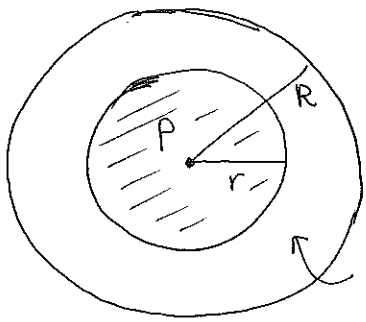
Spherical collapse



low density region to keep the mean density fixed at $\bar{\rho}$

$\rho > \bar{\rho}$

$$M = \frac{4\pi}{3} R^3 \bar{\rho} = \frac{4\pi}{3} r^3 \rho = \frac{4\pi}{3} r^3 (1+\delta)\, \bar{\rho}$$

$$\boxed{\left(\frac{R}{r}\right)^3 - 1 = \delta} \qquad \rho \equiv \bar{\rho}\,(1+\delta)$$

Let's now assume that the mass within r remains constant meaning ~~it~~ never crosses r → no "shell crossing". In this case there is conservation of energy

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = E = \text{constant}.$$

Clearly if $E > 0$ → (dr/dt) will be always positive and collapse will never occur

if $E < 0$ → there will be a turn around radius.

We have assumed no cosmological constant; if $\Lambda \neq 0$ then conservation of energy is written as

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} - \frac{\Lambda r^2}{6} = E$$

Now it depends on the combination E, Λ ; but its possible to have $E<0$ and expansion forever (Λ gives repulsion → opposes gravity).

Take $\Lambda = 0$ for simplicity and let's look at conditions for collapse. Let's assume some initial time where the shell "moves" or "expands" with the Hubble flow.

$(v_i = 0)$ $\longrightarrow$ $$\left.\frac{dr_i}{dt}\right|_{t=t_i} = H_i r_i$$

$$K_i = \frac{1}{2}\dot{r}_i^2 = \frac{1}{2} H_i^2 r_i^2$$

$$|U_i| = \frac{GM}{r_i} = G\frac{4\pi}{3}\bar{\rho}_i r_i^2 (1+\bar{\delta}_i) = \frac{1}{2} H_i^2 r_i^2 \Omega_i (1+\bar{\delta}_i)$$

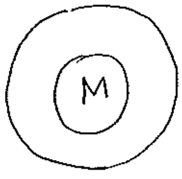
using Friedmann Eq.

$$E = K_i - |U_i| = K_i - K_i \Omega_i (1+\bar{\delta}_i) = K_i \Omega_i \left[\frac{1}{\Omega_i} - (1+\bar{\delta}_i)\right]$$

Condition for collapse is $E<0$

$$\boxed{(1+\bar{\delta}_i) > \frac{1}{\Omega_i}}$$

Let's see this calculation again for arbitrary geometry



$$|U| = \frac{GM}{r} = \frac{4\pi r^3}{3}\rho\frac{G}{r} = \frac{4\pi r^3}{3}\frac{G}{r}\bar{\rho}(1+\delta)$$

Take Friedmann Eq $$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\bar{\rho}$$ matter

$$(\Omega - 1) = \frac{k}{a^2 H^2}$$

$$|U| = \frac{G}{r}\,\frac{4\pi r^3}{3}\,\frac{3}{8\pi G}\left(H^2 + \frac{k}{a^2}\right)(1+\delta) \qquad K = \frac{\dot{r}^2}{2} = \frac{H^2 r^2}{2}$$

$$= \frac{1}{2} r^2 H^2 \Omega \,(1+\delta). \ \checkmark$$

Therefore for close $(\Omega > 1)$ or flat $(\Omega = 1)$ any initial overdense region (i.e. $\delta > 0$) will collapse.

In open universes only dense enough will do.

Maximum radious: $\quad E = -\dfrac{GM}{r_{max}} \quad (\dot{r} = 0)$

$$\cancel{K_i}\,\Omega_i \left(\frac{1}{\Omega_i} - (1+\delta_i)\right) = -\frac{GM}{r_{max}}\,\frac{r_i}{r_i} = -\frac{r_i}{r_{max}}\,\cancel{K}\,\Omega\,(1+\delta)$$

$$\boxed{\frac{r_{max}}{r_i} = \frac{1+\bar{\delta}_i}{(1+\bar{\delta}_i) - 1/\Omega_i}}$$

maximum radious → turn-around

Now we take ~~derivative of~~ $E = \text{const}$ ~~or equation of motion~~

$$\cancel{\ddot{r} = -\frac{GM}{r^2} = -\frac{4\pi G}{3}\rho r = -\frac{H^2}{2} r (1+\delta)}$$

$$\cancel{\rho = \bar{\rho}(1+\delta)}$$

$$\cancel{H^2 = \frac{8\pi G}{3}\bar{\rho}}$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = E$$

~~In addition the outer~~

Parametric Solution check

$$\frac{\dot{r}^2}{2} - \frac{GM}{r} = E$$

$$d\dot{r} = A \sin\theta \; d\theta$$

$$dt = B(1-\cos\theta)\, d\theta = B\frac{r}{A}\, d\theta$$

$$\frac{dr}{dt} = \frac{A^2}{Br}\sin\theta$$

$$\frac{1}{2}\frac{A^4}{B^2 r^2}\sin^2\theta - \frac{GM}{r} = E$$

$$\frac{A^4}{2B^2}\sin^2\theta - GMr = Er^2$$

$$\frac{A^4}{2B^2}(1-\cos^2\theta) - GMA(1-\cos\theta) = EA^2(1-2\cos\theta+\cos^2\theta)$$

1) $\frac{A^4}{2B^2} - GMA = EA^2 \quad \rightarrow \quad -EA^2 - GMA = EA^2 \qquad 2EA^2 = -GMA$ ✓

2) $GMA = -2EA^2 \quad \rightarrow \quad A = -GM/2E$

3) $-\frac{A^4}{2B^2} = EA^2 \quad \rightarrow \quad \boxed{A^2 = -2B^2E}$ ✓ $\qquad \boxed{A^3 = GMB^2}$

Let's now go back to the equation for $E = \text{const}$.

One parametric solution is

$$r = A(1-\cos\theta) \qquad \text{with} \qquad A^3 = GMB^2$$

$$t = B(\theta - \sin\theta) \qquad\qquad\qquad\qquad = -2EB^2$$

you can check that is satisfies eq of energy conservation

$\left(E = -\frac{GM}{2} \right) \rightarrow$ i.e. given E = constant

θ is a parameter that increases with time

$$\begin{cases} \theta = \pi & \rightarrow \text{turn around} \checkmark \\ \theta = 2\pi & \rightarrow \text{full collapse} \checkmark \end{cases}$$

from

$$\boxed{\delta = \left(\frac{R}{r}\right)^3 - 1}$$

+ the EdS solution $a \propto t^{2/3} \rightarrow \bar{\rho} = \frac{1}{6\pi G t^2} \sim \frac{1}{a^3}$

$$\boxed{\delta = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3} - 1}$$

$$\frac{M}{\frac{4\pi R^3}{3}} = \frac{1}{6\pi G t^2}$$

$$R^3 = MGt^2\,9/2$$

For Mat $H = \frac{2}{3t}$

$$\left(\frac{2}{3t}\right)^2 = \frac{8\pi G\bar{\rho}}{3}$$

$$\bar{\rho} = \frac{1}{6\pi G t^2} \checkmark$$

$$1 + \delta = \cancel{MG}\,\frac{9}{2}\,\frac{\cancel{B^2}(\theta - \sin\theta)^2}{\cancel{A^3}(1-\cos\theta)^3} \checkmark$$

Alternatively you can use

$$\ddot{r} = -\frac{GM}{r^2} = -\frac{4\pi r}{3} G \overset{\bar{\rho}(1+\delta)}{\rho} = -\frac{1}{2} H^2 (1+\delta) r$$

and write full equation for $\ddot{\delta}$ ... See Eq 7.6

using
$$R = a R_0$$
$$\dot{R} = H R \longrightarrow \text{after linearization}$$
$$\ddot{R} = -\frac{H^2}{2} R$$

$$\boxed{\delta_L = \frac{3}{5}\left[\frac{3}{4}(\varphi - \sin\varphi)\right]^{2/3}}$$

Also from using
$$\delta = \frac{3}{5}\delta_i\left(\frac{a}{a_i}\right) = \frac{3}{5}\delta_i\left(\frac{t}{t_i}\right)^{2/3}$$
$$+ \; B = \frac{3}{4}\frac{t_i}{\delta_i^{3/2}}$$

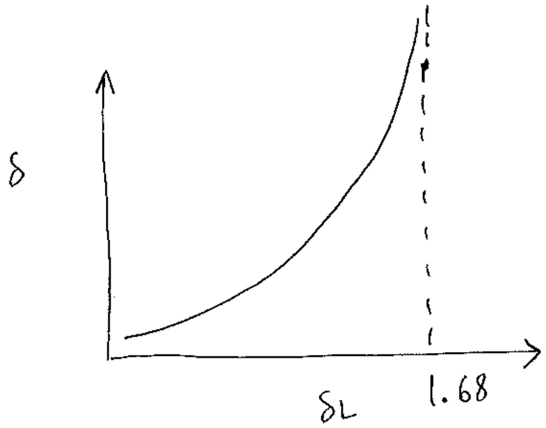
Overdensity at turn-around

$$(1+\delta)\Big|_{\text{turn around}} = \frac{9\pi^2}{16} \approx 5.55$$

$$(1+\delta_L)\Big|_{\text{turn around}} \approx 2.062$$

After the spherical ~~collapse~~ region turns around it continues to contract and according to this model at $\theta = 2\pi$ it collapses to a point of ∞ density. Moreover shell crossing can't be neglected as turning point get's closer to the centre

In reality what happens is violent relaxation.(*)



At collapse $\theta = 2\pi$

$$\delta \longrightarrow \infty$$

$$\boxed{\delta_L \longrightarrow 1.68}$$

See White, S. arXiv / 9602021.

28b

$r = A(1-\cos\theta)$

$t = B(\theta - \sin\theta)$

$$\frac{dr}{dt} = \frac{-A\sin\theta}{B(1-\cos\theta)} = +\frac{A^2\sin\theta}{B\,r} \checkmark$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = +\frac{A^4\sin^2\theta}{2B^2r^2} - \frac{GM}{r} = E$$

$$\frac{GM}{r} =$$

$$\frac{1}{2}\frac{A^4}{B^2}\sin^2\theta - GMr = Er^2$$

$$\frac{1}{2}\frac{A^4}{B^2}\sin^2\theta - GMA(1-\cos\theta) = EA^2(1-\cos\theta)^2$$

$$\frac{1}{2}\frac{A^4}{B^2}(1-\cos^2\theta) - GMA(1-\cos\theta) = EA^2(1-2\cos\theta+\cos^2\theta)$$

1) $\frac{1}{2}\frac{A^4}{B^2} - GMA = EA^2$

2) $-\frac{A^4}{2B^2} = EA^2$

$$-\frac{A^2}{2B^2} = E$$

$$\boxed{A^2 = -2EB^2} \checkmark$$

$$-EA^2 - GMA = EA^2$$

$$E\,2A^{\cancel{2}} = -GM\cancel{A}$$

$$\boxed{A = -\frac{GM}{2E}} \checkmark$$

$$-EA^2 - GMA = EA^2$$

$$-GM\cancel{A} = 2EA^{\cancel{2}} \checkmark$$

satisfaction ✓

$$+GM\cancel{A} = -2EA^{\cancel{2}} \quad (3)$$

$$\boxed{A = -\frac{GM}{2E}}$$

From $(E, M) \to A$ ✓

$$\frac{A^{\cancel{4}2}}{-\cancel{EA^2}} - GMA = EA^2$$

At turn around we have

$$r_{max} = 2A \quad \rightarrow \quad A = \frac{r_{max}}{2} = \frac{r_i}{2} \frac{1+\delta_i}{(1+\delta_i) - 1/\Omega_i}$$

$$B^2 = \frac{1}{GM} \left[\frac{r_{max}}{2} \right]^3$$

But we better use

$$E = K_i \Omega_i \left[1/\Omega_i - (1+\delta_i) \right] = \frac{1}{2} H_i^2 r_i^2 \Omega_i \left[1/\Omega_i - (1+\delta_i) \right]$$

Then

$$B^2 = \frac{A^2}{-2E} = \left(\frac{r_i}{2}\right)^2 \frac{(1+\delta_i)^2}{\left[(1+\delta_i) - 1/\Omega_i\right]^2} \cdot \frac{1}{H_i^2 r_i^2 \Omega_i \left[-1/\Omega_i + (1+\delta_i)\right]}$$

$$B^2 = \frac{(1+\delta_i)^2}{4 H_i^2 \Omega_i \left[(1+\delta_i) - 1/\Omega_i\right]^3} \qquad \text{Eq}(*)$$

<u>Linear solution</u>

$$\underset{\substack{\uparrow \\ \text{expand in } \theta}}{t \simeq} B \frac{\theta^3}{3!} \qquad \underset{\substack{\uparrow \\ \text{expand in } \theta}}{\delta \simeq} \frac{3\theta^2}{20} \quad \longrightarrow \quad \delta_L = \frac{3}{\underset{\substack{\downarrow \\ 5 \times 8^{2/3}}}{20}} \left(\frac{6t}{B}\right)^{2/3} = \frac{3}{5}\left[\frac{3}{4}(\theta - \sin\theta)\right]^{2/3}$$

From Eq(*) assuming $\Omega_i = 1 \rightarrow$ MAT dominated $H_i = \dfrac{2}{3t_i}$

$$B = \frac{1+\cancel{\delta_i}}{2 \cdot (2/3t_i) \underbrace{\Omega_i}_{=1} \left\{ 1 + \delta_i - \underbrace{1/\Omega_i}_{=1} \right\}^{3/2}} = \frac{3}{4} t_i \delta_i^{-3/2}$$

$$\delta_L = \frac{3}{20} \left\{ \frac{6t}{3t_i} 4 \delta_i^{3/2} \right\}^{2/3} = \frac{3}{5} \delta_i \left(\frac{t}{t_i}\right)^{2/3}$$

So we have "linear density threshold for turn-around" $\delta_{turn} = 1.06$

and "linear density threshold for collapse" $\delta_{coll} = 1.69$

Comparing to $1 + \delta_{turn} = 5.55$ and $1 + \delta_{coll} = 178$

we can appreciate how fast structure grows. when perturbation get non-linear. Paradigma: we can look at the linear density contrast at matter at early times ⇒ whenever the density contrast grows to 1.69 according to linear theory → matter concentration shall collapse to a virialized object with a characteristic overdensity of 178.

The ending result is that particles are in virial equilibrium (29)

$$E = U + K = -K \qquad (|U| = 2K)$$

$$U_{vir} = -2K_{vir} = 2E_{vir} = 2U_{turn}$$

$$\boxed{r_{vir} = \frac{r_{turn}}{2} = \frac{r_{max}}{2}}$$

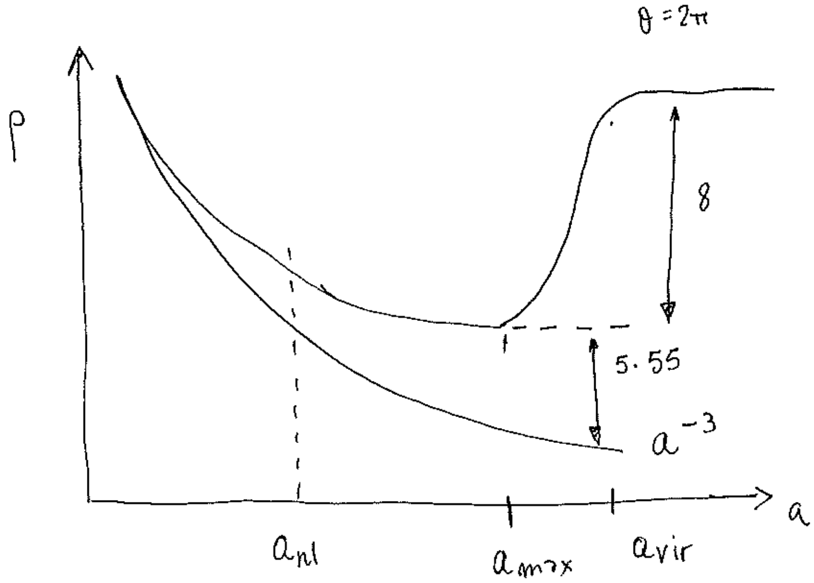
$$\rho_{vir} = 8\,\rho_{max}$$

$$(1+\delta)_{vir} = 8 \times 5.55 \times \left(\frac{a_{vir}}{a_{turn}}\right)^3 = 8 \; 5.55 \; 2^{2/3} \approx 178 \quad (18\pi^2)$$

↳ recall that $a \sim t^{2/3}$

$$\left(\frac{a_{vir}}{a_{turn}}\right) = \left(\frac{t_{vir}}{t_{turn}}\right)^{2/3}$$

$$= \left(\frac{\theta_v - \sin\theta_v}{\theta_t - \sin\theta_t}\right)^{2/3} = \left(\frac{2\pi}{\pi}\right)^{2/3}$$



$$\rho_{vir} = 8\,\rho_{turn}$$

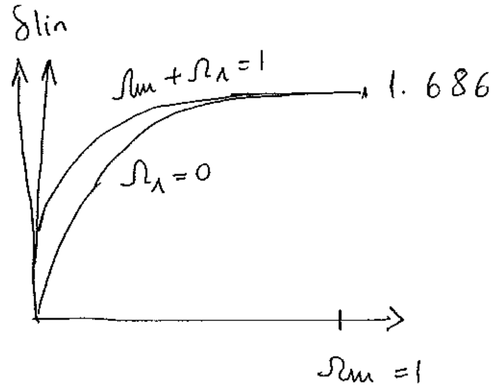
$$(1+\delta)_{vir}\,\rho_{vir} = 8\,(1+\delta)_{turn}\,\rho_{turn}$$

$$\frac{\rho_{turn}}{\rho_{vir}} = \left(\frac{a_{vir}}{a_{turn}}\right)^3$$

$$(1+\delta)_{vir} = 8 \; 5.55 \left(\frac{a_{vir}}{a_{turn}}\right)^3 = 18\pi^2$$

$$\delta_{lin} = 1.686 \quad \leftarrow \text{density contrast associated with virialization.}$$

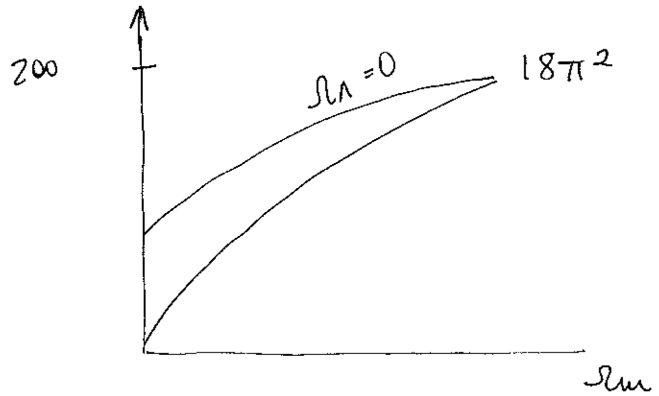
In fact it depends, but only slightly, on cosmology



see

Eke et al, MNRAS, 282, 263 (1996)

similarly the overdensity at virialization is



there are fitting formulas

$$\delta_c = \frac{3}{20} (12\pi)^{2/3} \, \Omega_m^{0.0055}$$

$$\Delta_{vir} = 18\pi^2 + 82x - 39x^2 \, ; \quad x \equiv \Omega_m - 1.$$

The idea to take home is that regions of space (spherical) whose over-densities equal (or exceed) the critical density for collapse by the present time will collapse into a bound structure (or halo).

Assuming Gaussian initial fluctuations; or a gaussian field → linearly extrapolate to today has a PDF

$$\boxed{P_R(\delta) = \frac{e^{-\delta^2/2\sigma^2(R)}}{\sqrt{2\pi\sigma^2(R)}}}$$

$$\frac{4\pi R^3}{3}\bar{\rho} = M$$

$$\sigma^2(R) \equiv \int d^3k \; P(k) \; W^2(kR).$$ with $P(k)$ linear power today

Given $\mathrm{Prob}_R(\delta)$ → one can easily compute

$$F(>\delta_c) = \int_{\delta_c}^{\infty} \mathrm{Prob}_R(\delta)\, d\delta$$ and identify

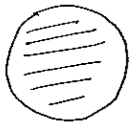
this with the fraction of mass M or larger

→ leads to the Press - Schechter Formalism for the abundance of halos (or clusters)

Mass Function

- Number density of objects with mass $(M, M+\delta M)$

- Given R, M, δ collapse will occur if $\delta_{lin} > \delta_c$



$$\delta_0 \longrightarrow \delta_{lin} > \delta_c$$

Idea of Press and Schechter (1974) was to identify the fraction of objects with mass $> M$ with the fraction of regions with $\delta > \delta_c$

$$F(>M) = \int_{\delta_c}^{\infty} P_R(\delta)\, d\delta = \frac{1}{\sqrt{2\pi}\,\sigma(R)} \int_{\delta_c}^{\infty} e^{-\delta^2/2\sigma^2(R)}\, d\delta$$

$$= \frac{1}{2} \operatorname{Erf}\left[\frac{\delta_c}{\sqrt{2}\,\sigma(R)}\right]$$

where

$$\sigma(R) \equiv \int d^3k \; P(k) \; W^2(kR)$$

$P(k)$: linear power spec.

W: interpolating function

$$M = V_R\, \bar{\rho} \propto \bar{\rho} R^3$$

Top Hat filter (spheres in real space)

$$V_R = \frac{4\pi R^3}{3} \qquad W(x) = \frac{3}{x^3}(\sin x - x\cos x)$$

fraction of objects with mass M and $M + \delta M$

$$f(M) = -\frac{\partial F}{\partial M} \, dM.$$

And comoving # density is $n \, dM$

$$n(M) \, dM = \frac{f(M)}{M/\bar{\rho}} = -\rho \frac{\partial F}{\partial M} \frac{dM}{M}$$

one can check that

$$\int_0^\infty n(M) \, M \, dM = \bar{\rho}$$

$$\int_0^\infty -\bar{\rho} \frac{\partial F}{\partial M} \frac{dM}{M} \, M = \bar{\rho}$$

$$F(>0) - \underbrace{F(>\infty)}_{=0} = 1 \quad \longrightarrow \quad \text{But } F(>0) \text{ gives } 1/2!$$

PS just multiplied by 2. (problem with sub structure)

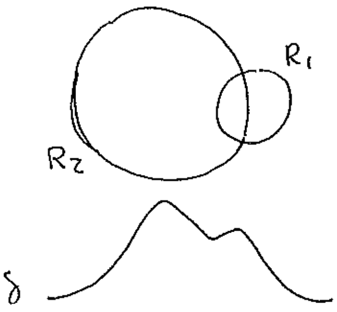
$$n_{ps}(M) \, dM = -2 \bar{\rho} \frac{\partial F}{\partial M} \frac{dM}{M}$$

one can write

$$F = \int_{\delta_c/\sigma}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \quad \longrightarrow \quad \frac{\partial F}{\partial M} = \frac{e^{-\delta_c^2/2\sigma}}{\sqrt{2\pi}} \frac{\delta_c}{\sigma^2} \frac{d\sigma}{dM}$$

$$\boxed{n_{ps} = -\frac{2\bar{\rho}}{M} \frac{\delta_c}{\sigma^2} \frac{e^{-\delta_c^2/2\sigma^2}}{\sqrt{2\pi}} \frac{d\sigma}{dM}}$$

Cloud-in-cloud problem



$\delta_1(\vec{x}) = \delta_S(\vec{x}, R_1) \quad t_2 \quad t_2 > t_1$

$\delta_2(\vec{x}) = \delta_S(x, R_2) \quad t_2 \quad R_2 > R_1$

If $\delta_1 > \delta_2 \Rightarrow$ then no problem: object collapses first ~~into~~ at time t_1 and scale R_1 then it merges to form a bigger object R_2 of M_2. (very high peak collapses first)

If $\delta_2 > \delta_1$ then is a problem because that point will be first part of M_2 and never of M_1 $\rightarrow$ so those point that on a larger radii belong to another scale must be excluded when computing density of peaks at scale R_1.

$\Rightarrow$ we need to partition the initial field.

$n(>M_1) > n(>M_2)$ if $M_2 > M_1$

it should be like this

$\delta_1 > \delta_2 \Rightarrow$ then no problem.

But if $\delta_2 > \delta_1 \Rightarrow$

What happens in a region of space δ x δ_1 δ_2

it will count for δ_1 (for M_1) and then it might or not not count for M_2 ($\delta_2 < \delta_1$). But if $\delta_2 > \delta_1$ it ~~will~~ should count for M_2 and you should stop over-counting.

— · —

Gaussian Fluctuations

$$\mathcal{P}[\delta_M > \delta_c] = \frac{1}{\sqrt{2\pi}\,\sigma_M} \int_{\delta_c}^{\infty} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2}\right] d\delta_M =$$

$$= F(>M, t)$$

the mass fraction contained on halos w/ $\delta_M > \delta_c$ at some given time

Smooth Density Field

- Variance of the density field $\langle \delta \rangle = \frac{1}{V} \int \delta(\vec{x})\, d^3x$ — Ergodic Theorem

then

$$\sigma^2 = \langle \delta^2 \rangle = \frac{1}{V} \int \delta^2(\vec{x})\, d^3x$$

From $\xi(r) = \langle \delta(\vec{x})\, \delta(\vec{x}+\vec{r}) \rangle = \frac{1}{(2\pi)^3} \int P(k)\, e^{i\vec{k}\cdot\vec{r}}\, d^3k$

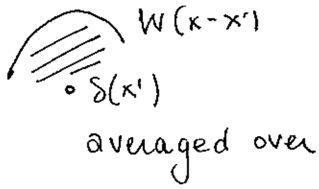
follows that $\sigma^2 = \int \frac{1}{(2\pi)^3} P(k)\, d^3k = \frac{1}{2\pi^2} \int P(k)\, k^2\, dk$

$$\boxed{\sigma^2 = \int \Delta^2(k)\, \frac{dk}{k}}$$

$\Delta(k) \equiv \frac{k^3 P(k)}{2\pi^2}$ unit-len power spectrum

- Smooth Field

$$\delta(\vec{x};R) = \int d^3x'\, \delta(\vec{x}')\, W(\vec{x}-\vec{x}';R)$$



$$\downarrow$$

$$\delta(\vec{k};R) = \delta(\vec{k})\, \tilde{W}(kR).$$

(convolution $\longleftrightarrow$ multiplication)

Note that for each filter one can define a mass M such that $M = \gamma_f\, \bar{\rho}\, R^3$ where γ_f depends on the shape of the filter.

Top Hat in Real Space $\begin{cases} \frac{3}{4\pi R^3} & r < R \\ 0 & r > R \end{cases}$ $\quad \gamma_t = \frac{4\pi}{3}$ — sphere

similar to the case with no smoothing

$$> \quad \sigma^2(R) = \langle \delta^2(\vec{x};R) \rangle = \frac{1}{2\pi^2} \int P(k)\, \tilde{W}^2(kR)\, k^2 dk .$$

in the limit $\lim_{R \to 0} \tilde{W}(kR) = 1$ so $\sigma^2(R \to 0) = \sigma^2$.

Mass Variance

$$M(\vec{x};R) = V_R \int \rho(\vec{x}')\, W(\vec{x}-\vec{x}';R)\, d^3x'$$

$$\bar{M}(R) = \langle M(\vec{x};R) \rangle$$

so

$$\sigma^2(M) = \left\langle \left\{ \frac{M(\vec{x};R) - \bar{M}(R)}{\bar{M}(R)} \right\}^2 \right\rangle$$

Note

if $\delta(\vec{x})$ is a Gaussian field so is $\delta(\vec{x};R)$. In particular

$$\mathcal{P}(\delta_M)\, d\delta_M = \frac{1}{\sqrt{2\pi}\,\sigma_M} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2} \right] d\delta_M .$$

where $\delta_M = \delta(\vec{x};M)$ and $\sigma_M = \sigma(M)$.

- Dark Matter Halos -

~~DM halos grow line~~

* Density perturbations grow linearly ~~until~~ until they reach some critical value after which they turn around and collapse to form virialized structure which then grow in mass / size by merging or accretion from vicinity

Peak Formalism → assumes that the mass that collapses to form halos can be identified by smoothing the "initial" ~~den~~ Gaussian density field w/ some filter ⇒ take all regions above a threshold. ("peaks")

$$\delta(\vec{x}, R) = \int d^3x' \, \delta(\vec{x}') \, W(\vec{x} + \vec{x}', R)$$

$$\eta_{pk}(x) = \sum_p \delta^{(D)}(x - \underbrace{x_p}_{\text{peak position}})$$

$$\eta(x) \equiv \vec{\nabla}\delta(\vec{x})$$

$$\eta(x_p) = 0 \quad \leftarrow \text{extrema}$$

$$\eta_i(x) \approx \sum_j \zeta_{ij}(\vec{x}) \, (\vec{x} - \vec{x}_p)_j$$

$\eta_i = \nabla_i \delta$

such that $\eta(x_p) = 0$ then we expand around it

$$\eta_i(x) = \sum_j \nabla_j (\nabla_i \delta) . (\vec{x} - \vec{x}_p)_j$$

$$\eta_i(x) = \sum_j \nabla_i \nabla_j \delta(x_p) \; (\vec{x} - \vec{x}_p)_j = \sum_j \zeta_{ij}(x_p)(\vec{x} - \vec{x}_p)_j$$

$$\zeta_{ij}(x_p) = \nabla_i \nabla_j \delta(x_p)$$

$$\downarrow$$

$$k_i k_j \; \delta(\vec{k}) \, e^{i\vec{k}.\vec{x}_p}$$

If you invert the 2nd derivatives

$$(\vec{x} - \vec{x}_p)_i = \sum_j \zeta^{-1}_{ij}(\vec{x}_p) \, \eta_j(\vec{x})$$

1) In order for a extrema ζ_{ij} has to be negative definite
2) Density peak of specific height $(\delta/\sigma) > \nu$ $\quad \sigma_0^2 = \langle \delta^2 \rangle$.

$$n_{pk}(\geq \nu) = \frac{(\sigma_2^2/3)^{3/2}}{(2\pi)^2} (\nu^2 - 1) \, e^{-\nu^2/2}$$

where

$\sigma_2^2 \sim \langle \zeta_{ij} \zeta_{ke} \rangle$

Press - Schecther Mass Function

- Postulate: the fraction of total mass in halos of mass $> M$ is equal to the $\mathcal{P}(\delta_M > \delta_c(t))$; the probability of δ smoothed on filter M being ~~[illegible]~~ over the collapse value.

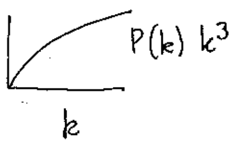
$$\mathcal{P}(\delta_M > \delta_c(t)) = \frac{1}{\sqrt{2\pi}\,\sigma_M} \int_{\delta_c}^{\infty} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2}\right] d\delta_M = \frac{1}{2}\,\mathrm{Erfc}\left[\frac{\delta_c}{2\sigma_M}\right]$$

$$\mathrm{Erfc}(x) = 1 - \mathrm{Erf}(x).$$

so

$$F(>M, t) = \frac{1}{2}\,\mathrm{Erfc}\left[\frac{\delta_c(t)}{2\,\sigma_M}\right]$$

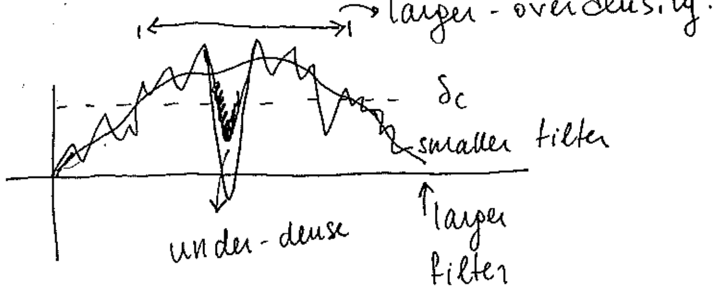
Problem: Since $\lim_{M \to 0} \sigma_M = \infty$ (basically $\int P(k)\,k^2\,dk$ diverges as $\int P(k)\,k^3\,d\ln k$ $\to$



then $F(>0, t) = 1/2$!

only half the mass is in halos.

This is because in this picture only $\delta > 0$ collapses while in real life you can have mass in under-dense regions enclosed in over-dense ones of bigger size.

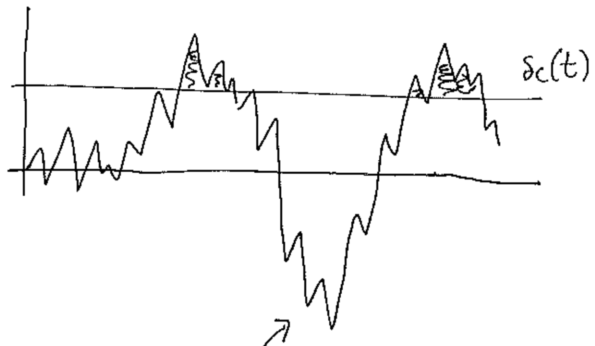


They introduced a "fudge factor" by saying that also mass in these underdense will eventually collapse into those virialized objects

$$\mathcal{P}(\delta < 0) = 1/2 \quad \text{for a Gaussian field.}$$

So the fixed this by postulating $\boxed{F(>M) = 2\,\mathcal{P}(\delta_M > \delta_c(t))}$.

Again the "cloud-in-cloud" problem



$F(>M)$ fraction of mass in halos more massive than M

there is mass in underdense regions also and it will of course collapse somewhere.

Now the Halo Mass Function : $n(M,t)$

$n(M,t)\,dM$: # of halos with mass in $\{M, M+dM = M'\}$ per unit volume

Fraction of mass in halos of mass M $= F(>M) - F(>M')$

$$= -\frac{\partial F}{\partial M}\,dM.$$

$\dfrac{\text{mass in halos}}{\text{"total"}} \times \dfrac{\text{"total"}}{\text{volume}} = \bar{\rho}$

$$M\,n(M,t)\,dM = \bar{\rho}\,\left|\frac{\partial F}{\partial M}\right|\,dM$$

to change to mass per unit volume

note abs value to kill - sign

$$n\, dM = \frac{\bar{\rho}}{M}\, 2\, \frac{\partial \mathcal{P}(\delta_M > \delta_c)}{\partial M}\, dM$$

$$n(M,t)\, dM = 2\, \frac{\bar{\rho}}{M}\, \frac{\partial \mathcal{P}}{\partial \sigma_M} \left| \frac{\partial \sigma_M}{\partial M} \right| dM$$

$$n(M,t)\, dM = 2\, \frac{\bar{\rho}}{M}\, \frac{\partial}{\partial \sigma} \left\{ \frac{1}{2} \operatorname{Erfc}\left(\frac{\delta}{\sqrt{2}\,\sigma} \right) \right\} \left| \frac{\partial \sigma}{\partial M} \right| dM$$

$$= \cancel{2}\, \frac{\bar{\rho}}{M}\, \frac{1}{\cancel{2}}\, \frac{-2}{\sqrt{\pi}}\, e^{-\delta^2/2\sigma^2} \left(\frac{-\delta}{\sqrt{2}\,\sigma^2} \right) \left| \frac{\partial \sigma}{\partial M} \right|$$

$$n(M,t)\, \delta M = \sqrt{\frac{2}{\pi}}\, \frac{\bar{\rho}}{M^2} \left(\frac{\delta_c}{\sigma} \right) e^{-\delta_c^2/2\sigma^2} \left| \frac{\partial \ln \sigma}{\partial \ln M} \right| dM$$

Defining $\nu \equiv \dfrac{\delta_c(t)}{\sigma(M)}$

$$n(M,t)\, dM = \frac{\bar{\rho}}{M^2}\, f_{PS}(\nu) \left| \frac{d \ln \nu}{d \ln M} \right| dM$$

$$f_{PS}(\nu) = \sqrt{\frac{2}{\pi}}\, \nu\, e^{-\nu^2/2}$$ ← multiplicity function.

time enters through $\delta_c(t) \equiv \dfrac{1.686}{D(t)}$ (as time passes by δ_c decreases and there is less damping → more halos of M!)

Cosmology is in $\sigma(M)$ through $P(k)$ and in δ_c through $D(z)$ -

Let's define M_* such that $\sigma(M_*) = \delta_c(t)$ at any given time

> For $M \ll M^*$ one can show that $n(M) \propto M^{-2}$

> For $M \gg M^*$ the abundance of halos is exp. suppressed.
[10^{13} Mo/h today]

Notes

1) The fudge factor used by PS was soon understood (Bond et al 1991) by the Excursion Set Formalism that takes care of not double counting (also known as Extended Press Schechter).

2) In real-life the collapse is NOT spherical but rather ellipsoidal (one axis collapses first into a pancake.
The case of ellipsoidal collapse has been worked out by Sheth Mo and Tormen MNRAS 323, 1-12 (2000)
Note that δ_{sc} does NOT depend on mass or initial size
in Ellipsoidal collapse δ_{EC} is a function of mass $\delta_{EC}(m)$

- less massive will need higher overdensity -

$$\frac{\delta_{ec}}{\delta_{sc}} = 1 + 0.47 \left[5(e^2 \pm p^2) \frac{\delta_{ec}^2}{\delta_{sc}^2} \right]^{0.6}$$

$$\int_{-\infty}^{+\infty} \frac{e^{-t^2}}{\sqrt{\pi}} dt = 1 = 2\int_0^{\infty} \frac{e^{-t^2}}{\sqrt{\pi}} dt = 2\left\{\int_0^{X} \frac{e^{-t^2}}{\sqrt{\pi}} dt + \int_X^{\infty} \frac{e^{-t^2}}{\sqrt{\pi}} dt\right\}$$

$$1 = 2\left\{\frac{1}{2}\,\mathrm{Erf}(x) + \int_X^{\infty} \frac{e^{-t^2}}{\sqrt{\pi}} dt\right\}$$

$$\int_X^{\infty} \frac{e^{-t^2}}{\sqrt{\pi}} dt = \frac{1}{2}\big[\underbrace{1 - \mathrm{Erf}(x)}_{\mathrm{Erfc}\,(x)}\big].$$

Recall

$$\mathrm{Erf}(x) \equiv \int_{-x}^{x} \frac{e^{-t^2}}{\sqrt{\pi}} dt$$

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{\delta_c}^{\infty} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) d\delta = \frac{\sqrt{2}\,\sigma}{\sqrt{2\pi}\,\sigma}\int_{\delta_c/\sqrt{2}\sigma}^{\infty} e^{-t^2} dt =$$

$$t^2 = \frac{\delta^2}{2\sigma^2}\ ; \quad t = \frac{1}{\sqrt{2}}\frac{\delta}{\sigma} \qquad dt = \frac{d\delta}{\sqrt{2}}\frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{\pi}}\int_{\delta_c/\sqrt{2}\sigma}^{\infty} e^{-t^2} dt = \frac{1}{2}\,\mathrm{Erfc}\left(\frac{\delta_c}{\sigma\sqrt{2}}\right)$$

$$\mathcal{P}(\delta_M > \delta_c) = \frac{1}{2}\,\mathrm{Erfc}\left[\frac{\delta_c}{\sqrt{2}\,\sigma_M}\right]$$

Now we can use that

$$\frac{\partial\,\mathrm{Erf}(z)}{\partial z} = \frac{2}{\sqrt{\pi}} e^{-z^2} \quad \text{or} \quad \frac{\partial\,\mathrm{Erfc}(z)}{\partial z} = -\frac{2}{\sqrt{\pi}} e^{-z^2}.$$

$$\frac{\delta_{ec}(R)}{\delta_c(t)} = \left[1 + 0.47 \left(\frac{S}{\delta_c^2(t)} \right)^{0.615} \right]$$

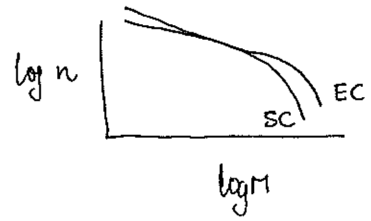
"moving" S : barrier

Numerically (can't be worked out exactly)

$$f_{EC} = 0.322 \left[1 + \frac{1}{\tilde{\nu}^{0.6}} \right] f_{PS}(\tilde{\nu})$$

$$\tilde{\nu} = 0.86\,\nu .$$

In the EC you get more ~~higher~~ large mass halos and less low mass one



→ much better fit to simulations.

Cluster Abundance

- How do you go from halo abundance to cluster abundance to cosmology :

The spherical collapse motivates fits to N-body simulations e.g

$$f_{ST} = A \sqrt{\frac{2q}{\pi}} \left\{ 1 + (q\nu^2)^{-p} \right\} \nu\, e^{-\nu^2 q/2}$$

$A = 0.322 \quad q = 0.707 \quad p = 0.3 \qquad \left(A = \frac{1}{2},\ p = 0,\ q = 1 \rightarrow \text{back PS} \right)$