

Robertson Walker Metric

①

General Relativity (GR) and Special Relativity (SR) use coordinates in space-time to describe events.

ds^2 : interval between events

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu$$

$\eta_{\mu\nu}$: Minkowsky metric $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $dx^\mu = (cdt, d\vec{x})$

In SR \rightarrow metric is fixed

In GR \rightarrow dynamical field solution of Einstein equations.

Cosmological principle \rightarrow homogeneity and isotropy ; functional form is fixed

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

"cosmic" time measured

by comoving observers at fix position

scale factor overall time dep. due to expansion of Universe

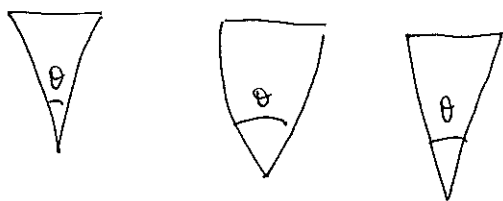
(r, θ, ϕ) : comoving coordinates.

$k \begin{cases} 0 & \text{flat} \rightarrow \text{no spatial curvature} \\ +1 & \text{closed} \\ -1 & \text{open} \end{cases}$

Conformal time $dt \equiv a^2 d\tau \rightarrow d\tau = \frac{dt}{a(t)}$

Another way of writing the spatial part of the metric

$$\chi \equiv \int \frac{dr}{\sqrt{1-kr^2}} \rightarrow \begin{matrix} (\sin \chi)^2 dx^3 \\ (\sinh \chi)^2 dx^3 \\ \chi^2 dx^3 \end{matrix}$$



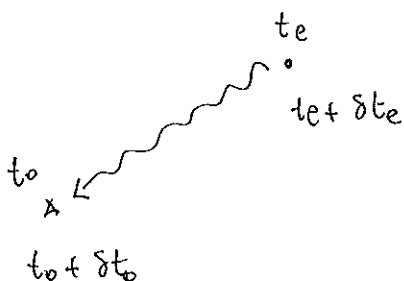
a given transverse distance subtends different angles in different geometries.

Kinematics of RW metric

redshift:

Let's consider propagation of light $\rightarrow ds^2 = 0$

$$dt^2 = \frac{a^2(t) dr^2}{1-kr^2}$$



$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_0^{r_s} \frac{dr}{\sqrt{1-kr^2}} \quad \leftarrow \text{first pulse}$$

$r_s = \text{fixed comoving coord.}$

$$\int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{dt}{a(t)} = \int_0^{r_s} \frac{dr}{\sqrt{1-kr^2}}$$

so

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{dt}{a(t)} = \int_{t_e + \delta t_e}^{t_o} \frac{dt}{a} + \int_{t_o}^{t_o + \delta t_o} \frac{dt}{a}$$

$$= \int_{t_e}^{t_o} \frac{dt}{a} - \int_{t_e}^{t_e + \delta t_e} \frac{dt}{a} + \int_{t_o}^{t_o + \delta t_o} \frac{dt}{a}$$

so

$$\int_{t_e}^{t_e + \delta t_e} \frac{dt}{a(t)} = \int_{t_o}^{t_o + \delta t_o} \frac{dt}{a(t)} \quad \rightarrow \quad \text{if we think on this as pulses with wavelength } \lambda \sim \Delta t$$

$$\frac{\delta t_e}{a(t_e)} = \frac{\delta t_o}{a(t_o)}$$

$$\frac{\lambda_{\text{emitted}}}{\lambda_{\text{obs.}}} = \frac{a_{\text{emitted}}}{a_{\text{obs.}}}$$

or

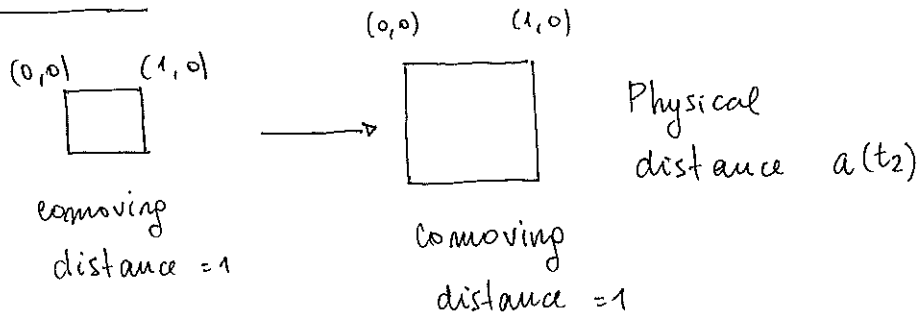
$$\frac{\lambda_{\text{obs.}}}{\lambda_e} = \frac{a_{\text{obs.}}}{a_e} \equiv 1 + z_e \quad \leftarrow \text{redshift}$$

↑
assuming $z=0$ at $\lambda_{\text{obs.}}$

"a" grows with time (universe expands) so ~~wavelength~~ wavelengths are stretched [i.e. redshifted]

Distances - 1

(3)



$$\lambda_{\text{com}} \cdot a = \lambda_{\text{phys}}$$

comoving distance that light

can travel

$$\eta \equiv \int_0^t \frac{dt'}{a(t')} \quad [c=1]$$

→ Regions separated by distances greater than η are NOT causally connected.

$\eta \leftrightarrow$ comoving horizon

η is also conformal time

$$\left\{ \begin{array}{l} \eta \propto a^{1/2} : \text{matter dominated universe} \\ \eta \propto a : \text{radiation dominated} \end{array} \right.$$

comoving distance between a distant emitter and us

$$\chi(a) = \int_{t(a)}^{t_0} c \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')}$$

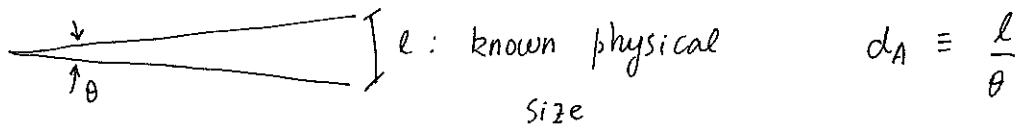
using $\frac{da}{dt} \equiv aH$

Typically we can see objects out to $z \lesssim 6$ so Universe is still matter dominated.

$$H \propto a^{-3/2}$$

$$\chi^{\text{Flat, MD}}(a) = \frac{2}{H_0} (1 - a^{1/2}) = \frac{2}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right) \sim \frac{z}{H_0} \quad (z \ll 1)$$

Angular Diameter Distance



d_A in an expanding universe

$l/a \leftarrow$ comoving size of object

$\chi(a) \leftarrow$ comoving distance

angle subtended

$$\theta = \frac{(l/a)}{\chi(a)} = \frac{l}{d_A} \rightarrow d_A^{\text{Flat}} = a \chi(a) = \frac{\chi}{1+z}$$

Angular distance is equal to χ at low z but it decreases at high z . Objects at large z appear larger than they would at intermediate z .

Non-zero curvature

$$d_A = \frac{a}{H_0 \sqrt{|\Omega_k|}} \begin{cases} \sinh \sqrt{\Omega_k} H_0 \chi & \Omega_k > 0 \\ \sin \sqrt{-\Omega_k} H_0 \chi & \Omega_k < 0 \end{cases}$$

if $\Omega_k \rightarrow 0$ $d_A \rightarrow a\chi$.

($d_A =$ physical distance?)

Distances - 2

$$d_A = \frac{S_k(x)}{1+z}$$

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In the ~~example~~ currently favoured cosmological model
"angular diameter distance" is a good approx. to
"proper distance" (i.e. distance when light left obj.)



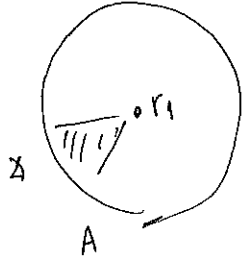
2 objects of same size \Rightarrow the one further out appears larger in an expanding universe

$$\theta_1 = \frac{l}{d_A(z_1)}$$

$$\theta_2 = \frac{l}{d_A(z_2)}$$

because after $z \sim 1.5$ d_A decreases w/ redshift.

Luminosity distance



L : intrinsic luminosity $\frac{\text{Energy}}{\text{time}}$

$$\frac{dt_e}{a_e} = \frac{dt_o}{a_{obs}}$$

$$P |_{\text{detector}} = \frac{dE_{obs}}{dt_{obs}} \cdot \frac{A}{A_{\text{true}}} \quad \leftarrow \text{area of detector}$$

$$E \sim \frac{1}{\lambda} \sim \frac{1}{a}$$

$$= \left(\frac{a_e}{a_{obs}} \right)^2 \frac{dE_e}{dt_e} \cdot \frac{A}{4\pi r_s^2 a_o^2}$$

L physical distance

$$E_o = E_e \cdot \frac{a_e}{a_o}$$

$$= \left(\frac{a_e}{a_o} \right)^2 L \frac{A}{4\pi a_o^2 r_s^2} \quad \frac{a_e}{a_o} = \frac{1}{1+z}$$

Flux at the detector

$$F = \frac{P}{A} = \frac{1}{(1+z)^2} \frac{1}{4\pi r_s^2} \frac{1}{a_o^2} = \frac{1}{4\pi d_L^2}$$

\uparrow
source comoving distance.

luminosity distance $d_L \equiv (1+z) a_o r_s$

Let's try to get rid of r_s

$$\int_0^{r_s} \frac{dr}{1-kr^2} = \int_{t_e}^{t_o} \frac{dt}{a(t)} = \begin{cases} \arcsin r_s \approx r_s + r_s^3/6 & k=+1 \\ r_s & k=0 \\ \operatorname{arcsinh} r_s \approx r_s - r_s^3/6 & k=-1 \end{cases}$$

Now we expand

Here $t = t_0 - \delta t$

$$a(t) = a_0 + \dot{a}_0 (t-t_0) + \frac{\ddot{a}_0}{2} (t-t_0)^2$$

$$\frac{1}{1+z} = \frac{a(t)}{a_0} = 1 + H_0 (t-t_0) - \frac{q_0 H_0^2}{2} (t-t_0)^2 \quad (*)$$

where $H = \frac{\dot{a}}{a}$ $q \equiv -\frac{\ddot{a}}{aH^2}$ \leftarrow it's called
deceleration parameter

Going back to

$$d_L = 4\pi a_0 r_s^2$$

Recast (*) as

$$\frac{a(t_0)}{a(t)} = \frac{a(t_0 - \delta t)}{a(t_0)} = 1 - \frac{\dot{a}}{a_0} \delta t + \frac{\ddot{a}}{2} \frac{\delta t^2}{a_0}$$

At the end

$$H_0 \delta t \approx \frac{z}{1+z} - q_0 \frac{z^2}{(1+z)^2}$$

you can get to

$$H_0 d_L = z + \frac{1}{2}(1-q_0) z^2 + \dots$$

* We need some kind of
standard candle \Rightarrow
objects of known L ,
measure $F \rightarrow$ compute d_L
as a function of their z

Dynamics

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Evolution of scale factor will be given by the solution to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

\uparrow Ricci tensor \uparrow Ricci scalar \uparrow stress-energy tensor

second derivatives of the potential

We can add a term $+ \Lambda g_{\mu\nu}$ for a cosmological constant so

$$T_{\mu\nu} \sim \rho \Lambda$$

$$\rho \Lambda = \frac{\Lambda}{8\pi G}$$

Equivalent to $\nabla^2 \phi = 4\pi G \rho$. Poisson equation

• Here stress-energy conservation means $\nabla_\nu (T_{\mu\nu}) = 0$

Homogeneity and isotropy impose

$$T^{\mu}_{\nu} = \begin{pmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix}$$

In general

$$T_{\mu\nu} = -p g_{\mu\nu} + (\rho + p) u_\mu u_\nu$$

for a perfect fluid (in our

case at rest in

com. coordinates

$$\text{so } u^0 = 1 \quad u_i = 0$$

From the conservation equation

$$d(\rho a^3) = -p da^3$$

\nwarrow work

"change in Energy in com volume"

Now we need the equation of state

$$p(t) = w \rho(t) \quad \text{with } w \text{ some constant}$$

$$d(\rho a^3) = -p da^3$$

↓

$$\rho = a^{-3(1+w)}$$

Let's consider some special cases

1) Matter : $w \approx 0 \rightarrow \rho \sim a^{-3}$ ✓ mass conservation

Recall this is relativistic pressure

$$\frac{p}{c^2} = w \rho \quad \text{in a normal fluid } p = v^2 \rho$$

$$w \approx \left(\frac{v}{c}\right)^2 \approx 0.$$

2) Radiation

$$p = \frac{1}{3} \rho \rightarrow \rho_R \sim \frac{1}{a^4}$$

3) Vacuum energy

$$p = -\rho \rightarrow w = -1 \rightarrow \rho_\Lambda = \text{constant}$$

Friedman equations (00 and ii components of Einstein eqs.)

(7)

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p$$

which can of course be written as

$$(1) \quad H \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}$$

$k = 0$ flat

$k = +1$ closed

$k = -1$ open

→ Gives the expansion rate H of the Universe

$$(2) \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p)$$

→ Shows how the Universe accelerates; given by $\rho + 3p$

(in GR pressure exerts gravity). Negative pressure gives

acceleration.

$$\frac{k}{a^2 H^2} = -1 + \frac{\rho}{\rho_{\text{crit}}} \quad \rho_{\text{crit}} = \frac{3H^2}{8\pi G} \quad \text{critical density}$$

today this is $\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G} = 1.87 \times 10^{-29} \text{ g } \frac{\text{cm}^2}{\text{cm}^3}$.

We define all densities w.r.t ρ_{crit}

$$\Omega_i = \frac{\rho_i}{\rho_{crit}}$$

$$\Omega_{TOT} = \sum_i \Omega_i$$

We can also write $\frac{k}{a^2 H^2} = \Omega_{TOT} - 1$ (*)

Recall

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{crit}} = \frac{\Lambda}{3H^2}$$

$$\boxed{\rho_\Lambda = \frac{\Lambda}{8\pi G}}$$

$$- \Omega_k \equiv \frac{k}{a^2 H^2} \quad (**)$$

From (*) you get

- $\Omega_{TOT} = 1$ flat
- $\Omega_{TOT} > 1$ closed.
- $\Omega_{TOT} < 1$ open

From (**) $\Omega_{TOT} + \Omega_k = 1$ always!

Going back to the deceleration parameter

$$q = - \frac{\ddot{a}}{aH^2} = + \frac{4\pi G}{3H^2} (\rho + 3p) =$$

$$\rho + 3p = \rho (1 + 3w) = \begin{cases} \rho_m \\ 2\rho_r \\ -2\rho_\Lambda \end{cases}$$

ii) Flatness Problem

Go back to
$$-\Omega_k = \frac{k}{a^2 H^2} = \Omega_{TOT}^{-1}$$

At early times $\frac{1}{a^2 H^2} \sim \frac{1}{a^2 \rho} \sim \{a, a^2\}$ so $\Omega_k \xrightarrow[t \rightarrow 0]{a \rightarrow 0} 0$
 curvature is negligible.

$\frac{1}{a^2 \rho} \leftarrow \begin{matrix} \frac{1}{a^3} \text{ MAT} \\ \text{or} \\ \frac{1}{a^4} \text{ RAD} \end{matrix}$

$$\Omega_k \sim \begin{cases} a & \text{MAT} \\ a^2 & \text{RAD} \end{cases} \quad \Omega_{TOT} \xrightarrow[a \rightarrow 0]{} 1.$$

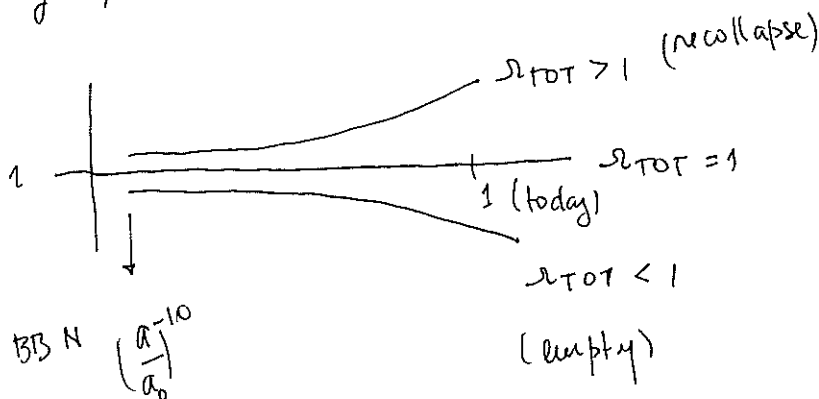
The other way around ;

$$\Omega_{TOT} (a)^{-1} = (\Omega_{TOT}^i)^{-1} \left(\frac{a}{a_i} \right) \quad \text{or} \quad (\Omega_{TOT}^i)^{-1} \left(\frac{a}{a_i} \right)^2$$

so

$$\Omega_{TOT} (a)^{-1} \approx (\Omega_{TOT}^i)^{-1} \left(\frac{a}{a_i} \right)^{1 \text{ or } 2}$$

Ω_{TOT} grows very fast with a , unless Ω_{TOT}^i was very close to 1. we would see large values of $\Omega_{TOT} \gg 1$ or $\ll 1$ today ; but observationally we don't



its a fine tuning problem \Rightarrow Universe is very close to flat

$$q = + \frac{4\pi G}{3H^2} (\rho_M + 2\rho_R - 2\rho_\Lambda) = + \frac{1}{2\rho_{crit}} (\rho_M + 2\rho_R - 2\rho_\Lambda) \quad (8)$$

$$= \frac{\Omega_M}{2} + \Omega_{RAD} - \Omega_\Lambda \approx \frac{\Omega_M}{2} - (1 - \Omega_M) = \frac{3}{2}\Omega_M - 1$$

Recall that

if that at late time $\Omega_M + \Omega_\Lambda = 1$ ($\Omega_{TOT} = 1$); $\Omega_{RAD} \sim 0$

$$q \approx \frac{3}{2}\Omega_M - 1$$

In general for one component $p = w\rho$ dominating the Universe

$$q = \frac{1}{2\rho_{crit}} (\rho + 3p) = \Omega_{TOT} \left(\frac{1+3w}{2} \right) > 0!$$

Recall that $q \sim -\ddot{a}$ so Universe accelerates for $q < 0$ or

$$w < -1/3 \quad (\text{not for matter!})$$

Static Universe (if we were to tune the value of Λ)

$$H = 0 \rightarrow \frac{8\pi G}{3}\rho = \frac{k}{a^2}$$

$$\ddot{a} = 0 \rightarrow \rho + 3p = 0 \rightarrow w = -1/3$$

$$\frac{3k}{8\pi G a^2} = \rho = -3p = -3p_\Lambda = 3p_\Lambda = \frac{3\Lambda}{8\pi G}$$

$$\Lambda = \frac{k}{a^2}$$

only Λ has pressure

Unstable to perturbations!

Some solutions to the Friedman eqs

(9)

i) Flat, MATTER dominated Universe $k=0$, $\Omega_M=1$

$$H^2 = \frac{8\pi G}{3} \rho_M \sim a^{-3} \quad \left(\frac{\dot{a}}{a}\right)^2 \sim a^{-3} \quad \dot{a} \sim \frac{1}{\sqrt{a}}$$

$$\sqrt{a} da \sim dt \quad d(a^{3/2}) \sim dt \quad \boxed{a(t) \sim t^{2/3}}$$

ii) Flat, RAD dominated

$$H^2 = \frac{8\pi G}{3} \rho_R \sim a^{-4} \quad \dot{a} \sim \frac{1}{a} \quad a da \sim dt$$

$$\boxed{a(t) \sim t^{1/2}}$$

iii) General case $p = w\rho$

$$\boxed{a \sim t^{2/3(1+w)^{-1}}}$$

iv) Flat, Λ dominated

$$H = H_0 \quad \dot{a} \sim a \quad \boxed{a \sim e^{H_0 t}} \quad \text{exponential expansion}$$

Acceleration of the Universe : SNIa results

(10)

- One way of measure the expansion of the Universe is to find standard variables ; in the case of SNIa their luminosity can be standardized so you can then measure Flux and d_L

$$d_L \equiv \frac{L}{4\pi F} = c(1+z) \int_0^z \frac{dz'}{H(z')}$$

$$H_0 = \dot{a}/a$$

$$q = -\frac{\ddot{a}}{aH^2}$$

$$j_0 \approx \ddot{a}''$$

$$d_L(z) = \frac{cz}{H_0} \left\{ 1 + \frac{1}{2} (1 - q_0)z - \dots \right\}$$

The acceleration gives a quadratic correction to Hubble's law (while a cubic term is change in acceleration)

Let's move ~~from~~ to magnitudes

M : absolute magnitude $\sim \log L$

m : apparent " $\sim \log F$

$M - m \equiv$ distance modulo \rightarrow log measurement of d_L

$M \equiv$ apparent magnitude an object would have at 10 pc

$$L(d) = \frac{L(10)}{(d/10)^2}$$

$$m = -2.5 \log_{10} L(d)$$

↑ luminosity at dist. d

$$m = -2.5 \log_{10} L(d) = -2.5 \log_{10} \left\{ L(l_0) \left(\frac{l_0}{d} \right)^2 \right\}$$

$$= \underbrace{-2.5 \log_{10} L(l_0)}_M - 5 \log_{10} (l_0/d_L)$$

$$m - M = 5 \log_{10} (d_L/l_0)$$

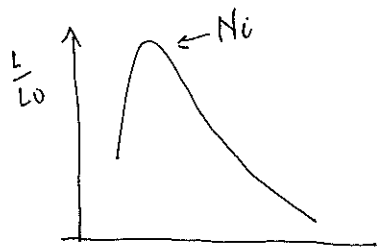
"standardizable candles".

Type Ia SN occur in binary systems (two stars orbiting each other). ~~The other star can be another~~ One star is a white dwarf; the other star can be another WD or a giant star or a smaller WD. Usually these WD are made of Carbon and Oxygen. The WD accretes from the companion star until the core reaches the ignition temp. for carbon fusion,* at about $1.44 M_{\odot}$ (sometimes referred as the Chandrasekhar limit).

(* and undergoes a thermonuclear reaction triggering a SN explosion.

The peak luminosity in all these systems are very similar as they originate from the same physical process.

In reality there is a variance of about 40% at peak 11
luminosity related to the amount of Ni in the
progenitor.. But this is very correlated with the width



of the light curve

decay \rightarrow the integral is a
measure of the bolometric mag.
(total luminosity)

This brings the 40% scatter to about 15%.

Less Nickel decays faster.

Horizon

At $t = t_0$ (present time) is the 3D separating particles that have been ~~in~~ already seen by an observer at t_0 from those that have not yet

"These are points that could have been in causal contact sometime in the life of the Omniverse"

$$ds^2 = \text{null geodesic}$$

$$\text{"comoving distance"} = \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} = c \int_0^{t_0} \frac{dt}{a}$$

← total comoving distance light could have traveled until t_0

"physical" horizon at t_0 is

$$d_H(t_0) = a(t_0) \int_0^{t_0} c \frac{dt}{a(t)}$$

conformal time.

← Farthest that a particle could have traveled by time t_0 .

if $a \sim t^n$ ($n = 2/3, 1/2$ for MAT, RAD) and $n < 1$

$$d_H(t_0) = \frac{ct}{1-n}$$

MAT = $d_H(t) = 3ct.$

RAD = $d_H(t) = 2ct.$

physical

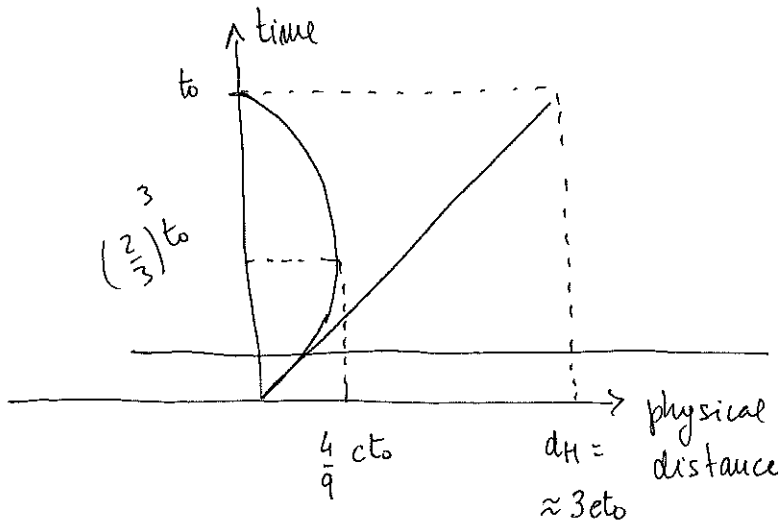
Notice this is larger than $c \cdot t$. But this is NOT the distance that light has traveled but rather the physical ~~to~~ position at t_0 of an object that sent a light ray at big bang arriving now at t_0
 → farthest object in the Universe for us.

Past light-cone:

This is our standard idea of a past lightcone,

where are the objects that we see now at t_0
 ↓
 at time $t < t_0$

As far as we can see now.



effectively this is cut by
 last scattering surface.

$$l(t) = a(t) \int_t^{t_0} \frac{c dt'}{a(t')} = 3c \left(t^{2/3} - t_0^{1/3} \right)$$

↑
 convert to
 physical
 at t .

⇓
 comoving distance to light
 emitted at t that gets
 to us at t_0 .

for MAT
 $a = t^{2/3}$

Maximum $\frac{\partial l}{\partial t} = 0 \rightarrow \frac{t}{t_0} = \left(\frac{2}{3}\right)^3$ $l_{\max} = \frac{4}{9} c t_0$

Hubble Radius : $c H^{-1}$: is a local quantity ; not integrated ~~(13)~~ (13)
 and note that $H \sim 1/t$ so H^{-1} is the local "time scale".

At time ~~to~~ t

$$v = H d \quad \text{so} \quad d = H^{-1} c$$

d_{Hubble} : surface at which recession speed is equal to speed of light.

$$v_H = H d_H = \frac{2}{3t} \cdot 3ct = 2ct \quad \text{recession speed of the horizon.}$$

Hubble radius shows up in equations of motion.

Physical distance of a worldline of an object with constant speed comoving position

$$\lambda_{\text{phys}} = a(t) \lambda_{\text{comoving}}$$

So

$$d_H(t) = \begin{cases} 3t & \text{MAT} & t \sim a^{3/2} \\ 2t & \text{RAD} & t \sim a^2 \end{cases}$$

$$H^{-1} = \begin{cases} \left(\frac{2}{3t}\right)^{-1} & \text{MAT} & t \sim a^{3/2} \\ \left(\frac{1}{2t}\right)^{-1} & \text{RAD} & t \sim a^2 \end{cases}$$

$$\lambda_{\text{phys}} \sim a \quad \text{always.}$$

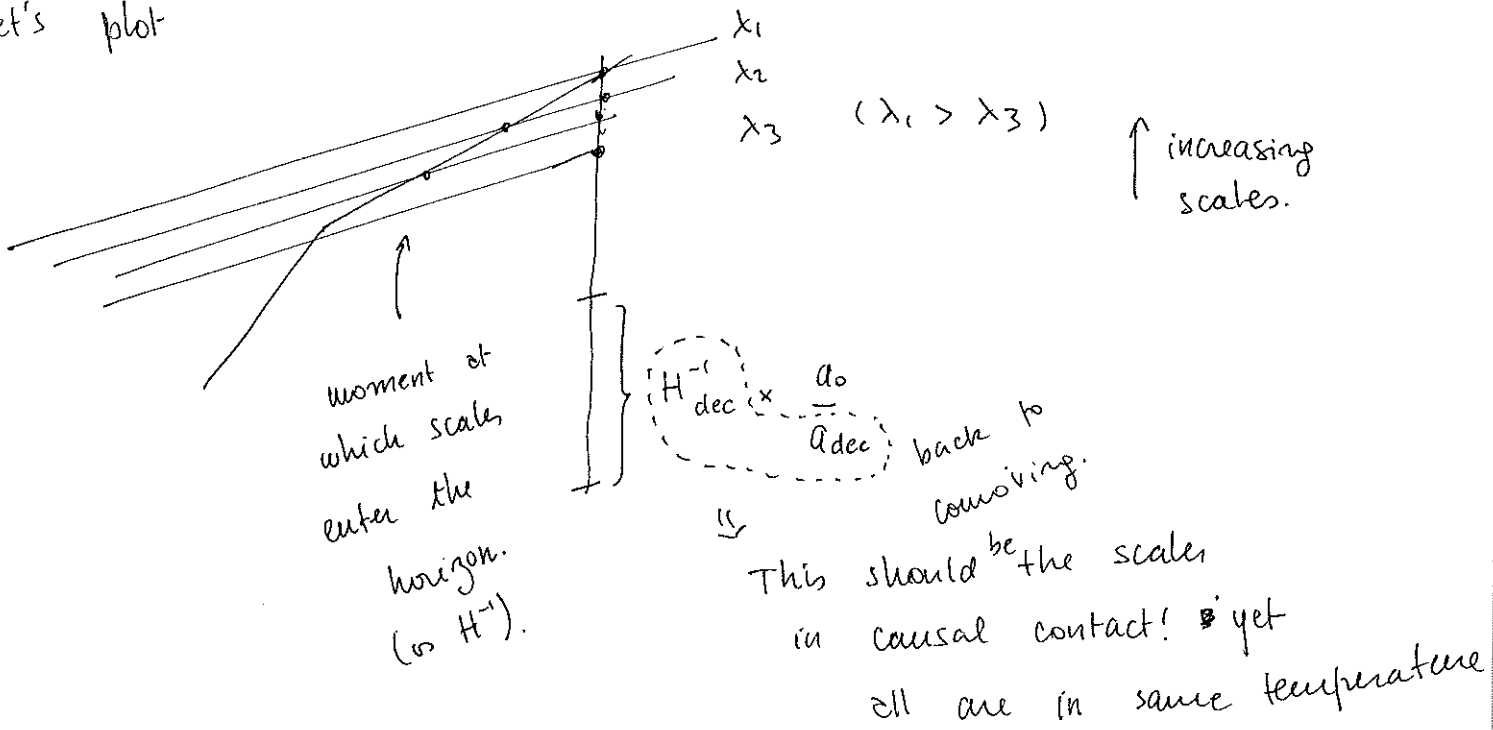
Note that if $a \sim t^n$

$$\lambda_{\text{phys}} \sim a \sim t^n \quad \text{with } n < 1$$

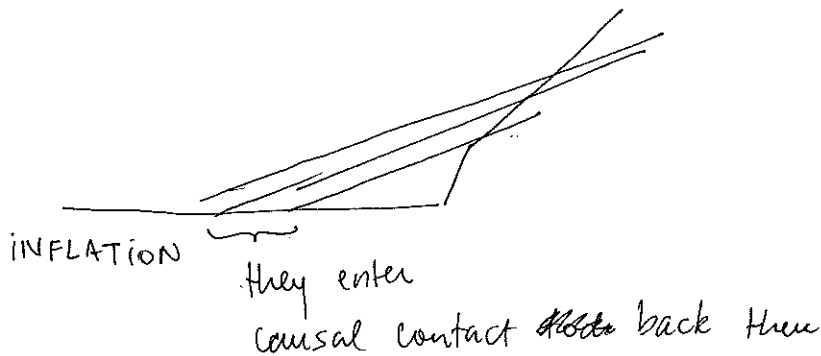
But $d_H, H^{-1} \sim t \rightarrow$ so the horizon grows faster than physical scales; this means that scales "enter the horizon" eventually.

Horizon Problem.

Let's plot



Problem with CMB temperature $\rightarrow H^{-1}_{\text{dec}}$.



$$t_{\text{dec}} \gg t_{\text{MATTER-RAD}}$$

Matter-RAD equality happened before decoupling of photons (at $z \sim 1100$)

~ 2740
↓

Particle horizon (Horizon) : maximum distance from which particles ⁽¹⁴⁾ could have travel to the observer in the age of the Universe.

Boundary between observable and unobservable Universe. It represents the furthest distance from which we can retrieve info ~~re~~ from the past. (about 46 billion light years)