Ejercicios de Cosmología (Segunda Parte)

Problema 1:
(2 puntos)

Consider the relic population of neutrinos. At $k_BT \gg 1$ MeV, neutrinos (and antineutrinos) interact quickly enough that they are populated at their thermal abundances. At $k_BT \approx 1$ MeV, the neutrinos stop interacting with the rest of the particles. After that time, the annihilation of the electrons and positrons heats the photons to a temperature that is $(11/4)^{1/3}$ higher than the temperature of the neutrinos.

The mass is negligible near the decoupling redshift, so one can use the relativistic limit $E = qc$. But at low redshift, we will assume the mass is large enough that the neutrinos are non-relativistic.

a) What is the velocity distribution of the massive neutrinos today? In other words, what is dn/dv? To compute this, use the fact that the neutrinos at high temperature are in a thermal distribution for a massless fermion and that the temperature at the decoupling redshift z_d is $(4/11)^{1/3}(1+z_d)2.725$ K. After decoupling, the momenta scale as $(1+z)^{-1}$. You should compute the momentum distribution at z_d and then convert is to the velocity distribution today. (Note that you do not need to compute z_d; it will cancel out).

b) Compute the mean velocity of the neutrinos today. The following integrals could be useful:

$$\int dx \frac{x^n}{e^x - 1} = n!\,\zeta(n+1) \tag{1}$$

$$\int dx \frac{x^n}{e^x + 1} = n!\,\zeta(n+1)(1 - 2^{-n}) \tag{2}$$

where $\zeta(m) = \sum_{k=1}^{\infty} k^{-m}$ is the Riemann zeta function. $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.202$, $\zeta(4) = \pi^4/90$.

Problema 2: Big Bang Nucleosíntesis.
(1 punto)

a) Suppose an extra neutrino species is added to the Universe. Would the predicted helium abundance go up or down?

b) Suppose the weak interactions were stronger than they actually are, so that the thermal equilibrium distribution between neutrons and protons were maintained until $k_B T = 0.25$ MeV. Would the predicted helium abundance be larger or smaller than in the standard model?

c) Suppose the proton-neutron mass difference were larger than the actual value. Would the predicted helium abundance be larger or smaller than in the standard BBN calculation?

Problema 3:
(2 puntos)

Consider the following reaction that takes place in the Universe:

$$p + n \to D + \gamma$$

What fraction of neutrons are in Deuterium when the Universe is 5-minutes old? Use that $\Omega_b h^2 = 0.02$, and that the baryons consist of 14% neutrons and 86% protons.

Problema 4:
(2 puntos)

Consider that we change particle physics so as to include a yet-undiscovered stable massive particle. For simplicity, we will make it spin-0 (meaning that $g = 1$) and call it X. X and its antiparticle $\bar{X}$ interact quickly enough in the early universe that their number densities reach thermal equilibrium. We will imagine that the mass m_X is large, of order the proton mass or larger.

a) If X interacts rarely enough, then it will decouple (interaction rate less than Hubble parameter) when the universe is still hotter than $m_X c^2$. In that case, X remains in a thermal distribution today. Show that this is a cosmological catastrophe by computing $\Omega_m h^2$.

b) If X interacts more quickly, then it remains able to follow the thermal equilibrium prediction as the temperature drops below the rest mass. In other words, as the temperature drops, the X and $\bar{X}$ can annihilate. However, as the number density drops, the annihilation reaction slows and freezes out. This leaves a relic population of X and $\bar{X}$ particles that might be the dark matter today.

Compute the relic abundance of X and $\bar{X}$ particles as a function of the annihilation cross-section σ_a and the mass m_X. You may assume that the reaction ends when the reaction rate is equal to the Hubble parameter. You may assume that the Hubble parameter is to be computed at a time when the temperature of the universe is $0.1 m_X c^2/k$. You may assume that $g_* = 100$; g_* is the total number of relativistic spin-states, including a penalty of 7/8 for fermions. Assume a zero chemical potential.

Compute the cross section the particle should have if we want it to be the dark matter today (i.e., $\Omega_X h^2 \approx 0.11$).

Problema 5:
(2 puntos)

The scale of the acoustic peaks in the CMB power spectrum is set by the comoving distance that a sound wave could travel between the time when the perturbation was created ($t \approx 0$) and the epoch of recombination ($t = t_*$). In other words,

$$s = \int_0^{t_*} dt\, c_s(z)\,(1+z)$$

where c_s is the sound speed. s is called the sound horizon.

a) The sound speed results from the competition between restoring forces and inertia. For a simple fluid, $c_s^2 = dp/d\rho$. Before recombination, the photons and baryons can be assumed to be locked together. For the photons, $p_\gamma = \rho_\gamma c^2/3$, but for the baryons $p_b \approx 0$.

Compute the sound speed of the combined fluid, $c_s^2 = (dp/da)/(d\rho/da)$ in terms of redshift and cosmological parameters. Remember that the photons have their pressure and density scale by a^{-4}, while the baryon density scales only as a^{-3}.

Show that for $\Omega_b h^2 = 0.02$ and $\Omega_{\rm rad} h^2 = 4.2 \times 10^{-5}$ the approximation that $c_s \approx c/\sqrt{3}$ is good to $\sim 30\%$ for $z > 1000$.

For the rest of the problem, assume $c_s = c/\sqrt{3}$.

b) Now compute the sound horizon s. Assume only matter and radiation. Assume recombination occurs at redshift z_*, which we'll take to be 1000. Argue why one cannot neglect the radiation contribution to $H(z)$ in this calculation. Assume $\Omega_m h^2 = 0.14$ and $\Omega_{\rm rad} h^2 = 4.2 \times 10^{-5}$.

c) Demonstrate that the position $\ell_{\rm acoustic}$ of the acoustic peaks depends primarily on Ω_K and only slightly on Λ and h, by varying the relevant parameters.

Problema 6:
(1 punto)

a) Calculate the mean free path length of a photon in the early universe as a function of redshift, assuming that the dominant opacity is Thompson scattering and that all of the electrons in the universe are ionized. Assume $\Omega_b h^2 = 0.02$, $\Omega_m h^2 = 0.14$, and treat all the baryons as hydrogen. Compare your result to the Hubble distance (i.e. to $c/H(z)$).

b) At $z = 1000$, what ionization fraction would be needed to allow the mean free path to be equal to the Hubble distance (which is the rough criteria for the photons to stream freely past the electrons).

That this number is less than 1 means that the recombination of the electrons and protons is important to the physics of the CMB. In particular, because recombination sweeps the ionization fraction from 1 to about 10^{-4} in about 10% of the Hubble time (then), it means that the photons we see last scattered in a rather thin shell in redshift.

Alternative cosmologies might keep the hydrogen ionized even at $z < 1000$ (by some large amount of energy injection, of course). In such cosmologies, the photons still eventually decouple from the electrons (you can use part (a) to say when this is), but they do so over an entire Hubble time. The resulting last-scattering surface is very thick, resulting in a significant weakening of the CMB anisotropies.