BIASING : Clustering of tracers

It's been observationally known for quite some time (since the 80's) that the clustering of objects such as galaxies, or cluster of galaxies, is not universal, that is, it depends on the type of object and thus not all objects can trace the clustering of dark matter. The difference in clustering strength of a given tracer of large-scale structure (LSS), when compared to dark matter is known as <u>bias</u>.

The discussion here applies in general to any type of object, could be a dark matter halo, a galaxy or cluster of galaxies, though generically we will use "galaxy" for definiteness. Galaxies differ in their clustering from dark matter because they live in special regions compared to the DM, i.e. they are spatially segregated compared to DM, that is, they are <u>not</u> a random sample of dark matter particles (in which case the clustering will be the same).

The simplest model that incorporates the main ideas of galaxy bias is the so-called <u>local bias model</u>. Imagine we are interested in large-scale clustering and therefore we smooth the density fluctuations δ and galaxy fluctuations δ_g with some low-pass filter (e.g. the top-hat function we defined earlier) so that fluctuations are small. Local bias assumes that large-scale clustering of δ_g can be thought of as a local function of δ, that is: $\delta_g(\vec{x}) = f[\delta(\vec{x})]$ for some undetermined function f.

Since fluctuations are small, we are allowed to expand the function f in Taylor series leading to

$$\delta_g = \sum_{k=0}^{\infty} \frac{b_k}{k!}\, \delta^k = b_1 \delta + \frac{b_2}{2}(\delta^2 - \sigma^2) + \cdots$$

where the linear term ($k=1$) is known as linear bias (b_1), the quadratic ($k=2$) contribution quadratic bias (b_2), etc. The constant ($k=0$) term b_0 is obtained by imposing that δ_g is a fluctuation and thus $\langle \delta_g \rangle = 0$ (leading to $b_0 = -\frac{b_2}{2}\sigma^2 + \cdots$)

If we calculate now the low-order cumulants, to leading order in PT we have,

$$\boxed{\sigma_g^2 = \langle \delta_g^2 \rangle \simeq b_1^2 \langle \delta^2 \rangle = b_1^2 \sigma^2}$$

$$\langle \delta_g^3 \rangle \simeq b_1^3 \langle \delta^3 \rangle + \frac{3}{2} b_2 b_1^2 \underbrace{\langle \delta^2 (\delta^2 - \sigma^2) \rangle}_{\substack{\text{to leading order} \\ 3\sigma^4 - \sigma^4 = 2\sigma^4}}$$

$$\langle \delta_g^3 \rangle = b_1^3 \langle \delta^3 \rangle + 3 b_2 b_1^2 \sigma^4$$

$$\Rightarrow \boxed{S_{3g} = \frac{\langle \delta_g^3 \rangle}{\langle \delta_g^2 \rangle^2} = \frac{b_1^3 \langle \delta^3 \rangle + 3 b_2 b_1^2 \sigma^4}{b_1^4 \sigma^4} = \frac{1}{b_1} S_3 + 3\frac{b_2}{b_1^2}}$$

Similarly, we can calculate correlators in Fourier space

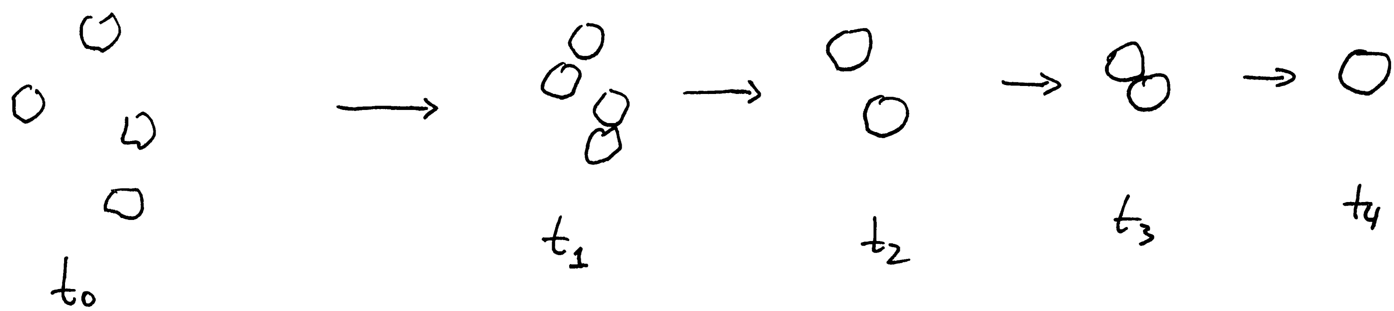
$$\boxed{\begin{aligned} P_g(k) &\simeq b_1^2 P(k) \\ B_g(k_1, k_2, k_3) &= b_1^3 B(k_1, k_2, k_3) + \frac{b_2}{b_1^2}\left[P(k_1) P(k_2) + \text{cyc.} \right] \end{aligned}}$$

$$\Rightarrow \boxed{Q_g = \frac{B_g(k_1, k_2, k_3)}{P_g(k_1) P_g(k_2) + \text{cyc.}} = \frac{1}{b_1} Q + \frac{b_2}{b_1^2}}$$

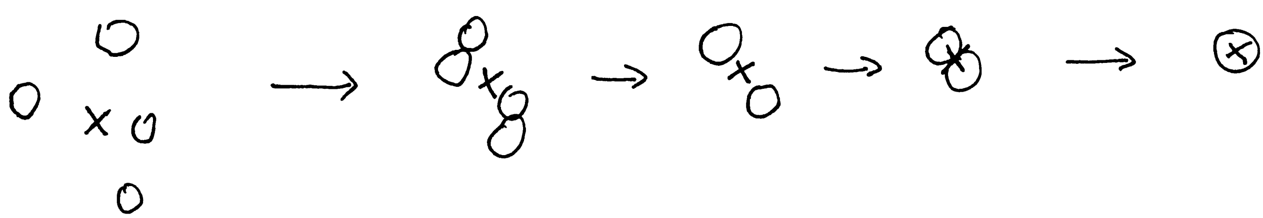
This last expression explains how one can measure b_1, b_2 - Measure Q_g, and predict Q from cosmology (remember that Q is independent of σ_8 to lowest order in PT). Since one can compare Q_g to Q for <u>different triangle shapes</u>, one is able to determine b_1 and b_2/b_1^2 by such comparisons. Notice that at the 2pt level, the power spectrum $P_g \propto b_1^2 \sigma_8^2$ so one cannot break the degeneracy between b_1 & σ_8. That's broken by the bispectrum and is one example of how to use non-Gaussian information. Notice this is not possible to do with S_3 since by measuring S_{3g} one has only one number (there is no triangle shape here). Also, doing it at different smoothing scales R determines nearly degenerate combinations $\frac{S_3}{b_1} + 3\frac{b_2}{b_1^2}$ since S_3 varies very little with R (as opposed to Q with triangle shape).

Now lets examine the limitations of local bias. To do so we ask a simple question: suppose at some time t^*, galaxies do obey perfectly the local bias relation. Is local bias preserved under time-evolution?

The question requires a bit of thought before it can be answered, as we need to define time-evolution for tracers such as galaxy fluctuations. A simple way to do so is to think of the constituent matter a galaxy is made of and trace it back or forward in time by following its center of mass at all times in such a way as the number of objects is <u>conserved</u>. At different times galaxies may merge and their number will change, but we can always keep track of the center of mass of the constituents of galaxy at time t^* and call it the same object at another time - Schematically, if we have merging things will look like as time goes on as:



if we are interested in clustering at time t_4, we can always trace back its constituents and declare they make the same object at <u>all</u> times, so the center of mass (marked as × here) will look like



As long as we do this trick, by definition, the number of "objects" is conserved, so it obeys the continuity equation,

$$\frac{\partial \delta_g}{\partial \tau} + \vec{\nabla} \cdot \left[(1+\delta_g)\, \vec{v}_g \right] = 0$$

To find a closed equation for the evolution of δ_g, we need information on $\vec{v}_g$ - The simplest proposal is that galaxy velocities are <u>unbiased</u>, that is $\vec{v}_g = \vec{v}$, equal to that in the dark matter- this is well-motivated because in the large-scale limit (which we are interested here) Galilean invariance imposes $\vec{v}_g = \vec{v}$ in the $k \to 0$ limit (corrections to this statement scale as k^2) - Thus we have the simple evolution to solve for,

$$\frac{\partial \delta_g}{\partial \tau} + \vec{\nabla} \cdot \left[(1+\delta_g)\, \vec{v} \right] = 0$$

which we can solve using perturbation theory (PT). In linear PT,

$$\frac{\partial \delta_g^{(1)}}{\partial \tau} = -\vec{\nabla} \cdot \vec{v}^{(1)} = -\theta^{(1)} = \frac{\partial \delta^{(1)}}{\partial \tau} = \mathcal{H} f \frac{D}{D_*} \delta_*$$

where $\delta_* = \delta(t = t^*)$. Recall that $\mathcal{H} = \frac{\partial \ln a}{\partial \tau}$ and $f = \frac{d \ln D}{d \ln a}$

$$\Rightarrow \frac{\partial \delta_g^{(1)}}{\partial \ln D} = \delta_* \frac{D}{D_*} = \Theta^{(1)} = -\frac{\theta^{(1)}}{\mathcal{H} f}$$

$$\Theta \equiv \vec{\nabla} \cdot \vec{u} \qquad \vec{u} \equiv -\frac{\vec{v}}{\mathcal{H} f}$$

$$\Rightarrow \frac{\partial \delta_g^{(1)}}{\partial \ln D} = \Theta^{(1)} \quad \Rightarrow \quad \delta_g^{(1)} = \delta_{g*}^{(1)} + \int_{D_*}^{D} d\ln D \; \Theta^{(1)}$$

since $\Theta^{(1)} \propto D \Rightarrow$

$$\delta_g^{(1)} = \underbrace{\delta_{g*}^{(1)}}_{\underbrace{b_1^* \delta_*}_{b_1^* \frac{D_*}{D} \delta^{(1)}}} + \underbrace{\Theta^{(1)}}_{\frac{D}{D_*} \delta_* = \delta^{(1)}} \left(1 - \frac{D_*}{D}\right)$$

$$\Rightarrow \delta_g^{(1)} = \delta^{(1)} \left[1 + (b_1^* - 1) \frac{D_*}{D} \right] \qquad (t \geq t^*)$$

So we see that @ $t > t^*$ we can identify the time-dependent linear bias $b_1 \equiv \delta_g^{(1)} / \delta^{(1)}$

$$\boxed{b_1 = 1 + \frac{(b_1^* - 1)}{D/D_*}}$$

which says that as $t > t^*$ (and thus $D > D_*$) the linear bias decays, and in the long-time limit <u>objects become unbiased</u> ($b_1 \to 1$ as $t \gg t^*$). This should not be surprising since we are assuming that galaxies follow matter velocities, so they will end up tracing matter at late times.

So far, apart from time evolution of bias, at linear level the local form of bias is preserved at $t > t^*$ (i.e. we could identify a linear bias parameter by taking δ_g/δ and seeing that this ratio was space-independent).

Let's go to second-order in PT and find bias evolution:

$$\frac{\partial \delta_g^{(2)}}{\partial \tau} + \theta^{(2)} = - \vec{\nabla} \cdot \left[\delta_g^{(1)} \vec{v}^{(1)} \right]$$

Again, we write $\vec{v}^{(1)} = -\mathcal{H} f \vec{u}$ with $\vec{\nabla} \cdot \vec{u} = \delta^{(1)} = \frac{D}{D_*} \delta_*^{(1)}$

and also $\theta^{(2)} = -\mathcal{H} f \, \Theta^{(2)}$ with $\Theta^{(2)}_{(\vec{k},\tau)} = \frac{D^2}{D_*^2} \int \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\, G_2(\vec{k}_1,\vec{k}_2)\, \delta_1^*(k_1)\delta_1^*(k_2)\, d^3k_1 d^3k_2$

Dividing by $\mathcal{H} f$ we then have:

$$\frac{\partial \delta_g^{(2)}}{\partial \ln D} = \Theta^{(2)} + \vec{\nabla} \cdot \left[\delta_g^{(1)} \vec{u} \right]$$

which we can solve again right away,

$$\delta_g^{(2)} = \delta_{g*}^{(2)} + \int_{D_*}^{D} d\ln D \left\{ \Theta^{(2)} + \vec{\nabla} \cdot \left[\delta_g^{(1)} \vec{u} \right] \right\}$$

$$= \delta_{g*}^{(2)} + \int_{D_*}^{D} d\ln D \underbrace{\left\{ \Theta^{(2)} + \vec{\nabla} \cdot \left[\delta^{(1)} \vec{u} \right] \right\}}_{\propto D^2} + \int_{D_*}^{D} d\ln D \, (b_1^* - 1) \underbrace{\vec{\nabla} \cdot \left[\delta_*^{(1)} \vec{u} \right]}_{\propto D}$$

$$\delta_g^{(1)} = \underbrace{\delta^{(1)}}_{\propto D} + \underbrace{(b_1^* - 1)\, \delta_*^{(1)}}_{\propto \text{ const.}}$$

we can now integrate right away

$$\delta_g^{(2)} = \delta_{g*}^{(2)} + \frac{1}{2}\left[\Theta^{(2)} - \Theta_*^{(2)} \right] + \frac{1}{2} \left\{ \vec{\nabla} \cdot \left[\delta^{(1)} \vec{u} \right] - \vec{\nabla} \cdot \left[\delta_*^{(1)} \vec{u}_* \right] \right\} +$$

$$+ (b_1^* - 1) \left\{ \vec{\nabla} \cdot \left[\delta_*^{(1)} \vec{u} \right] - \vec{\nabla} \cdot \left[\delta_*^{(1)} \vec{u}_* \right] \right\}$$

and impose initial conditions @ $t = t^*$ of the local bias form:

$$\delta_{g*}^{(2)} = \underbrace{b_1^* \, \delta_*^{(2)}}_{\substack{\text{linear bias} \\ \text{acting on} \\ \text{2nd order matter} \\ \text{perturbations}}} + \underbrace{\frac{b_2^*}{2} \left[\delta_*^{(1)} \right]^2}_{\substack{\text{quadratic local bias} \\ \text{acting on linear} \\ \text{matter perturbations}}}$$

And to look at quadratic bias @ t, we need to substract at time t the induced second-order perturbation due to the linear bias @ t acting on 2nd-order matter perturbations, so the relevant quantity is

$$\delta_g^{(2)} - b_1 \delta^{(2)} = \frac{b_2^*}{2}[\delta_*^{(1)}]^2 + b_1^* \delta_*^{(2)} - b_1 \delta^{(2)} + \frac{1}{2}[\Theta^{(2)} - \Theta_*^{(2)}]$$
$$+ \frac{1}{2}\left\{ \nabla\cdot[\delta^{(1)}\vec{u}] - \nabla\cdot[\delta_*^{(1)}\vec{u}_*]\right\} +$$
$$+ (b_1^* - 1)\left\{ \nabla\cdot[\delta_*^{(1)}\vec{u}] - \nabla\cdot[\delta_*^{(1)}\vec{u}_*]\right\}$$

To finish the calculation we need to write out $\delta^{(2)}$ and $\Theta^{(2)}$ in terms of $\delta^{(1)}$ and $\vec{u}$. The results that we know of in Fourier space can be easily translated, e.g.

$$\delta^{(2)}_{(\vec{k})} = \int \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\, d^3k_1\, d^3k_2\ F_2(\vec{k}_1,\vec{k}_2)\ \delta_1(\vec{k}_1)\,\delta_1(\vec{k}_2)$$
$$= \int \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\, d^3k_1 d^3k_2 \left[1 + \frac{\vec{k}_1\cdot\vec{k}_2}{k_2^2} + \frac{2}{7}[(\hat{k}_1\cdot\hat{k}_2)^2 - 1]\right] \delta_1(\vec{k}_1)\,\delta_1(\vec{k}_2)$$

which in configuration space reads

$$\delta^{(2)} = [\delta^{(1)}]^2 + \vec{u}\cdot\vec{\nabla}\delta^{(1)} + \frac{2}{7}\mathcal{G} \equiv \mathcal{D} + \frac{2}{7}\mathcal{G}$$

where
$$\begin{cases} \mathcal{G} \equiv [\nabla_{ij}\nabla^{-2}\delta^{(1)}]^2 - [\delta^{(1)}]^2 \\ \mathcal{D} \equiv [\delta^{(1)}]^2 + \vec{u}\cdot\vec{\nabla}\delta^{(1)} \end{cases}$$

Similarly $\Theta^{(2)} = \mathcal{D} + \frac{4}{7}\mathcal{G}$ (recall velocities have double contribution from tidal fields than densities)

So this means:

$$\nabla\cdot[\delta^{(1)}\vec{u}] = [\delta^{(1)}]^2 + \vec{u}\cdot\vec{\nabla}\delta^{(1)} \equiv \mathcal{D}$$

So our object of interest simply reads,

$$\delta_g^{(2)} - b_1\delta^{(2)} = \frac{b_2^*}{2}[\delta_*^{(1)}]^2 + \delta_*^{(2)} - \delta^{(2)} + \frac{1}{2}[\Theta^{(2)} - \Theta_*^{(2)}]$$
$$+ \frac{1}{2}(\mathcal{D} - \mathcal{D}_*) + (b_*-1)\left\{ \delta_*^{(2)} - \frac{D_*}{D}\delta^{(2)} + \frac{D_*}{D}\mathcal{D} - \mathcal{D}_* \right\}$$

look at contributions not proportional to (b_*-1) or b_2^*:

$$\delta_*^{(2)} - \frac{1}{2}\Theta_*^{(2)} - \frac{1}{2}\mathcal{D}_* = \cancel{\mathcal{D}_*} + \cancel{\frac{2}{7}\mathcal{G}_*} - \frac{1}{2}\cancel{\mathcal{D}_*} - \cancel{\frac{2}{7}\mathcal{G}_*} - \frac{1}{2}\cancel{\mathcal{D}_*} = 0\,!$$

and same for $\delta^{(2)} - \frac{1}{2}\Theta^{(2)} - \frac{1}{2}\mathcal{D} = 0$

So what remains on top of b_2^* term is (b_1^*-1) contribution,

$$\{\ \} = \cancel{\mathcal{D}_*} + \frac{2}{7}\mathcal{G}_* - \cancel{\frac{D_*}{D}\mathcal{D}} - \frac{D_*}{D}\frac{2}{7}\mathcal{G} + \cancel{\frac{D_*}{D}\mathcal{D}} - \cancel{\mathcal{D}_*}$$
$$= -\frac{2}{7}\frac{D_*}{D}\mathcal{G}\left(1-\frac{D_*}{D}\right) \qquad \text{since } \mathcal{G}_* = \mathcal{G}\left(\frac{D_*}{D}\right)^2$$

Then we have the simple result:

$$\delta_g^{(2)} - b_1\delta^{(2)} = \frac{b_2^*}{2}[\delta_*^{(1)}]^2 - \frac{2}{7}\underbrace{(b_1^*-1)\frac{D_*}{D}}_{b_1-1}\mathcal{G}\left(1-\frac{D_*}{D}\right)$$

$$\Rightarrow \boxed{\delta_g^{(2)} - b_1\delta^{(2)} = \frac{b_2^*}{2}\left(\frac{D_*}{D}\right)^2[\delta^{(1)}]^2 - \frac{2}{7}(b_1-1)\,\mathcal{G}\left(1-\frac{D_*}{D}\right)}$$

From here, we identify the local bias quadratic bias evolution:

$$b_2 = b_2^*\left(\frac{D_*}{D}\right)^2$$

and a non-local contribution

$$\boxed{\delta_g^{(2)}\Big|_{\text{non-local}} = -\frac{2}{7}(b_1-1)\left\{[\nabla_{ij}\nabla^{-2}\delta^{(1)}]^2 - [\delta^{(1)}]^2\right\}\left(1-\frac{D_*}{D}\right)}$$

So, local bias <u>is not</u> preserved by time evolution.

The lesson is that we should add to quadratic order (and all orders) the invariants (scalars) that are built out of second-derivatives of the potential

$$[\nabla_{ij}\nabla^{-2}\delta]^2 - [\delta^{(1)}]^2 \underset{\nabla^2\Psi = \delta}{=} (\nabla_{ij}\Psi)^2 - (\nabla^2\Psi)^2$$

[Show results]