General Relativity - Background evolution

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \; T_{\mu\nu} + \Lambda \; g_{\mu\nu}$$

the metric reads (assuming Universe is homogeneous + isotropic).

$$ds^2 = c^2 \underbrace{dt^2}_{\text{proper time.}} - a^2(t) \left[f(r) \, dr^2 + g(r) \, d\psi^2 \right]$$

solution to EQ.

$$f = \frac{1}{1 - kr^2} \qquad k = \begin{cases} 0 & \text{flat} \\ -1 & \text{closed} \\ 1 & \text{open.} \end{cases}$$

spherical coordinates.

$$d\psi = r^2 \, d\theta^2 + r^2 \sin^2\theta \; d\phi^2$$

$$g_{\mu\nu} = \begin{pmatrix} c^2 & & & \\ & -a^2/(1-kr^2) & & \\ & & -a^2 r^2 & \\ & & & \ddots \; -a^2 r^2 \sin^2\theta \end{pmatrix}$$

going back to Einstein equations

$$T^\mu{}_\nu = \begin{pmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix} \xrightarrow{\partial_\mu T^\mu{}_\nu} d(\rho a^3) = -p \, da^3$$

$d(\rho a^3)$: change in energy; da^3: change in volume.

$$\rho \sim a^{-3(1+w)} \qquad p = w\rho$$

$$w = \begin{cases} 0 & \text{matter} \\ 1/3 & \text{rad} \\ -1 & \Lambda \end{cases}$$

00: Friedmann Eq.

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

ii:

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

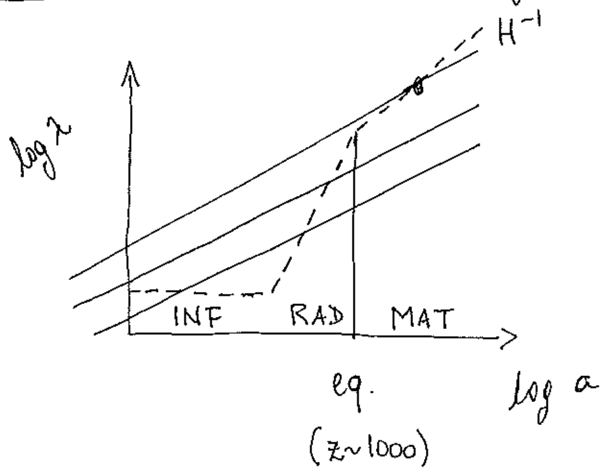
$\longrightarrow$ determines whether Universe accelerates or not

$H = \frac{\dot a}{a}$ $\leftarrow$ Hubble function $\leftrightarrow$ determines the cosmic history. (e.g. determine "distances" $\rightarrow$ to supernovae)

$H^2 = \frac{8\pi G}{3}\rho$

<u>Evolution of scales</u>

physical separations $\lambda \sim a$



$H^{-1} \sim$ const (Inflation)

$H^{-1} \sim a^2$ (RAD)

$H^{-1} \sim a^{3/2}$ (MAT)

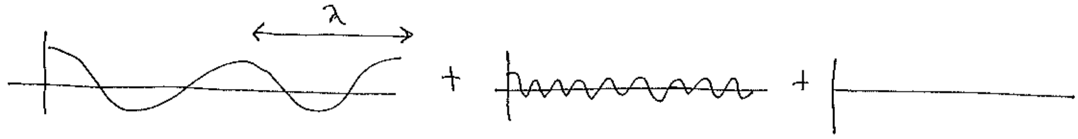
we will now discuss the evolution of perturbations to ρ

$\rho = \rho(1+\delta)$ $\rightarrow$ if scales are larger than H^{-1} $\rightarrow$ GR

smaller $\rightarrow$ Newtonian

Explain definition of scale or separation

ρ



Similarly you can think of doing spheres of size λ or correlation between points separated by λ.

After inflation modes of cosmological scales are outside the horizon (or Hubble radius) and re-enter later when universe is dominated by RAD or MAT. depending on scale of interest. Evolution is then divided as

$$\begin{cases} t < t_{enter}(\lambda) & \text{when } \lambda > H^{-1} \quad \text{must use GR} \\ t > t_{enter}(\lambda) & \text{when } \lambda < H^{-1} \quad \text{Newton analysis} \end{cases}$$

Let's now focus on matter perturbations during MAT era.

<u>Newtonian Linear PT</u>

$$\underset{\text{physical}}{r(t)} = a(t) \underset{\text{comoving}}{x(t)} \longrightarrow \underset{\text{physical}}{\vec{v}} = \underset{\text{Hubble flow}}{\vec{r}\, H} + \underset{\text{"peculiar"}}{\vec{v}_p} = \frac{d\vec{x}}{d\tau}$$

$$= \mathcal{H} \vec{x} + \vec{v}_p$$

$$\mathcal{H} = \frac{1}{a}\frac{da}{d\tau} = aH.$$

$a\, d\tau = dt \qquad \tau$ conformal time.

Definicion $\quad \dfrac{\dot{a}}{a} \equiv H$

$$\frac{\partial \rho}{\partial t} \cancel{+ \vec{v}_0 \cdot \vec{\nabla}\rho}$$

$$\cancel{d\rho + 3 a^2 \, da \, \rho = 0}$$

$$\cancel{\frac{d\rho}{dt} + 3 a^2 \frac{da}{dt} \rho} \qquad \cancel{H = \frac{1}{a}\frac{da}{dt}}$$

$$\cancel{a^2 \frac{d\rho}{dt} = -3 a^3 H \rho}$$

Equations of motion

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_{\vec{r}} [\rho \vec{v}] = 0 \qquad \text{conservation of mass}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} \vec{v} = -\frac{\vec{\nabla} p}{\rho} - \vec{\nabla}_r \Phi_{tot} \qquad \text{conservation of momentum}$$

$$\nabla_r^2 \Phi_{tot} = 4\pi G \rho \qquad \text{Poisson Eq.}$$

total grav. potential (background + pert.)

They can be derived from the Vlasov equation (see Bernardeau et al 2002)

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{\vec{p}}{am} \cdot \vec{\nabla} f - a m \nabla \phi \frac{\partial f}{\partial \vec{p}} = 0$$

$$\int d^3p \, f(x, p, \tau) = \rho(x, \tau)$$

$$\int d^3p \, \frac{\vec{p}}{am} f(x, p, \tau) \equiv \vec{u}(\vec{x}, \tau) \, \rho(x, \tau)$$

$$\vdots$$

We have to transform to ~~phys~~ comoving coordinates.

$$\Phi_{tot} = \Phi_b + \phi$$

$$\Phi_b = \frac{2\pi G}{3} r^2 \bar{\rho}(t) \quad \text{follows from} \quad \nabla_r^2 \Phi_b = 4\pi G \bar{\rho}$$

from $(\vec{r}, t) \longrightarrow (\vec{x}, \tau)$. Now thing are functions of $\vec{x}, \tau$

$$f = f(\vec{x}, \tau)$$

$$\vec{r} = a\vec{x} \qquad dt = a\, d\tau$$

$t = t(\tau, x)$

$$\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial \tau}{\partial t}\right|_{\vec{r}} \frac{\partial}{\partial \tau} + \left.\frac{\partial \vec{x}}{\partial t}\right|_{\vec{r}} \frac{\partial}{\partial \vec{x}}$$

$$\boxed{\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \frac{1}{a}\frac{\partial}{\partial \tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}}$$

$$\frac{\partial x}{\partial t} a + x \frac{\partial a}{\partial t} = 0$$

$$\frac{\partial x}{\partial t} = -x \frac{\partial a}{\partial t}\frac{1}{a}$$

$$= -x H = -x\frac{\mathcal{H}}{a}$$

$$\mathcal{H} = aH$$

$r = r(\tau, x)$

$$\left.\frac{\partial}{\partial r}\right|_t = \underbrace{\left.\frac{\cancel{\partial \tau}}{\partial r}\right|_t}_{=0} \frac{\partial}{\partial \tau} + \underbrace{\left.\frac{\partial x}{\partial \vec{r}}\right|_t}_{= 1/a} \cdot \frac{\partial}{\partial x}$$

$$\boxed{\vec{\nabla}_{\vec{r}} = \frac{1}{a}\vec{\nabla}}$$

Now take

$$\Phi_{tot} = \phi_b + \phi$$

$$\rho = \rho(1+\delta) = \rho + \delta\rho$$

$$p = p + \delta p$$

$$H = H + \delta H$$

$$\vec{V} = H\vec{r} + \vec{v}_p$$

$$\downarrow$$

$$\delta H = \frac{1}{3}\vec{\nabla}_r v_p$$

continuity mass $\longrightarrow$ $$\boxed{\frac{\partial \delta}{\partial \tau} + \vec{\nabla}\left[(1+\delta)\vec{v}\right] = 0.}$$

momentum $\longrightarrow$ $$\frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\left[\Phi_{tot} + \frac{1}{2}\frac{\partial \mathcal{H}}{\partial \tau}x^2\right] - \frac{\vec{\nabla}p}{\rho}$$

Recall that $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho$ $\quad \mathcal{H} = aH \quad \frac{\partial \mathcal{H}}{\partial \tau} = \frac{\partial \dot{a}}{\partial \tau} = a \cdot \ddot{a}$

$$\frac{\partial \mathcal{H}}{\partial \tau} = -\frac{4\pi G}{3}\rho a^2$$

$(\rho + 3p)$ more generally

you get to

$$\boxed{\frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\nabla\Phi - \frac{\vec{\nabla}p}{\rho}}$$

in the same way; using $H^2 = \frac{8\pi G}{3}\rho \quad \Omega_m = \frac{\rho}{\rho_{crit}} \quad \rho_{crit} = \frac{3H^2}{8\pi G}$

$\downarrow$

$$\boxed{\nabla_r^2\Phi = \frac{3}{2}\mathcal{H}^2\Omega_m\delta} \longleftrightarrow \nabla_r^2\phi = 4\pi G\delta\rho$$

$$\frac{1}{a^2}\nabla^2\Phi = 4\pi G\rho_m\delta_m$$

$$= 4\pi G\rho_{crit}\Omega_m\delta_m$$

$$= \cancel{4\pi} G\frac{3H^2}{8\pi G}\Omega_m\delta_m$$

$$\nabla^2\Phi = \frac{3}{2}\mathcal{H}^2\Omega_m\delta_m$$

In general $\quad \nabla_r^2\Phi = 4\pi G\delta\rho$

$$\nabla^2\Phi = 4\pi G a^2\delta\rho$$

Using (3b)

$$\frac{\partial}{\partial t} = \frac{1}{a}\frac{\partial}{\partial \tau} - \frac{\mathcal{H}}{a}\,\vec{x}\cdot\vec{\nabla} \quad ; \quad \vec{\nabla}_r = \frac{1}{a}\vec{\nabla} \quad ; \quad \begin{array}{l}\rho = \bar{\rho}(1+\delta) \\ \vec{v} = H\vec{r} + \vec{v}_p\end{array} \quad ; \quad \begin{array}{l} r = ax \\ \bar{\rho} \sim a^{-3}\end{array}$$

$$\frac{\partial \rho}{\partial t} = (1+\delta)\frac{\partial \bar{\rho}}{\partial t} + \bar{\rho}\frac{\partial \delta}{\partial t} = -3H\rho + \bar{\rho}\frac{\partial \delta}{\partial t} \qquad (1)$$

$$\vec{\nabla}_r(\rho\vec{v}) = \rho\,\vec{\nabla}_r\vec{v} + \vec{v}\,\bar{\rho}\,\vec{\nabla}_r(1+\delta)$$

$$\vec{v} = H\vec{r} + \vec{v}_p \quad Eq(*)$$

$$\nabla_r\vec{v} = 3H + \vec{\nabla}_r\vec{v}_p$$

$$= 3H\rho + \rho\,\vec{\nabla}_r\vec{v}_p + \vec{v}\,\bar{\rho}\,\vec{\nabla}_r(1+\delta) \qquad (2)$$

$$(1)+(2) = \bar{\rho}\frac{\partial\delta}{\partial t} + \rho\,\vec{\nabla}_r\vec{v}_p + \vec{v}\,\bar{\rho}\,\vec{\nabla}_r(1+\delta)$$

use Eq(*)

$$\bar{\rho}\frac{1}{a}\frac{\partial\delta}{\partial\tau} \cancel{-\bar{\rho}\frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\delta} + \frac{\bar{\rho}}{a}(1+\delta)\vec{\nabla}\vec{v}_p + \cancel{\bar{\rho}Ha\hat{x}\,\vec{\nabla}_r\delta} + \vec{v}_p\frac{\bar{\rho}}{a}\vec{\nabla}\delta$$

$$\frac{\bar{\rho}}{a}\left\{\frac{\partial\delta}{\partial\tau} + (1+\delta)\vec{\nabla}\vec{v}_p + v_p\vec{\nabla}(1+\delta)\right\} = 0$$

$$\boxed{\frac{\partial\delta}{\partial\tau} + \vec{\nabla}\left((1+\delta)\vec{v}_p\right) = 0}$$

✓

Solutions to linear eqs.

$$\frac{\partial \delta}{\partial \tau} + \nabla \left[(1+\delta) \vec{v} \right] = 0$$

$$\frac{\partial \vec{v}}{\partial \tau} + \mathcal{H} \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \vec{\nabla} \phi - \frac{\vec{\nabla} p}{\rho}$$

$$\nabla^2 \phi = \frac{3}{2} \mathcal{H}^2 \Omega_m \delta .$$

to linear order:

(1) $$\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot \vec{v} = 0 \qquad \theta \equiv \vec{\nabla} \cdot \vec{v} \qquad \vec{w} = \vec{\nabla} \times \vec{v}$$

(2) $$\frac{\partial \theta}{\partial \tau} + \mathcal{H} \theta = \underbrace{- \frac{3}{2} \mathcal{H}^2 \Omega_m(\tau) \delta}_{\nabla_x^2 \phi = 4\pi G a^2 \delta \rho} - \frac{\nabla^2 p}{\bar{\rho}}$$

(3) $$\frac{\partial \vec{w}}{\partial \tau} + \mathcal{H} \vec{w} = 0 .$$

To see what's going on with Eq(3) let's assume $\Omega_m = 1$ (EdS). We need $\mathcal{H} = \mathcal{H}(\tau)$

Recall Friedman Eqs in conformal coordinates

$$\frac{\partial \mathcal{H}}{\partial \tau} = - \frac{\Omega_m(\tau)}{2} \mathcal{H}^2(\tau) + \frac{\Lambda}{3} a^2 = \left(\Omega_\Lambda - \frac{\Omega_m}{2} \right) \mathcal{H}^2 . \checkmark$$

$$(\Omega_{tot} - 1) \mathcal{H}^2 = k \qquad \Omega_m = \Omega_M + \Omega_\Lambda$$

but we can readily see that $\frac{W'}{W} = - \mathcal{H} < 0$ always

meaning that any initial vorticity decays w/ time!
it's not sourced by density perturbations.

Notice that

$$\mathcal{H}\, d\tau = \frac{da}{a} = d\ln a \quad ✓ \qquad \frac{da}{a} = \frac{\dot{a}\, dt}{a} = \underbrace{a H}_{\mathcal{H}} \underbrace{\frac{dt}{a}}_{d\tau} = d\ln a.$$

$$\frac{1}{w} \frac{\partial w}{\partial \ln a} = -1 \qquad \frac{\partial \ln w}{\partial \ln a} = -1 \qquad \ln w = -\ln a \qquad \boxed{w \simeq \frac{1}{a}}$$

Solutions for velocity divergence and density.

For simplicity let's do the fourier transform: $\vec{\nabla} \to i\vec{k}$, $\nabla^2 \to -k^2$

$$\frac{\partial \delta_k}{\partial \tau} + \theta_k = 0$$

$$\frac{\partial \theta_k}{\partial \tau} + \mathcal{H}\theta_k = \frac{3}{2}\Omega_m \mathcal{H}^2 \delta_k + k^2 \frac{p_k}{\bar{\rho}}$$

$\mathcal{H}\theta_k$: (suppresion of growth) friction term

$\frac{3}{2}\Omega_m \mathcal{H}^2 \delta_k$: Gravity (enhances growth)

$k^2 \frac{p_k}{\bar{\rho}}$: suppresion of growth at high k (small scales) pression

k is a frequency
low k → large λ

a) Pressureless

$$\frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_k}{\partial \tau} = \frac{3}{2}\Omega_m \mathcal{H}^2 \delta_k$$

local in space.

$$\delta_k(\tau) = D(\tau)\, A_k \qquad \left(\text{or} \quad \delta_R = D(\tau)\, \delta_R(\tau=0)\right)$$

$$\frac{d^2 D}{d\tau^2} + \mathcal{H}\frac{dD}{d\tau} = \frac{3}{2}\Omega_m(\tau)\, \mathcal{H}^2 D$$

D is the growth factor

Note:
Equations are separable as long as $\frac{f}{\Omega^2} \sim 1$ => this is a quite good approx.

Solution for EdS

$$\boxed{\mathcal{H} = \frac{2}{\tau} \qquad \Omega_m = 1 \qquad a \sim \tau^2}$$

$$\frac{d^2 D}{d\tau^2} + \frac{2}{\tau}\frac{dD}{d\tau} = \frac{3}{2}\overbrace{\cancel{\Omega_m}}^{=1}\frac{4}{\tau^2} D$$

$$H^2 = \frac{8\pi G}{3}\frac{\rho}{a^3}$$

take $D = \tau^n$

$$\frac{\mathcal{H}^2}{a^2} = \frac{1}{a^3}$$

$$\left[\frac{1}{a}\frac{\partial a}{\partial \tau}\right]^2 = \frac{1}{a}$$

$$n(n-1) + 2n - 6 = 0$$

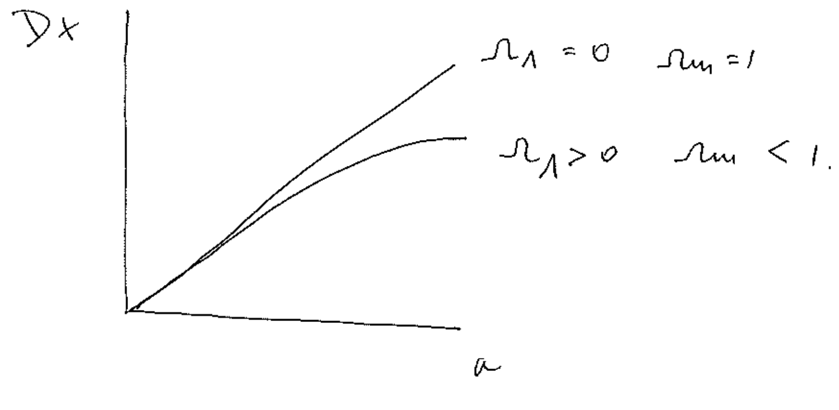
$$\frac{\partial a}{\partial \tau} = a^{1/2}$$

$$n = \frac{-1 \pm 5}{2} = \begin{cases} -3 \\ 2 \end{cases}$$

$$a \sim \tau^2$$

$$D_+ \sim \tau^2 \sim a$$

$$D_- \sim \tau^{-3} \sim a^{-3/2}$$



slow down of growth.

Typical Exam Question: by how much perturbations grow from z_1 to z_2 in an EdS Universe.

Solution for EdS $\quad \mathcal{H} = \frac{2}{\tau} \quad \Omega_m = 1 \quad a \sim \tau^2$

$$\begin{cases} D_+ \sim a & \text{"growing mode" solution} \\ D_- \sim a^{-3/2} & \text{"decaying mode"} \end{cases}$$

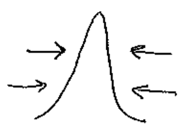
$$\boxed{\delta_k(\tau) = A_k \cdot a + B_k\, a^{-3/2}}$$

$$\theta_k(\tau) = -\frac{\partial \delta_k}{\partial \tau} = -a\mathcal{H}\frac{\partial \delta_k}{\partial a} = -a\mathcal{H}\left\{ A_k - \tfrac{3}{2} B_k\, a^{-3/2-1} \right\};$$

$$\boxed{\theta_k(\tau) = -\mathcal{H}\left\{ A_k\, a - \frac{3}{2} B_k\, a^{-3/2} \right\}}$$

Growing mode

$$\frac{\theta}{\mathcal{H}} = -\delta$$



$a \sim \tau^2$

$\ln a = 2\ln\tau + A$

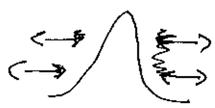
$\frac{da}{a} = \frac{2}{\tau} d\tau$

$\downarrow$

$da = a\,\mathcal{H}\, d\tau$

Decaying mode

$$\frac{\theta}{\mathcal{H}} = \frac{3}{2}\delta$$



Solution for the linear growth if only matter and vacuum energy (Ω_m, Ω_Λ) are present

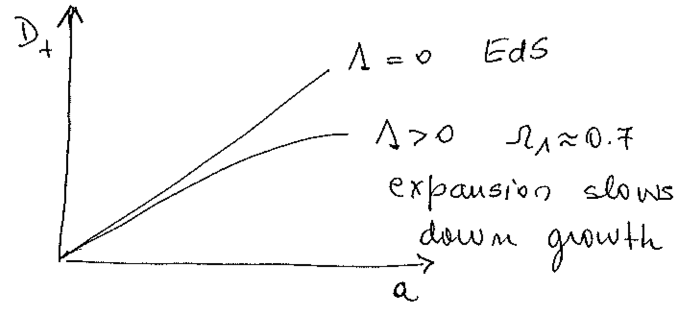
$$D_1^{(+)} = a^3 H(a)\,\frac{5\Omega_m(a)}{2}\int_0^a \frac{da}{a^3 H^3(a)} \qquad D_1^{(-)} = \frac{\mathcal{H}}{a} = H(a)$$

where $\quad H(a) = \left\{ \Omega_m^0\, a^{-3} + (1 - \Omega_m^0 - \Omega_\Lambda^0)\, a^{-2} + \Omega_\Lambda^0 \right\}^{1/2}.$

in the general case

$$\frac{\theta}{\mathcal{H}} \longrightarrow \theta/f\mathcal{H} \sim \delta$$

$$f = \frac{d \ln D}{d \ln a} \approx \Omega_m^{0.6}$$



2nd order PT:

From Eqs. of motion

$$\frac{\partial \delta}{\partial \tau} + \vec{\nabla}\left[(1+\delta)\vec{v}\right] = 0$$

$$\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta + (\vec{v}.\vec{\nabla})\vec{v} = -\nabla^2\phi = -\frac{3}{2}\mathcal{H}^2\Omega\,\delta.$$

the nonlinear terms are $\vec{\nabla}(\delta\vec{v})$ and $\vec{\nabla}(\vec{v}.\nabla)\vec{v}$. And we want to transform to fourier space where

$$\nabla.\vec{v} = \theta$$

$$\delta_{\vec{k}} = \int \frac{d^3x}{(2\pi)^3}\, e^{-i\vec{k}.\vec{x}}\, \delta_{\vec{x}}$$

$$\nabla.\vec{v} \Rightarrow i\vec{k}\cdot\vec{v}(\vec{k}) = \theta(\vec{k})$$

$$\boxed{\vec{v} = -i\,\frac{\vec{k}}{k^2}\theta(\vec{k})}$$

We want to write the above eqs. in Fourier Space so we need to multiply by $\int \frac{d^3x}{(2\pi)^3}\, e^{-i\vec{k}.\vec{x}}\,\{\ldots\}$

$$\int \frac{d^3x}{(2\pi)^3} \, e^{-i\vec{k}.\vec{x}} \quad \vec{\nabla}(\delta\vec{v})$$

$$\frac{1}{\mathcal{H}}\frac{\partial}{\partial \tau} = a\frac{\partial}{\partial a}$$

$$\mathcal{H}\,\partial\tau = \partial \ln a$$

$$\delta(x) = \int d^3k_1 \;\; \delta(\vec{k}_1)\; e^{i\vec{k}_1.\vec{x}}$$

$$\vec{v}(x) = \int d^3k_2 \;\; \vec{v}(\vec{k}_2)\; e^{i\vec{k}_2.\vec{x}}$$

$$\vec{\nabla} \to \int dk_1\,dk_2 \;\; \delta(\vec{k}_1)\,\vec{v}(\vec{k}_2)\cdot i\,(\vec{k}_1+\vec{k}_2) \;\; e^{i(\vec{k}_1+\vec{k}_2).\vec{x}}$$

Doing the outer integration in x

$$\int d^3k_1\,d^3k_2 \;\; \delta(\vec{k}_1)\;\underbrace{\vec{v}(\vec{k}_2)}_{\downarrow}\cdot i\,(\vec{k}_1+\vec{k}_2) \;\; \underbrace{\int \frac{d^3x}{(2\pi)^3}\, e^{i(\vec{k}_1+\vec{k}_2).\vec{x}}}_{\Downarrow}$$

$$-i\,\frac{\vec{k}_2}{k_2^2}\,\theta(\vec{k}_2) \qquad\qquad \delta_D(\vec{k}_1+\vec{k}_2)$$

$$\frac{\partial\delta_{\vec{k}}}{\partial\tau} + \theta_k = -\int dk_1\,dk_2 \;\; \underbrace{\frac{\vec{k}_2\cdot(\vec{k}_1+\vec{k}_2)}{k_2^2}}_{\Downarrow\; \alpha(\vec{k}_1,\,\vec{k}_2)} \;\; \delta_D(\vec{k}-k_{12}) \;\; \delta(k_1)\,\theta(k_2)$$

$$\frac{\partial\delta_{\vec{k}}}{\partial\tau} + \theta_k = -\int d^3k_1\,d^3k_2 \;\; \delta_D(\vec{k}-\vec{k}_{12})\;\alpha(\vec{k}_1,\vec{k}_2)\;\delta(k_1)\,\theta(k_2).$$

$$\frac{\partial\theta_k}{\partial\tau} + \mathcal{H}\,\theta_k + \frac{3}{2}\Omega_m\mathcal{H}^2\delta_k = -\int d^3k_1\,d^3k_2 \;\; \delta_D(\vec{k}-\vec{k}_{12})\,\beta(\vec{k}_1,\vec{k}_2)\;\theta(k_1)\,\theta(k_2)$$

Now notice $\displaystyle \frac{\partial}{\partial\tau}\left(\frac{\theta}{\mathcal{H}}\right) = \frac{1}{\mathcal{H}}\frac{\partial\theta}{\partial\tau} - \frac{\theta}{\mathcal{H}^2}\underbrace{\frac{\partial\mathcal{H}}{\partial\tau}}_{= -\frac{\mathcal{H}^2}{2}} \longrightarrow \frac{1}{\mathcal{H}^2}\frac{\partial\theta}{\partial\tau} = \frac{1}{\mathcal{H}}\frac{\partial(\theta/\mathcal{H})}{\partial\tau} - \frac{\theta}{\mathcal{H}2}$

using $\mathcal{H} = \frac{2}{\tau}$

$\alpha(\vec{k}_1, \vec{k}_2) = \dfrac{\vec{k}_{12} \cdot \vec{k}_1}{k_1^2}$
$\beta(\vec{k}_1, \vec{k}_2) = \dfrac{k_{12}^2 \, (\vec{k}_1 \cdot \vec{k}_2)}{2 k_1^2 k_2^2}$

Solution to 2nd order

$$\frac{1}{\mathcal{H}}\frac{\partial \delta_k}{\partial \tau} + \frac{\theta_k}{\mathcal{H}} = -\int dk_1 dk_2\, \alpha(\vec{k_1},\vec{k_2})\, \delta_D(\vec{k}-\vec{k}_{12})\, \delta(\vec{k_1})\,\theta(\vec{k_2})/\mathcal{H}$$

$$\frac{1}{\mathcal{H}}\frac{\partial}{\partial \tau}\left(\theta/\mathcal{H}\right) + \frac{1}{2}\frac{\theta}{\mathcal{H}} + \frac{3}{2}\Omega_m\, \delta = -\int dk_1\, dk_2\, \beta(k_1 k_2)\, \frac{\theta(\vec{k_1})}{\mathcal{H}}\frac{\theta(\vec{k_2})}{\mathcal{H}}$$

we seek for solutions of the form

$$\delta = \sum a^n \delta_n$$

$$\theta/\mathcal{H} = -\sum a^n \theta_n$$

at linear order we know (already solved) $\delta_1 = \theta_1$

$$\frac{1}{\mathcal{H}}\frac{\partial}{\partial \tau} = a\frac{\partial}{\partial a}$$

$$\delta = a\,\delta_1 + a^2\,\delta_2 \qquad \theta/\mathcal{H} = -a\,\delta_1 - a^2\theta_2$$

$$2\delta_2 - \theta_2 = +\alpha.\ \delta_1\delta_1 \longrightarrow 2\delta_2 - \theta_2 = \alpha\ \delta_1\delta_1$$

$$-2\theta_2 - \frac{1}{2}\theta_2 + \frac{3}{2}\delta_2 = -\beta\ \delta_1\delta_1 \longrightarrow -3\delta_2 + 5\theta_2 = 2\beta\ \delta_1\delta_1$$

$$\theta_2 = 2\delta_2 - \alpha\ \delta_1\delta_1$$

$$-3\delta_2 + 10\,\delta_2 - 5\alpha\,\delta_1\delta_1 = 2\beta\ \delta_1\delta_1$$

$$\delta_2 = \left(\frac{5\alpha + 2\beta}{7}\right)\delta_1\,\delta_1$$

$$\boxed{\delta_2(\vec{k}) = \int d^3k_1\, d^3k_2\ \frac{5\,\alpha(\vec{k_1},\vec{k_2}) + 2\,\beta(\vec{k_1},\vec{k_2})}{7}\ \delta_1(\vec{k_1})\ \theta_1(\vec{k_2}).}$$

$$F_2(\vec{k_1},\vec{k_2}) = \frac{5}{7} + \frac{1}{2}\cos.\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \frac{2}{7}\cos^2.$$

In fact F_n kernels can be found to all orders

$$\delta_n = \int F_n(\vec{q}_1, \ldots \vec{q}_n)\, d^3q_1 \ldots d^3q_n\, \delta_D(k - k_{1-n})$$

and similar expression for θ_n.

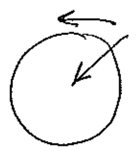
Back to 2nd order

$$F_2(\vec{k}_1, \vec{k}_2) = \frac{5}{7} + \frac{1}{2}\, \hat{k}_1 \hat{k}_2 \left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \frac{2}{7}\left(\hat{k}_1^i \hat{k}_1^j - \frac{1}{3}\delta_{ij}\right) \times \left(\hat{k}_2^i \hat{k}_2^j - \frac{1}{3}\delta_{ij}\right)$$

first term is a spherically averaged solution.

$$\left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij}\right)\delta_1(\vec{k}) \longrightarrow \left(\nabla_i \nabla_j \phi - \frac{1}{3}\nabla^2\phi\, \delta_{ij}\right)$$

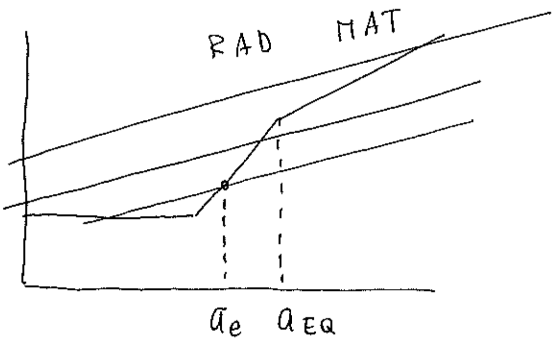
this is a <u>tidal force</u> tensor due to tidal gravitational forces



middle term is a dipole from Euler (x) to Lagrange (fluid element) transformation -

Parallel modes grow stronger ($\sim \cos^2$) $\Rightarrow$ filaments?

Evolution during ~~matter~~ RAD domination.



at scales $\lambda < H^{-1}$ Rad is smooth $\delta_R \ll 1$ but $(\lambda_J \gg H^{-1})$

but ρ and hence H is dominated by $\bar{\rho}_R$!

$$\frac{\partial^2 \delta}{\partial \tau} + \mathcal{H} \frac{\partial \delta}{\partial \tau} = 4\pi G a^2 \delta\rho_{TOT} = 4\pi G a^2 \left(\bar{\rho}_M \delta + \underbrace{\cancel{\bar{\rho}_R \delta_R}}_{\sim 0} \right) \quad (2)$$

But $\mathcal{H}^2 = \frac{8\pi G a^2}{3} (\bar{\rho}_R + \bar{\rho}_M)$ (1)

Let's change to $x = \frac{a}{a_{EQ}}$ $\quad x \ll 1 \rightarrow$ RAD

$x \gg 1 \rightarrow$ MAT

$\mathcal{H} \equiv \frac{1}{a} \frac{da}{d\tau}$

$$x = \frac{\bar{\rho}_M}{\bar{\rho}_R} \longrightarrow dx = \frac{da}{a_{EQ}} = \frac{a}{a_{EQ}} \frac{da}{a} = x \mathcal{H} d\tau$$

$$\frac{\partial}{\partial \tau} = \mathcal{H} x \frac{\partial}{\partial x} \checkmark$$

$$(1) \longrightarrow 4\pi G a^2 \bar{\rho}_M = \frac{3}{2} \mathcal{H}^2 \frac{x}{1+x}$$

From Friedmann Eq.

$$\frac{\partial \mathcal{H}}{\partial x} = -\frac{\mathcal{H}}{2} \frac{x+2}{x(x+1)}$$

Use
$$\left\{ \frac{d\bar{\rho}_m}{dx} = -\frac{3\rho}{x} \right.$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\downarrow$$

$$\frac{1}{a^2} \frac{\partial \mathcal{H}}{\partial \tau} = -\frac{4\pi G}{3}(.)$$

so we end up in

$$\boxed{2x(1+x) \frac{\partial^2 \delta}{\partial x^2} + (3x+2) \frac{\partial \delta}{\partial x} = 3\delta} \quad \text{Eq}(**)$$

The growing mode solution can be guessed easily by setting $\frac{\partial^2 \delta}{\partial x^2} = 0$

$$\underline{\delta_k^+ = A_k \, (3x+2).}$$

For $x \gg 1$ we get the usual sol in MAT era. $\delta \propto a$

For $x \ll 1$ growth is suppressed b/c the extra contribution to $\mathcal{H}$ speeds up expansion of universe. $\rightarrow$ less growth.

For $\lambda > H^{-1}$ $\delta \propto a^2$ during RAD $\rightarrow$ to do the matching we need

$(1+x)^{1/2} \sim 1+x/2$

$\frac{2+x/2}{x/2} \sim \frac{4}{x}$ $(x \ll 1)$.

$$\underline{\delta_k^- = B_k^- \left[(3x+2) \ln \frac{\sqrt{1+x}+1}{\sqrt{1+x}-1} - 6\sqrt{1+x} \right]}$$

which gives

$$\delta_k = \begin{cases} B_k^- \left(2 \ln \left(\frac{4}{x}\right) - 6\right) & x \ll 1 \\ \frac{8}{15} B_k^- \frac{1}{x^{3/2}} & x \gg 1 \end{cases}$$

Not totally trivial to get this.

$x^{-3/2}$ decaying mode in MAT era!

How much a mode that enters ~~well~~ well in RAD grows until EQ?

$$x_{enter} \equiv x_e \ll 1 \longrightarrow x = 1$$

Derivation of Eq (**) in P9 (9b)

$$\frac{\partial^2 \delta}{\partial \tau^2} = \frac{\partial}{\partial \tau}\left(\mathcal{H} x \frac{\partial \delta}{\partial x}\right) = \mathcal{H} x \frac{\partial}{\partial x}\left(\mathcal{H} x \frac{\partial \delta}{\partial x}\right)$$

$$= \mathcal{H} x^2 \frac{\partial \delta}{\partial x}\frac{\partial \mathcal{H}}{\partial x} + \mathcal{H}^2 x \frac{\partial \delta}{\partial x} + \mathcal{H}^2 x^2 \frac{\partial^2 \delta}{\partial x^2}$$

Using (1):

$$\boxed{\frac{\partial \mathcal{H}}{\partial x} = -\frac{\mathcal{H}}{2}\frac{x+2}{x(x+1)}}$$

← Using Friedmann equation you get this (with $\rho + 3p$)

$$\mathcal{H}\frac{\partial \delta}{\partial \tau} \downarrow$$

$$\mathcal{H}^2 x^2 \frac{\partial^2 \delta}{\partial x^2} + \mathcal{H}^2 x \frac{\partial \delta}{\partial x} + \mathcal{H}^2 x^2 \frac{\partial \delta}{\partial x}\left\{-\frac{\mathcal{H}}{2}\frac{x+2}{x(x+1)}\right\} + \mathcal{H}^2 x \frac{\partial \delta}{\partial x}$$

$$\cancel{\mathcal{H}^2} x^2 \frac{\partial^2 \delta}{\partial x^2} + \underbrace{\left(2x - x^2 \frac{x+2}{2x(x+1)}\right)}_{\frac{x(3x+2)}{2(x+1)}} \frac{\partial \delta}{\partial x}\cancel{\mathcal{H}^2} = \frac{3}{2}\cancel{\mathcal{H}^2}\frac{x}{x+1}\delta$$

$$2(x+1)\, x \frac{\partial^2 \delta}{\partial x^2} + (3x+2)\frac{\partial \delta}{\partial x} = 3\delta$$

$$\mathcal{H}^2 = \frac{8\pi G a^2(\bar{\rho}_m + \bar{\rho}_{rad})}{3} \quad + \quad \frac{d\rho_m}{dx} = -\frac{3\rho_m}{x}.$$

you get (1)

$$\rho = \frac{\rho_0}{a^3} \qquad \frac{d\rho}{da} = -\frac{3}{a^4}\rho_0 = -\frac{3\rho}{a} \qquad x = \frac{a}{a_{eq}}$$

$$\frac{1}{a_{eq}}\frac{d\rho}{dx} = -\frac{3\rho}{a} \longrightarrow \boxed{\frac{d\rho}{dx} = -\frac{3\rho}{x}}$$

9c

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$p = \frac{1}{3}\rho$

$p = 0$

$$\frac{1}{a}\frac{d}{dt}\left(\frac{da}{dt}\right) = -\frac{4\pi G}{3}(\rho + 3p)$$

$\mathcal{H} = \frac{da}{d\tau}\frac{1}{a} =$

$$\frac{1}{a^2}\frac{d}{d\tau}\left[\mathcal{H}\right] = -\frac{4\pi G}{3}\text{"}\left(\rho(\cancel{1}+3w)\Big|_{RAD} + \dots\Big|_{MAT}\right)$$

$\mathcal{H} = \frac{a}{a}\frac{da}{dt} = aH$

$dt = a\,d\tau$

$$\frac{1}{a^2}\frac{d\mathcal{H}}{d\tau} = -\frac{4\pi G}{3}\left(2\rho_{RAD} + \rho_M\right)$$

$$4\pi G a^2 \bar{\rho}_M = \frac{3}{2}\mathcal{H}^2\frac{x}{1+x}$$

$\rho + 3w\rho$

$\rho(1 + 3w)$

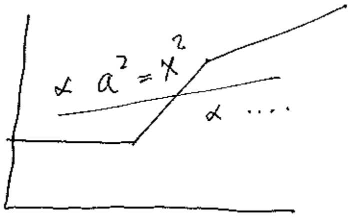
$2\rho_{RAD} + \rho_M$

$$\frac{1}{a^2}\frac{\partial\mathcal{H}}{\partial\tau} = -\frac{4\pi G}{3}\bar{\rho}_M\left(2\frac{\bar{\rho}_{RAD}}{\bar{\rho}_M} + 1\right)$$

$$\frac{\partial\mathcal{H}}{\partial\tau} = -\left(\frac{4\pi G}{3}a^2\bar{\rho}_M\right)\left(1 + \frac{2}{x}\right)$$

$$\cancel{\mathcal{H}}x\frac{\partial\mathcal{H}}{\partial x} = -\frac{1}{\cancel{3}}\frac{\cancel{3}\mathcal{H}^{\cancel{2}}}{2}\frac{\cancel{x}}{1+x}\frac{2+x}{\cancel{x}}$$

$$\boxed{\frac{\partial\mathcal{H}}{\partial x} = -\frac{\mathcal{H}}{2}\frac{2+x}{x(1+x)}}$$



Solution is

$$B = -x_e^2$$
$$A = \frac{x_e^2}{2}\left[2\ln(4/x_e) - 5\right]$$

$$\begin{cases} \delta^{\text{outside}} = \delta^{\text{inside}} \\ \dfrac{\partial \delta^{\text{out}}}{\partial x} = \dfrac{\partial \delta^{\text{ins}}}{\partial x} \end{cases}$$

$$\delta_{out} \equiv x^2$$

$$\delta_{in} = \underset{\uparrow \text{ Grow. mode}}{A_k(3x+2)} + B_k\left(2\ln\frac{4}{x} - 6\right)$$

$$\delta = x_e^2\left[\ln(4/x_e) - \frac{5}{2}\right](3x+2) - x_e^2\left[(3x+2)\ln\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1}\right) - 6\sqrt{1+x}\right]$$

$$\approx x_e^2 \ln(x_e^{-1})(3x+2)$$

$(x_e \ll 1)$ ↑ Full Dec. mode

so

$$\frac{\delta(x=1)}{\delta(x_e)} = \frac{5\, x_e^2 \ln(x_e^{-1})}{x_e^2} = 5\ln\left(\frac{a_{EQ}}{a_{enter}}\right)$$

Perturbations grows logarithmically!

if the evolution had been in MAT all the time

$$\frac{\delta(x=1)}{\delta(x_e)} \sim \frac{a_{EQ}}{a_{ent.}}$$

$$\boxed{\text{suppression} \sim \ln\left(\frac{a_{EQ}}{a_{enter}}\right) \cdot \frac{a_{enter}}{a_{EQ}}.}$$

with respect to matter dominated era

Note that this factor depends on the wavelength of the modes through a_{enter}! But this is easy to get.

$$\lambda_{\text{comoving}}\, a_e(k) = \frac{2\pi}{k} a_e(k) = H^{-1}(t_e) \propto a_e^2(k)$$

so $a_e(k) \propto \dfrac{1}{k} \longrightarrow \boxed{\dfrac{a_e}{a_{EQ}} = \dfrac{k_{eq}}{k}}$

Evolution during radiation

(10b)

Matching at x_e ($<<1$) well in rad. dominated

$$\delta^{outside} = \delta^{inside} \qquad \delta^{out}(x) = x^2$$

$$\frac{\partial \delta}{\partial x}^{outside} = \frac{\partial \delta}{\partial x}^{inside} \qquad \delta^{in} = A(3x+2) + B\left(2\ln\frac{4}{x} - 6\right)$$

(1)

$$x_e^2 = 2A - 6B + 2B\ln\frac{4}{x_e}$$

$$2x_e = 3A - 2B\frac{1}{x_e}$$

$A \propto x_e^2$ (from 1st eq) $\Rightarrow$ take 2nd eq and drop A $\rightarrow$

$$\boxed{B = -x_e^2}$$

$$x_e^2 - 6x_e^2 = 2A - 2x_e^2 \ln 4/x_e$$

$$x_e^2\left(-5 + 2\ln 4/x_e\right) = 2A$$

$$\boxed{A = \frac{x_e^2}{2}\left(2\ln 4/x_e - 5\right)}$$

$$\delta_{in}(x) \simeq x_e^2 \ln x_e^{-1}(3x+2) \qquad \text{from replacing into (1)}$$

$$\delta(x_e) = x_e^2 \quad \text{and} \quad \delta(x=1) = 5x_e^2 \ln x_e^{-1}$$

The amplification of the mode is

This is important

$$\frac{\delta(x=1)}{\delta(x_e)} = \frac{5\, x_e^2 \ln x_e^{-1}}{x_e^2} = 5 \ln\left(\frac{a_{EQ}}{a_e}\right)$$

only logarithmic

$$x = \frac{a}{a_{EQ}}$$

If the mode were all the time in matter $\delta(x) = x$

$$\frac{\delta(x=1)}{\delta(x_e)} = \frac{1}{x_e} = \frac{a_{EQ}}{a_e}$$

$$\boxed{\text{suppression} \sim \ln\left(\frac{a_{EQ}}{a_e}\right) \cdot \frac{a_e}{a_{EQ}}}$$

growth w/ respect to matter dom. mode.

The super simple solution

$$\frac{\partial^2 \delta}{\partial \tau} + \mathcal{H} \frac{\partial \delta}{\partial \tau} = 4\pi G a^2 (\bar{\rho}_M \delta + \rho_R \delta_R)$$

ρ_R is dominant so we can assume $\rho_n \sim 0$

these are very small in particular b/c they undergo acoustic oscillations

$$\delta'' + \mathcal{H} \delta' = 0 \qquad \text{Eq(1)}$$

$$\delta \equiv c_1 + c_2 \int \frac{d\tau}{a} \qquad \text{Eq(2)}.$$

easy to check that

$$\frac{\partial \delta}{\partial \tau} = \frac{1}{a} \quad (*) \qquad \frac{\partial^2 \delta}{\partial \tau^2} \approx \frac{-1}{a^2} \frac{\partial a}{\partial \tau} \approx -\frac{1}{a} \mathcal{H} \quad (**)$$

(*) and (**) in Eq(1) leads to Eq(2) ✓

During RAD $a \propto \tau$

$$\int \frac{d\tau}{\tau} \sim \ln \tau \sim \ln a \quad \rightarrow \quad \boxed{\delta_m = c_1 + c_2 \ln a}$$

$$\Delta_R \propto k^{n_s - 1}$$

$$\ln \Delta_R = \ln A + n_s - 1 \ \ln k$$

$$\frac{\partial \ln \Delta_R}{\partial \ln k} = n_s - 1$$

$$\boxed{n_s \equiv 1 + \frac{\partial \ln \Delta_R}{\partial \ln k}}$$

$$\mathcal{P}_R = 4$$

$$\Delta_R = 4\pi k^3 \mathcal{P}_R \rightarrow k^{n_s - 1} \qquad k^{\cdots} -$$

$$P_R \simeq k^{n_s - 4}.$$

$$-k^2 \Phi_k = \frac{3}{2} \Omega_m \mathcal{H}^2 \delta = \frac{3}{2} \mathcal{H}^2 \delta \sim \text{constant}.$$

$$\Delta_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2_{t_*}$$ so at H^{-1} all scales have the same spectrum amplitude.

These fluctuations are converted ~~d~~ into $\mathcal{R}$ $(\delta\phi \to \mathcal{R})$

$$\mathcal{R}_k = -\left[\frac{H}{\dot\phi}\,\delta\phi_k\right]_{t_*}$$

$\dot{\mathcal{R}}_k = 0$ ~~n~~ these don't evolve $\quad \Delta_{\mathcal{R}} = 4\pi k^3 P_{\mathcal{R}}(k)$

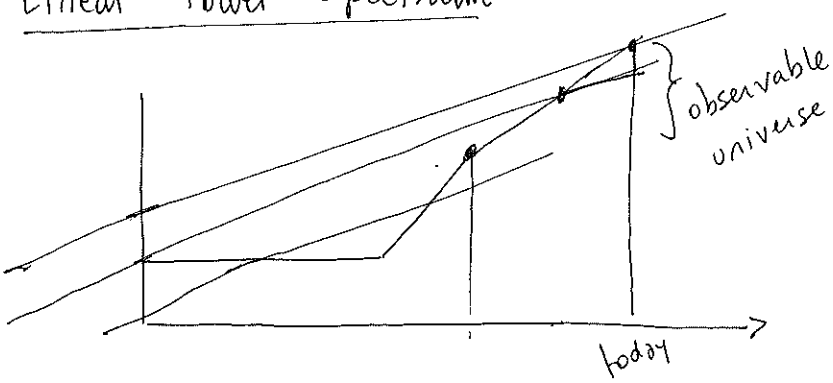
$$\Phi_k = -\frac{3(1+w)}{5+3w}\mathcal{R}_k .$$

Harrison Zeldovich spectrum is scale invariant

$$\Delta_{\mathcal{R}} = \left\{\left(\frac{H}{2\pi}\right)^2_{t_*}\left(\frac{H}{\dot\phi}\right)_{t_*}\right\}\Delta_{\delta\phi} \sim \text{constant} \overset{\text{if}}{\sim} f(k) \text{ eg } k^{\overset{\text{"power"}}{n_s - 1}}$$

$$\frac{\partial \ln \Delta_{\mathcal{R}}}{\partial \ln k} \equiv n_s - 1 \longrightarrow \boxed{n_s = 1 + \frac{\partial \ln \Delta_{\mathcal{R}}}{\partial \ln k}}$$

Linear Power Spectrum



During inflation perturbations in the inflaton field at horizon exit are all equal; with same amplitude

$$\Delta\phi(k) = \left(\frac{H}{2\pi}\right)^2_{t^*} \sim$$

Harrison-Zeldovich power spectrum

So at H^{-1} all have the same Δ if H is not constant

spectral tilt $\quad n_s = 1 + \dfrac{\partial \ln \Delta_R}{\partial \ln k} \longrightarrow \Delta_k \propto k^{n_s-1}$.

These fluctuations are converted into "curvature perturbations"

$\delta\phi \longrightarrow R$

$$R_k = -\left[\frac{H}{\dot\phi}\,\delta\phi_k\right]_{t^*} \propto k^{(n_s-1)}$$

At super hubble scales $\dot R_k = 0$!! $\qquad \Delta_R = 4\pi k^3 P_R(k) = \left[\left(\frac{H}{\dot\phi}\right)^2\left(\frac{H}{2\pi}\right)^2\right]_{t^*}$

Once the scale becomes sub-horizon again we can convert or translate these fluctuations into perturbations of the of the gravitational potential. Φ

Using GR one gets that $\Phi_k = \dfrac{-3(1+w)}{5+3w} R_k$

w equation of state of matter or radiation.

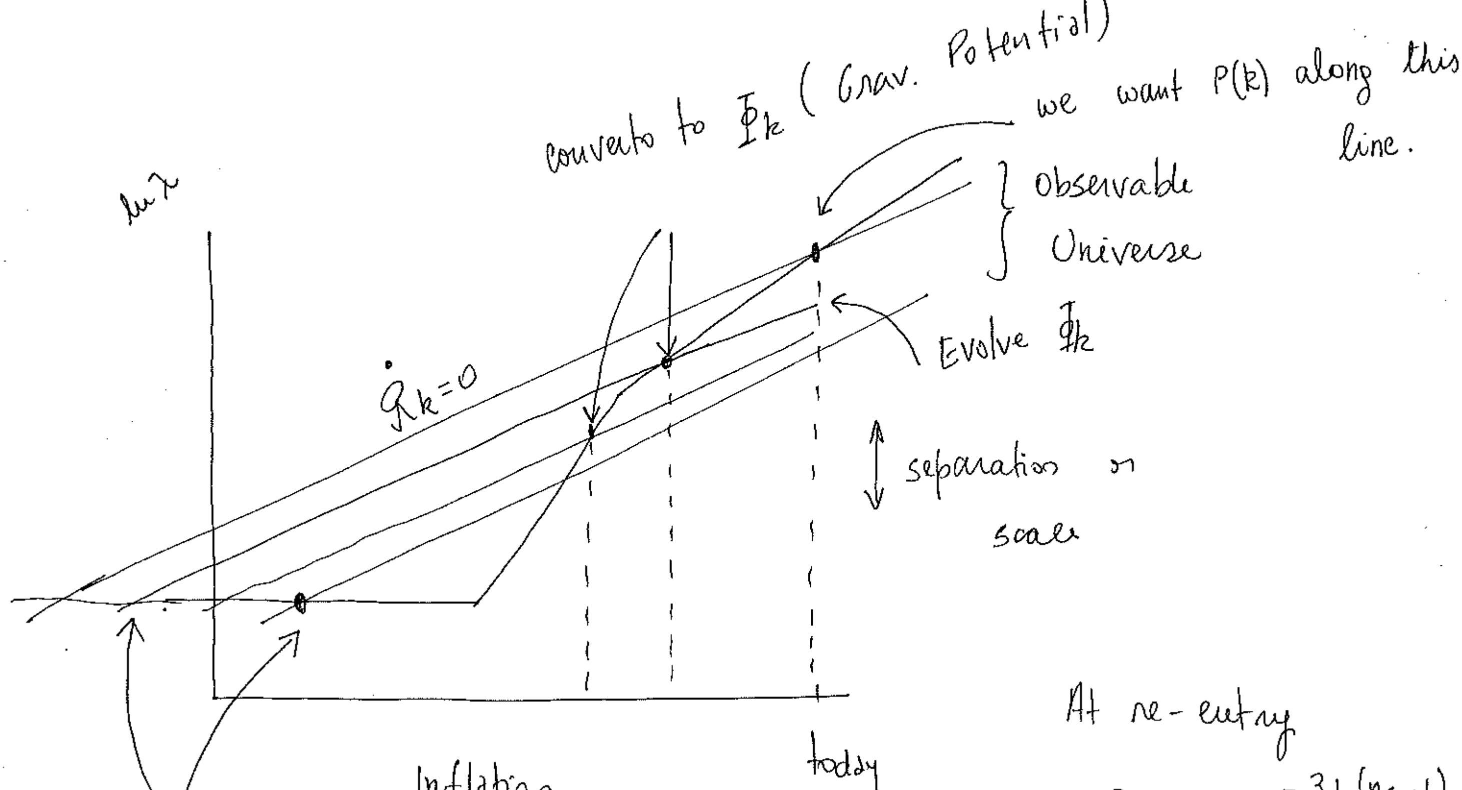
$$P_\delta(k) = k^4 P_\Phi = k^4 P_R \sim k^4 \underbrace{k^{-3+n_s-1}}_{\substack{P_\phi \\ \uparrow \\ \text{inflation}}} \sim k^{n_s}$$

$$R_k = \left[\frac{H}{\dot\phi} \, \delta\phi_k \right]_{t*}$$

$$P_R(k) = \frac{H^2}{\dot\phi^2} P_\phi(k)$$

$$\downarrow$$

$$\left(\frac{H^2}{2\pi}\right)_{t*} \sim k^{-3+n_s-1}$$



At re-entry

$P_\Phi = k^{-3+(n_s-1)}$

$\Delta\phi(k) = \left(\frac{H}{2\pi}\right)\Big|_{t_*}$ all

scales same Δ

ϕ_k suppress

Φ_k does not evolve

$$P_{\mathcal{R}}(k) \propto k^{-3+n-1} \longrightarrow \boxed{P_{\Phi} \propto k^{-3+(n_s-1)}}$$

For scales that enter "today"

For scales that enter <u>during MAT era</u> $\mathcal{H} = \frac{2}{\tau}$

$$-k^2 \Phi_k = \frac{3}{2} \mathcal{H}^2 \underbrace{\Omega_m}_{=1} \delta_k = \frac{3}{2} \mathcal{H}^2 \delta \propto \frac{3}{2} \mathcal{H}^2 a \propto \text{const !}$$

because $\mathcal{H}^2 \sim \frac{1}{a}$.

so for scales entering in MAT $\boxed{P_{\Phi} = k^{-3+(n-1)}}$ $(k < k_{eq})$

For modes that <u>enter during</u> RAD

$H^2 \sim 1/a^4$ during rad

$$-k^2 \phi_k = \frac{3}{2} \mathcal{H}^2 \delta_k \propto (H^2 a^2) \ln a \propto \frac{\ln a}{a^2}$$

$\nabla^2 \phi = \frac{3}{2} \mathcal{H}^2 \delta$

$$\ln \left[\frac{a_{EQ}}{a_{enter}} \right] \left(\frac{a_{enter}}{a_{EQ}} \right)^2$$

pert. in δ grow log. but in ϕ they are suppressed

$$\boxed{P_{\Phi} \propto P_{\mathcal{R}} \left[\ln \left(k/k_{eq} \right) \left(\frac{k_{eq}}{k} \right)^2 \right]^2} \qquad k > k_{eq}$$

$$k_{eq} = \Big(\frac{14}{\underset{0.27}{\Omega_m} \underset{0.7}{h^2}} \Big)^{-1} \text{Mpc}^{-1} \sim 0.0135 \ h/\text{Mpc}$$

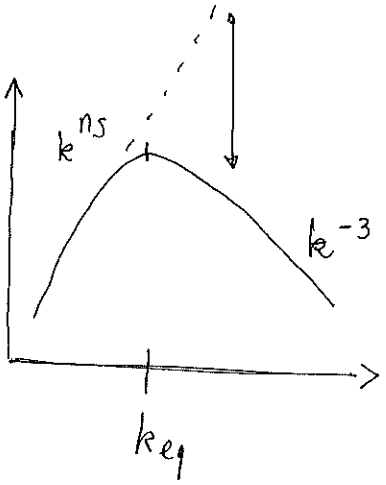
$$\delta_k = -\frac{2}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \phi_k$$

$$\boxed{P_{\delta}(k) \propto k^4 P_{\phi} \propto k^{n_s} T^2(k)}$$

linear power spectrum <u>today</u>

When

$$T(k) = \begin{cases} 1 & k \ll k_{eq} \\ \ln\left(\frac{k}{k_{eq}}\right)\left(\frac{k_{eq}}{k}\right)^2 & k > k_{eq} \\ 0 & k \gg k_{FS} \end{cases}$$



- At large scales one sees the "<u>primordial</u>" power spectrum! (inflation)
- k_{eq} depends on ~~Ω~~ $\Omega_m h$ $\Rightarrow$ way of getting Ω_m.
- nonlinear effects will play a role.

BBKS

$$T(q) = \frac{\ln(1+2.34q)}{2.34q}\left[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.7q)^4\right]^{-1/4}$$

$q = \frac{k}{\Gamma h}\,\mathrm{Mpc}^{-1}$; $\Gamma \equiv \Omega_m h \ \exp[-\Omega_B$ satisfies $T(q)\begin{cases} 1 & q \to 0 \\ \frac{\ln(q)}{q^2} & q \gg 1 \end{cases}$

In reality what you have to do is to solve the full Boltzmann eqs for all the species (e.g. Seljak & Zaldarriaga 469 pag 437 1996) $\longrightarrow$ $\boxed{\text{CMB fast}}$

Show slides $\longrightarrow$ linear vs. nonlinear from real data

→ Say that those are lines of constant comoving scale

$$\frac{\delta(x=1)}{\delta(x_e)} = 5 \ln\left(\frac{a_{EQ}}{a_e}\right) \quad \text{+ evolution of fluctuations in RAD}$$

$$\mathcal{H}^2 \sim \frac{1}{a^2} \quad \text{in RAD}$$

$x = 1$ is $a = a_{EQ}$

$$\frac{\mathcal{H}^2(x=1)}{\mathcal{H}^2(x_e)} = \left(\frac{a_e}{a_{EQ}}\right)^2$$

$$\underset{\substack{\uparrow \\ x=1}}{\phi_k(t_{EQ})} = \phi_k(t_e) \cdot \ln\left(\frac{a_{EQ}}{a_e}\right) \cdot \left(\frac{a_e}{a_{EQ}}\right)^2$$

~~Now~~ And because ϕ does not evolve with redshift beyond a_{EQ} $\quad \phi_k(t) \equiv \phi_k(t_{EQ})$

$$\lambda_{physical} = a_e \, \lambda_{comoving} = H^{-1}(a_e) \qquad H^{-1} \sim a^2 \ (\text{RAD})$$

$$a_e \underbrace{\frac{2\pi}{k}}_{=\lambda_{com}} = H^{-1}(a_e) = H^{-1}_{EQ} \frac{a_e^2}{a_{EQ}^2}$$

$$a_{EQ} \frac{2\pi}{k_{EQ}} = H^{-1}(a_{EQ})$$

$$\boxed{\frac{k_{EQ}}{k} = \frac{a_e(k)}{a_{EQ}}}$$

So

$$\Phi_k(t) = \Phi_k(t_e) \cdot \left[\ln\left(k/k_{EQ}\right) \left(\frac{k_{EQ}}{k}\right)^2 \right]$$

$$\downarrow$$

$$P_\phi \propto \underbrace{P_R}_{k^{n_s-1-3}} \left[\ln\left(k/k_{EQ}\right) \cdot \left(\frac{k_{EQ}}{k}\right)^2 \right]^2$$

$$P_\delta \propto k^4 P_\phi \quad \text{so}$$

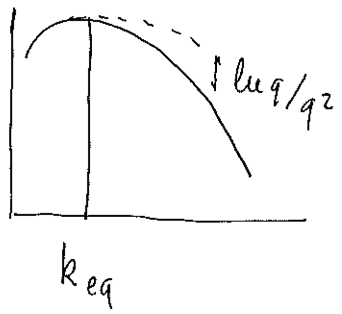
$$P_\delta \simeq k^{n_s} \left[\underbrace{\ln\left(\frac{k}{k_{EQ}}\right)\left(\frac{k_{EQ}}{k}\right)^2}_{\text{"transfer function"}} \right]^2$$

Summary so far

(1) The simplest models of inflation predict the fluctuations in the fields are Gaussian (inflaton / curvature / grav. potential / density) with $P \sim k^{n_s}$.

(2) Gaussian fields are easy to treat. Fluctuations at different points are minimally correlated $\Rightarrow$ only the 2-point CF is nonzero! or similarly the $P(k)$.

(3) Using linear theory perturbations we derived $P_s(k)$ "today" or after all scales of interest enter the horizon



$$k_{eq} \simeq 0.06\ \Omega_m\ h/\text{Mpc}$$

$$T(q) \simeq \left[1 + \{aq + (bq)^{3/2} + (cq)^2\}^{\nu}\right]^{-1/\nu}$$

$$q = \frac{k}{\Gamma} \qquad \Gamma = \Omega_m h \quad \text{shape parameter.}$$

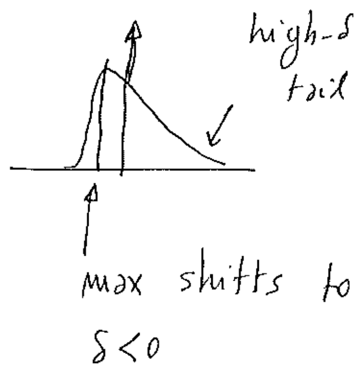
(4) ~~But~~ Linear theory preserves Gaussian statistics but gravity leads to big void regions and large localized overdensities $\Rightarrow$ linear theory should brake down

(5) Linear theory valid at large separation where gravity is weak (or early times where fluctuations are small) $\quad 4\pi k^3 \Delta(k) \approx 1 \qquad k_{nl} \sim 0.2\ h/\text{Mpc}$

in fact we saw this with simulations!

6) Using nonlinear evolution we computed the skewness

$$S_3 \equiv \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^2}$$ and it was positive



also the kurtosis

7) Hierarchical clustering.