	17	18	19	20
1	.00014	.00005	.00002	.00001	.00000	.00000	.00000	.00000	.00000	.00000
2	.20100	.16014	.12715	.10066	.07950	.06265	.04927	.03869	.03033	.02374
3	.67502	.62209	.57136	.52323	.47795	.43564	.39630	.35991	.32636	.29553
4	.91747	.89126	.86304	.83337	.80275	.77158	.74019	.70887	.67784	.64728
5	.98648	.97885	.96935	.95807	.94517	.93081	.91517	.89844	.88079	.86237
6	.99865	.99732	.99530	.99250	.98882	.98425	.97875	.97235	.96506	.95693
7	.99993	.99979	.99953	.99908	.99837	.99736	.99598	.99419	.99195	.98924
8	1.00000	.99999	.99997	.99993	.99984	.99968	.99944	.99907	.99856	.99788
9		1.00000	1.00000	1.00000	.99999	.99997	.99994	.99989	.99980	.99968
10					1.00000	1.00000	1.00000	.99999	.99998	.99996
11								1.00000	1.00000	1.00000

N / c	21	22	23	24	25	26	27	28	29	30
1	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
2	.01857	.01450	.01132	.00882	.00687	.00535	.00416	.00323	.00251	.00195
3	.26729	.24147	.21793	.19650	.17702	.15935	.14334	.12885	.11575	.10392
4	.61733	.58811	.55970	.53216	.50554	.47987	.45517	.43145	.40870	.38693
5	.84335	.82386	.80401	.78392	.76368	.74338	.72309	.70288	.68280	.66290
6	.94802	.93837	.92805	.91712	.90565	.89368	.88128	.86851	.85541	.84203
7	.98605	.98236	.97817	.97349	.96832	.96269	.95661	.95010	.94318	.93588
8	.99700	.99590	.99456	.99296	.99110	.98895	.98651	.98378	.98076	.97745
9	.99949	.99924	.99890	.99846	.99792	.99725	.99645	.99551	.99441	.99315
10	.99993	.99989	.99982	.99973	.99960	.99943	.99921	.99894	.99861	.99821
11	.99999	.99999	.99998	.99996	.99994	.99990	.99985	.99979	.99971	.99960
12	1.00000	1.00000	1.00000	1.00000	.99999	.99999	.99998	.99997	.99995	.99992
13					1.00000	1.00000	1.00000	1.00000	.99999	.99999
14									1.00000	1.00000

N / c	31	32	33	34	35	36	37	38	39	40
1	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
2	.00151	.00117	.00091	.00070	.00054	.00042	.00033	.00025	.00020	.00015
3	.09325	.08363	.07497	.06717	.06016	.05386	.04820	.04312	.03856	.03448
4	.36612	.34624	.32729	.30923	.29205	.27570	.26018	.24544	.23145	.21819
5	.64323	.62382	.60470	.58590	.56744	.54934	.53161	.51427	.49733	.48078
6	.82843	.81463	.80069	.78663	.77250	.75831	.74410	.72990	.71572	.70159
7	.92822	.92022	.91192	.90332	.89447	.88538	.87608	.86658	.85690	.84707
8	.97384	.96995	.96578	.96134	.95664	.95168	.94648	.94104	.93539	.92952
9	.99172	.99012	.98834	.98638	.98423	.98191	.97939	.97670	.97382	.97077
10	.99773	.99717	.99652	.99578	.99494	.99399	.99294	.99178	.99050	.98910
11	.99946	.99930	.99910	.99886	.99857	.99824	.99785	.99741	.99692	.99636
12	.99989	.99985	.99980	.99973	.99965	.99954	.99942	.99928	.99911	.99891
13	.99998	.99997	.99996	.99994	.99992	.99990	.99986	.99982	.99977	.99971
14	1.00000	1.00000	1.00000	.99999	.99999	.99998	.99997	.99996	.99995	.99993
15				1.00000	1.00000	1.00000	.99999	.99999	.99999	.99999

TABLE 1—(*Continued*)

N / c	41	42	43	44	45	46	47	48	49	50
1	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
2	.00012	.00009	.00007	.00005	.00004	.00003	.00002	.00002	.00001	.00001
3	.03081	.02753	.02459	.02196	.01960	.01750	.01561	.01393	.01242	.01108
4	.20562	.19373	.18247	.17181	.16174	.15222	.14323	.13474	.12672	.11916
5	.46464	.44891	.43359	.41868	.40418	.39008	.37639	.36310	.35020	.33769
6	.68752	.67354	.65965	.64588	.63223	.61872	.60536	.59215	.57911	.56623
7	.83711	.82702	.81684	.80657	.79623	.78583	.77539	.76492	.75442	.74392
8	.92345	.91719	.91075	.90415	.89739	.89048	.88344	.87628	.86899	.86160
9	.96754	.96413	.96056	.95682	.95293	.94888	.94467	.94033	.93584	.93122
10	.98759	.98596	.98421	.98233	.98033	.97822	.97598	.97363	.97115	.96856
11	.99573	.99504	.99428	.99344	.99253	.99154	.99047	.98933	.98810	.98679
12	.99868	.99842	.99813	.99779	.99742	.99701	.99655	.99605	.99550	.99490
13	.99963	.99955	.99945	.99933	.99919	.99904	.99886	.99866	.99844	.99820
14	.99991	.99988	.99985	.99982	.99977	.99972	.99966	.99959	.99951	.99941
15	.99998	.99997	.99996	.99995	.99994	.99993	.99991	.99988	.99986	.99983

N / c	51	52	53	54	55	56	57	58	59	60
1	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
2	.00001	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000
3	.00988	.00880	.00785	.00699	.00623	.00555	.00494	.00440	.00392	.00349
4	.11203	.10530	.09896	.09298	.08735	.08205	.07706	.07236	.06793	.06377
5	.32556	.31381	.30242	.29140	.28073	.27041	.26042	.25077	.24144	.23242
6	.55353	.54101	.52868	.51654	.50459	.49283	.48128	.46992	.45876	.44780
7	.73342	.72294	.71247	.70203	.69162	.68126	.67094	.66068	.65049	.64035
8	.85412	.84654	.83889	.83116	.82337	.81552	.80762	.79968	.79171	.78370
9	.92648	.92161	.91662	.91152	.90632	.90102	.89562	.89013	.88455	.87889
10	.96586	.96304	.96011	.95708	.95393	.95069	.94734	.94390	.94036	.93674
11	.98540	.98392	.98237	.98073	.97900	.97720	.97531	.97334	.97129	.96916
12	.99425	.99356	.99280	.99200	.99113	.99022	.98924	.98821	.98712	.98598
13	.99792	.99762	.99729	.99693	.99654	.99611	.99565	.99515	.99462	.99406
14	.99931	.99919	.99906	.99891	.99875	.99857	.99837	.99815	.99791	.99765
15	.99979	.99975	.99970	.99964	.99958	.99951	.99943	.99934	.99925	.99914

N / c	61	62	63	64	65	66	67	68	69	70
1	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
2	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
3	.00310	.00276	.00246	.00219	.00195	.00173	.00154	.00137	.00122	.00108
4	.05986	.05617	.05271	.04946	.04640	.04352	.04082	.03828	.03589	.03365
5	.22371	.21529	.20717	.19933	.19176	.18445	.17741	.17061	.16406	.15774
6	.43705	.42649	.41614	.40599	.39603	.38628	.37672	.36736	.35819	.34921
7	.63029	.62030	.61040	.60057	.59083	.58119	.57163	.56217	.55280	.54354
8	.77567	.76761	.75955	.75148	.74340	.73533	.72726	.71919	.71115	.70311
9	.87316	.86736	.86150	.85557	.84958	.84355	.83746	.83133	.82516	.81895
10	.93302	.92921	.92533	.92136	.91731	.91320	.90901	.90475	.90042	.89604
11	.96695	.96466	.96230	.95986	.95735	.95476	.95211	.94938	.94659	.94373
12	.98477	.98351	.98218	.98080	.97936	.97786	.97630	.97469	.97301	.97128
13	.99345	.99281	.99212	.99140	.99063	.98983	.98898	.98809	.98716	.98619
14	.99737	.99707	.99674	.99639	.99602	.99562	.99519	.99474	.99425	.99374
15	.99902	.99889	.99874	.99858	.99841	.99823	.99803	.99781	.99758	.99733

TABLE 1—(*Continued*)

N	71	72	73	74	75	76	77	78	79	80
c										
1	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
2	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
3	.00096	.00086	.00076	.00068	.00060	.00053	.00047	.00042	.00037	.00033
4	.03155	.02958	.02772	.02598	.02435	.02282	.02138	.02003	.01877	.01758
5	.15165	.14578	.14013	.13468	.12943	.12438	.11951	.11482	.11031	.10597
6	.34043	.33183	.32342	.31519	.30714	.29928	.29159	.28407	.27672	.26955
7	.53437	.52531	.51635	.50750	.49875	.49011	.48158	.47316	.46485	.45664
8	.69510	.68712	.67916	.67123	.66333	.65546	.64764	.63985	.63211	.62441
9	.81271	.80644	.80014	.79382	.78748	.78112	.77475	.76836	.76197	.75557
10	.89159	.88709	.88253	.87792	.87326	.86856	.86381	.85902	.85419	.84932
11	.94080	.93781	.93476	.93165	.92848	.92525	.92197	.91864	.91525	.91182
12	.96950	.96765	.96576	.96380	.96180	.95974	.95762	.95546	.95324	.95098
13	.98518	.98412	.98302	.98187	.98069	.97946	.97819	.97687	.97552	.97412
14	.99321	.99264	.99204	.99142	.99076	.99008	.98936	.98861	.98783	.98702
15	.99707	.99678	.99648	.99616	.99582	.99546	.99508	.99468	.99426	.99382

N	81	82	83	84	85	86	87	88	89	90
c										
1	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
2	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
3	.00030	.00026	.00023	.00021	.00018	.00016	.00015	.00013	.00011	.00010
4	.01647	.01542	.01444	.01353	.01267	.01186	.01110	.01040	.00973	.00911
5	.10178	.09776	.09389	.09017	.08659	.08314	.07983	.07664	.07357	.07063
6	.26253	.25569	.24900	.24247	.23609	.22986	.22379	.21786	.21207	.20643
7	.44855	.44056	.43269	.42493	.41727	.40973	.40229	.39497	.38775	.38064
8	.61675	.60914	.60159	.59408	.58662	.57922	.57188	.56459	.55735	.55018
9	.74917	.74276	.73636	.72996	.72356	.71717	.71079	.70442	.69806	.69172
10	.84442	.83949	.83452	.82953	.82451	.81947	.81440	.80932	.80421	.79909
11	.90833	.90480	.90123	.89761	.89395	.89025	.88651	.88273	.87892	.87507
12	.94867	.94630	.94390	.94144	.93894	.93640	.93381	.93118	.92851	.92580
13	.97268	.97119	.96967	.96811	.96650	.96486	.96317	.96145	.95969	.95789
14	.98618	.98531	.98440	.98346	.98249	.98149	.98046	.97939	.97830	.97717
15	.99336	.99287	.99237	.99184	.99129	.99071	.99011	.98949	.98884	.98818

N	91	92	93	94	95	96	97	98	99	100
c										
1	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
2	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
3	.00009	.00008	.00007	.00006	.00006	.00005	.00004	.00004	.00003	.00003
4	.00853	.00798	.00747	.00699	.00654	.00612	.00573	.00536	.00502	.00469
5	.06779	.06507	.06245	.05994	.05752	.05520	.05297	.05082	.04876	.04678
6	.20092	.19555	.19031	.18520	.18022	.17536	.17062	.16600	.16150	.15712
7	.37364	.36674	.35995	.35327	.34669	.34021	.33384	.32757	.32140	.31533
8	.54306	.53600	.52901	.52207	.51520	.50839	.50164	.49496	.48834	.48178
9	.68539	.67908	.67279	.66651	.66026	.65403	.64783	.64165	.63549	.62937
10	.79395	.78880	.78364	.77847	.77329	.76810	.76291	.75771	.75251	.74731
11	.87119	.86728	.86334	.85937	.85538	.85136	.84731	.84324	.83915	.83504
12	.92305	.92026	.91743	.91457	.91167	.90874	.90578	.90278	.89975	.89670
13	.95605	.95418	.95226	.95032	.94833	.94632	.94426	.94218	.94006	.93791
14	.97601	.97482	.97359	.97234	.97105	.96974	.96839	.96702	.96561	.96417
15	.98748	.98677	.98602	.98526	.98447	.98366	.98282	.98196	.98107	.98016

TABLE 2

95%-POINTS $\epsilon_{N,.95}$ AND 99%-POINTS $\epsilon_{N,.99}$ FOR KOLMOGOROV'S STATISTIC

(1) N	(2) $\epsilon_{N,.95}$	(3) $\epsilon_{N,.99}$	(4) $\bar{\epsilon}_{N,.95}$	(5) $\bar{\epsilon}_{N,.99}$	(6) $\frac{\bar{\epsilon}_{N,.95}}{\epsilon_{N,.95}}$	(7) $\frac{\bar{\epsilon}_{N,.99}}{\epsilon_{N,.99}}$
2	.8419	.9293	.9612	1.1509	1.142	1.238
3	.7076	.8290	.7841	.9397	1.108	1.134
4	.6239	.7341	.6791	.8138	1.088	1.109
5	.5633	.6685	.6074	.7279	1.078	1.089
10	.4087	.4864	.4295	.5147	1.051	1.058
15	.3375	.4042	.3507	.4202	1.039	1.040
20	.2939	.3524	.3037	.3639	1.033	1.033
25	.2639	.3165	.2716	.3255	1.029	1.028
30	.2417	.2898	.2480	.2972	1.026	1.025
40	.2101	.2521	.2147	.2574	1.022	1.021
50	.1884	.2260	.1921	.2302	1.019	1.018
60	.1723	.2067	.1753	.2101	1.018	1.016
70	.1597	.1917	.1623	.1945	1.016	1.015
80	.1496	.1795	.1518	.1820	1.015	1.014
90	.1412		.1432		1.014	
100	.1340		.1358		1.013	

4. *Examples*

4.1. *Determination of sample size needed.*

4.11. We wish to approximate $F(x)$ empirically by $F_N(x)$ so that the error is everywhere less than .15, on the 90% probability level. How large must be the sample size N? To answer this question, we find by interpolation in Table 1 that $P\{D_{65} < .15\} > .900$, so that $N = 65$ is sufficient.

4.12. An approximation to $F(x)$ by $F_N(x)$ is desired on the 99% probability level with an error less than .05 everywhere; what sample size is needed? An inspection of Table 1 shows that N must be >100, hence the asymptotic formula (1.1) will be used. The asymptotic 99% point, according to Section 3, is $1.6276 \cdot N^{-1/2}$, hence by setting this equal to .05 and solving for N we find $N = 1060$.

4.2. *Estimating probabilities.*

In Table 3, column (2) contains an ordered sample of a random variable X, consisting of the values X_i, $i=1, 2, \cdots, 40$. The values in columns (3) and (4) are

$$L(i) = \max\left(0, \frac{i}{40} - .2101\right)$$

and

$$U(i) = \min\left(1, \frac{i}{40} + .2101\right)$$

for $i=0, 1, 2, \cdots, 40$, where .2101 is the value of $\epsilon_{40,\ .95}$ from Table 2. It can be asserted with probability .95 that the true continuous probability distribution function is everywhere contained in the "confidence band" defined by

$$L(i) < F(x) < U(i) \qquad \text{for } X_i \leqq x \leqq X_{i+1}. \tag{4.2}$$

Therefore, any number of statements of the following kinds may be made *simultaneously* on a probability level of at least .95: $P\{X<.7867\} = P\{X<X_{14}\}$ is a number between .1149 and .5351; $P\{.7867<X<1.5137\}=P\{X_{14}<X<X_{34}\}$ is a number between $L(34)-U(14) = .0798$ and $U(34)-L(14)=.8601$; $P\{X>1.5677\}=1-P\{X<X_{37}\}$ is less than $1-L(37)=.2851$. Each of these statements separately could be made on a probability level higher than .95.

4.3. *Testing a completely specified hypothesis.*

We wish to test on the .95 probability level the hypothesis H_0 that the sample in column (2) of Table 4.2 above was obtained from a normal population with expectation 1 and standard deviation $1/\sqrt{6}$; we agree to reject H_0 if the probability function

$$F_0(x) = \frac{\sqrt{6}}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-1/2\ 6(X-1)^2} dX = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{6}(x-1)} e^{-u^2/2} du \tag{4.31}$$

is not entirely contained in the confidence band (4.2).

For this purpose we may use the graphical procedure, in which the confidence band (4.2) is plotted, then a large number of values of $F_0(x)$ are computed from (4.31) and a graph of $F_0(x)$ is sketched, and finally H_0 is rejected when this graph reaches or crosses the lower or upper boundary of the confidence band. The obvious disadvantage of this procedure is that it requires the computation of many values of $F_0(x)$.

TABLE 3

DATA FOR EXAMPLES IN SECTIONS 4.2 AND 4.3

(1) i	(2) X_i	(3) $L(i)$	(4) $U(i)$
1	.0475	.0000	.2351
2	.2153	.0000	.2601
3	.2287	.0000	.2851
4	.2824	.0000	.3101
5	.3743	.0000	.3351
6	.3868	.0000	.3601
7	.4421	.0000	.3851
8	.5033	.0000	.4101
9	.5945	.0149	.4351
10	.6004	.0399	.4601
11	.6255	.0649	.4851
12	.6331	.0899	.5101
13	.6478	.1149	.5351
14	.7867	.1399	.5601
15	.8878	.1649	.5851
16	.8930	.1899	.6101
17	.9335	.2149	.6351
18	.9602	.2399	.6601
19	1.0448	.2649	.6851
20	1.0556	.2899	.7101
21	1.0894	.3149	.7351
22	1.0999	.3399	.7601
23	1.1765	.3649	.7851
24	1.2036	.3899	.8101
25	1.2344	.4149	.8351
26	1.2543	.4399	.8601
27	1.2712	.4649	.8851
28	1.3507	.4899	.9101
29	1.3515	.5149	.9351
30	1.3528	.5399	.9601
31	1.3774	.5649	.9851
32	1.4209	.5899	1.0000
33	1.4304	.6149	1.0000
34	1.5137	.6399	1.0000
35	1.5288	.6649	1.0000
36	1.5291	.6899	1.0000
37	1.5677	.7149	1.0000
38	1.7238	.7399	1.0000
39	1.7919	.7649	1.0000
40	1.8794	.7899	1.0000

Another procedure is based on the fact that $F_0(x)$ can leave the confidence band (4.2) if and only if it leaves this confidence band at one of the sample points X_i, $i=1, \cdots, N$, that is if at least one of the inequalities

$$L(i) < F_0(X_i) < U(i-1), \qquad i = 1, 2, \cdots, N \tag{4.32}$$

is violated. It would, therefore, be sufficient to compute $F_0(X_i)$ for all sample points X_i, and to reject H_0 if at least one of the inequalities (4.32) is not satisfied. Even this procedure has the disadvantage that it may require the computation of all the $L(i)$, $U(i-1)$ and $F_0(X_i)$.

Compared with the preceding two, the following method saves a considerable amount of computation:

We consider the sample values ordered increasingly, as in column (2) of Table 4.2, and compute

$$L(1) = .0000, \qquad F_0(X_1) = .0098, \qquad U(0) = .2101.$$

Since these three numbers satisfy (4.32), the smallest X_i for which (4.32) could be violated must be such that either $L(i) \geqq F_0(X_1)$ or $F_0(X_i) \geqq U(1)$, that is either $i/40 - .2101 \geqq .0098$ or $F_0(X_i) \geqq 1/40 + .2101$, hence either $i \geqq 8.796$ or $X_i \geqq .7052$; this means that either $i \geqq 9$ or, according to column (2) of Table 4.2, $i \geqq 14$ is the earliest sample value to check for (4.32). We compute for $i=9$:

$$L(9) = .0149, \qquad F_0(X_9) = .1603, \qquad U(8) = .4101.$$

Since these three numbers satisfy (4.32), the next smallest X_i for which (4.32) could be false must be such that either $L(i) \geqq F_0(X_9)$ or $F_0(X_i) \geqq U(9)$, that is either $i/40 - .2101 \geqq .1603$ or $F_0(X_i) \geqq 9/40 + .2101$, hence either $i \geqq 14.82$ or $X_i \geqq .9052$; this means either $i \geqq 15$ or, according to column (2), $i \geqq 17$. We therefore compute for $i=15$

$$L(15) = .1649, \qquad F_0(X_{15}) = .3918, \qquad U(14) = .5601$$

and note that (4.32) is verified.

The next smallest X_i for which (4.32) could be false must be such that either $L(i) \geqq F_0(X_{15}) = .3918$ or $F_0(X_i) \geqq U(15) = .5851$, that is $i \geqq 24.08$ or $X_i \geqq 1.0877$, hence $i \geqq 25$ or $i \geqq 21$. We compute for $i=21$

$$L(21) = .3149, \qquad F_0(X_{21}) = .5867, \qquad U(20) = .7101,$$

and see that (4.32) is verified.

Continuing this procedure, we finish up by calculating only the values

i	$L(i)$	$U(i-1)$	$F_0(X_i)$	$U(i)$
1	.0000	.2101	.0098	.2351
9	.0149	.4101	.1603	.4351
15	.1649	.5601	.3918	.5851
21	.3149	.7101	.5867	.7351
27	.4649	.8601	.7468	.8851
34	.6399	1.0000	.8958	1.0000

and do not reject H_0 since (4.32) is satisfied for all these i. If at some step of this procedure (4.32) had not been satisfied, we would have rejected H_0 and stopped computing. This method appears particularly useful for large samples.

5. *Other distribution-free statistics*

5.1. A number of distribution-free statistics have been studied which lend themselves for treating problems such as those illustrated in the preceding section. Without attempting an enumeration of such statistics and the techniques based on them, we should like to mention some of the more important among them and compare them briefly with Kolmogorov's statistic D_N.

5.2. *The Chi-square.*

This well-known and extensively tabulated statistic is being used for testing completely specified hypotheses such as the one exemplified in 4.3. The χ^2 statistic becomes approximately distribution-free for $N \to \infty$ but is not distribution-free for finite N, and little is known about the manner in which its actual distribution for finite N and given $F(x)$ is approximated by its limiting distribution. By contrast, D_N is a distribution-free statistic for finite N and its exact probability distribution is tabulated for finite N (Table 1 of this paper) and for the asymptotic case [2].

Not enough is known about the power of either test to justify the preference for using the χ^2 or D_N for testing a completely specified hypothesis. The χ^2 technique, however, requires grouping of data, while in applying D_N one uses the individual observations; this suggests that the D_N test may utilize the information better than the χ^2 test.

The χ^2 statistic has the advantage that it can be used for testing the composite hypothesis that $F(x)$ belongs to a parametric family of distributions. This is due to the fact that under fairly general assumptions it is known how the probability distribution of χ^2 is approximately affected when parameters are estimated from the sample (loss of one degree of freedom for each parameter estimated). No such knowledge is available for D_N.

The statistic D_N can be used for estimating an unknown $F(x)$ by a confidence band as illustrated in 4.2. Confidence regions obtained by using the χ^2 have no simple intuitive meaning.

5.3. *Confidence bands with variable width.*

Wald and Wolfowitz [8] have developed a theory of distribution-free confidence bands more general than those defined by D_N. These confidence bands could, in particular, be constructed so that their width decreases towards the lower and the upper end of the distribution, which would be an improvement on D_N. Numerical tabulations, however, are not available for this theory, either for finite sample sizes or for the asymptotic case.

5.4. *One-sided confidence bands.*

A one-sided confidence band was proposed by Smirnov [9] who also gave an asymptotic expression for the corresponding probability distribution. The exact probability distribution for finite sample size N was derived by Wald and Wolfowitz [8]. An alternative expression for the exact probability distribution was proposed by Birnbaum and Tingey [10] and was used to tabulate the 10%, 5%, 1% and .1% points for $N=5$, 8, 10, 20, 40, 50. Since for $N=50$ Smirnov's asymptotic expression is already very good, the probability distribution for one-sided confidence bands is at present tabulated well enough for practical use. It can be used for a one-sided test of a completely specified hypothesis or for estimation of an unknown $F(x)$ by a one-sided confidence contour.

5.5. *Smirnov's statistic.*

Modifying a statistic proposed by Cramér and von Mises, Smirnov [11] introduced the distribution-free statistic

$$\omega_n^2 = \int_{-\infty}^{+\infty} [F_n(x) - F(x)]^2 dF(x)$$

and derived an asymptotic expression for its probability distribution. This statistic could be used for testing completely specified hypotheses. No tabulation of its probability distribution is available.[3]

[3] At the time of the printing of this paper, a table of the limiting distribution of $n\omega_n^2$ was published in T. W Anderson and D. A. Darling, "Asymptotic Theory of Certain 'Goodness of Fit' Criteria Based on Stochastic Processes," *Annals of Mathematical Statistics*, 23 (1952), 193–212.

5.6. *Sherman's statistic.*

The distribution-free statistic

$$\omega_n = \frac{1}{2}\sum_{i=1}^{n+1}\left| F(X_i) - F(X_{i-1}) - \frac{1}{n+1}\right|$$

where $X_0 = -\infty$, $X_{n+1} = +\infty$, was introduced and studied by Sherman [12]. He derived its exact probability distribution for finite sample size n, and showed that this distribution is asymptotically normal. No tabulation is available for finite sample size. For large samples Sherman's statistic can be used to test completely specified hypotheses. The calculation of ω_n appears more time-taking than the use of D_N illustrated in 4.3. Not enough is known about the power of either test to justify a preference for a test based on Kolmogorov's or on Sherman's statistic.

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APPENDIX

A 1. *Kolmogorov's formulas*

The following recursion formulas for computing Prob $\{D_N < c/N\}$

are those given by Kolmogorov in [1] except for minor changes in notation:

$$\text{(A 1.1)} \qquad \text{Prob}\left\{D_N < \frac{c}{N}\right\} = \frac{N!}{N^N} e^N R_{0,N}(c),$$

where $R_{i,k}(c)$ is defined for all integers i, all non-negative integers k, and $c=1, 2, \cdots, N$, and

$$\text{(A 1.2)} \qquad R_{0,0}(c) = 1, \qquad R_{i,0}(c) = 0 \qquad \text{for } i \neq 0$$

$$\text{(A 1.3)} \qquad R_{i,k}(c) = 0 \qquad \text{for } |i| \geqq c$$

$$\text{(A 1.4)} \qquad R_{i,k+1}(c) = e^{-1} \sum_{s=0}^{2r-1} R_{i+1-s,k}(c) \frac{1}{s!} \qquad \text{for } |i| \leqq c-1.$$

The change of notations for passing from (A 1.1)–(A 1.4) to Massey's formulas in [6] may be summarized in the following "dictionary":

(A 1.1)–(A 1.4)	Massey
c	k
N	n
k	m
$c+i$	j
$i+c+1-s$	h
i	$j-k$
s	$j-h+1$
$e^k R_{i,k}(c)$	$U_j(m)$

A 2. *Truncation and truncation error*

In the following all derivations are carried out for c fixed; the argument c will, therefore, be omitted.

We "truncate" the right-hand sums in (A 1.4) by retaining only the terms for $s=0, 1, \cdots, r$, where $r<2c-1$, so that the $R_{i,k}(c)$ are replaced by quantities $S_{i,k}$ defined by the recursive formulas

$$\text{(A 2.1)} \qquad S_{0,0} = 1, \qquad S_{i,0} = 0 \qquad \text{for } i \neq 0$$

$$\text{(A 2.2)} \qquad S_{i,k} = 0 \qquad \text{for } |i| \geqq c$$

$$\text{(A 2.3)} \qquad S_{i,k+1} = e^{-1} \sum_{s=0}^{r} S_{i+1-s,k} \frac{1}{s!} \qquad \text{for } |i| \leqq c-1.$$

The resulting "truncation error" $R_{i,k}-S_{i,k}$ satisfies the inequality

$$0 \leqq R_{i,k} - S_{i,k} \leqq 1 - \left(e^{-1}\sum_{v=0}^{r} \frac{1}{v!}\right)^k = M_k. \tag{A 2.4}$$

This inequality follows by induction from (A 1.4), (A 2.3) and the easily (again by induction) verified fact that $0 \leqq R_{i,k}(c) \leqq 1$.

Example: for $k \leqq 100$, $r=12$, inequality (A 2.4) yields the upper bound for the truncation error: $M_k < k \cdot 10^{-10} \leqq 10^{-8}$.

A 3. *Round-off error*

To perform the computations on a machine with a capacity of t decimal digits, we introduce auxiliary numbers τ, $\phi(s)$, and a, defined by

$$\tfrac{1}{2}10^{-t} = \tau,$$

$$\frac{1}{s!} = \phi(s)10^{-u_s} + E_s = \left(\sum_{j=1}^{t} a_j 10^{-j}\right)10^{-u_s} + E_s, \tag{A 3.01}$$

where

$$|E_s| \leqq 10^{-u_s}\tau \quad \text{and} \quad a_1 \geqq 1,$$

$$e^{-1} = a + E, \quad \text{where} \quad |E| \leqq \tau. \tag{A 3.02}$$

Whenever $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, and u, v are t-digit numbers, $u \times v$ will denote the result of computing the product uv exactly and then rounding off to t digits after the decimal point, so that

$$u \times v = uv + G, \quad \text{where} \quad |G| \leqq \tau.$$

Whenever $0 \leqq f \leqq 1$, we will denote by $\{f\}$ the result of rounding f off to t digits after the decimal point so that

$$\{f\} = f + F, \quad \text{where} \quad |F| \leqq \tau.$$

We now calculate the numbers $T_{i,k}$ defined by the recursive relationships

$$T_{0,0} = 1, \qquad T_{i,0} = 0 \qquad \text{for } i \neq 0 \tag{A 3.1}$$

$$T_{i,k} = 0 \qquad \text{for } |i| \geqq c \tag{A 3.2}$$

$$T_{i,k+1} = \left\{a\left[T_{i+1,k} + T_{i,k} + \sum_{s=2}^{r}\left\{(T_{i+1-s,k} \times \phi(s))10^{-u_s}\right\}\right]\right\}. \tag{A 3.3}$$

For the "round-off error" $S_{i,k}-T_{i,k}$ we have the inequality

$$\text{(A 3.4)} \qquad |S_{i,k} - T_{i,k}| < \beta(1 + \alpha + \alpha^2 + \cdots + \alpha^{k-1}) = \mu_k$$

where

$$\alpha = a\sum_{s=0}^{r} \phi(s)10^{-u_s}$$

$$\beta = a\sum_{s=2}^{r}|E_s| + e|E| + \tau\left[a\left(\sum_{s=2}^{r} 10^{-u_s} + r - 1\right) + 1\right]$$

and E_s, u_s, E, are defined by (A 3.01) and (A 3.02). Inequality (A 3.4) follows by induction from (A 2.3) and (A 3.3).

Example: for $r=12$, $t=10$, one obtains from (A 3.4) the estimate $\mu_k < 3.33k\cdot 10^{-10}$, hence for $k \leqq 100$ the round-off error is always less than $3.33\ 10^{-8}$.

A 4. *Computation of Table 2*

It is not difficult to show that the probability distribution of D_N is given by

$$\text{(A 4.1)} \qquad P\left(D_N < \frac{1}{2N} + v\right) = N!\int_{1/2N-v}^{1/2N+v}\int_{3/2N-v}^{3/2N+v}\cdots \cdot\int_{(2N-1)/2N-v}^{(2N-1)/2N+v} g(u_1, u_2, \cdots, u_N)du_N \cdots du_2du_1$$

for $0 \leqq v \leqq (2N-1)/2N$,[4] where

$$g(u_1, u_2, \cdots, u_N) = \begin{cases} 1 & \text{for} \quad 0 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_N \leqq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

For small values of N, (A 4.1) can be evaluated by quadrature. In particular one obtains

$$P\left(D_2 < \frac{1}{4} + v\right) = \begin{cases} 2(2v)^2 & \text{for} \quad 0 \leqq v \leqq \dfrac{1}{4} \\ -2v^2 + 3v - \dfrac{1}{8} & \text{for} \quad \dfrac{1}{4} \leqq v \leqq \dfrac{3}{4}, \end{cases}$$

[4] It is easily seen that $P(D_N < u) = 0$ for $0 \leqq u \leqq 1/2N$, so that the case $-(1/2N) \leqq v \leqq 0$ need not be considered.

$$P\left(D_3 < \frac{1}{6} + v\right) = \begin{cases} 6(2v)^3 & \text{for } 0 \leqq v \leqq \frac{1}{6} \\ -12v^3 + 8v^2 + v - \frac{1}{9} & \text{for } \frac{1}{6} \leqq v \leqq \frac{2}{6} \\ -4v^3 + \frac{11}{3} v - \frac{11}{27} & \text{for } \frac{2}{6} \leqq v \leqq \frac{3}{6} \\ -2v^3 - 5v^2 + \frac{25}{6} v - \frac{17}{108} & \text{for } \frac{3}{6} \leqq v \leqq \frac{5}{6}. \end{cases}$$

Similar expressions have been obtained for $N=4$ and 5. For larger N the evaluation of (A 4.1) soon seems to become prohibitive.

For $N=2, 3, 4, 5$ the values of $\epsilon_{N, .95}$ and $\epsilon_{N, .99}$ given in Table 2 were obtained by equating the polynomials obtained from (A 4.1) to .95 and .99, respectively, and solving the resulting algebraic equations of degree N. For $N \geqq 10$ the tabulated values of $\epsilon_{N, .95}$ and $\epsilon_{N, .99}$ were obtained by inverse interpolation from Table 1.