Robertson-Walker Metric

In general relativity (GR), as in special relativity (SR), one uses coordinates to describe events in space-time. In SR, we have that the interval ds^2 between events is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \equiv \eta_{\mu\nu}\, dx^\mu dx^\nu$$

where we defined the Minkowsky metric $\eta_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$ and $dx^\mu = (c\,dt, dx, dy, dz)$. Whereas in SR $\eta_{\mu\nu}$ is fixed, in GR the metric becomes a dynamical field, the solution of Einstein's equations.

Before we discuss solutions of Einstein's equations, it turns out that the symmetries implied by the cosmological principle, homogeneity and isotropy, already determined the functional form of the metric (up to the sign of the curvature),

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \underset{\uparrow\atop H \,\&\, I}{=} c^2 dt^2 - a^2(t)\Big[\overbrace{\frac{dr^2}{\underbrace{1-kr^2}} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)}^{\text{comoving coordinates}} \Big]$$

this is the "cosmic time", i.e. proper time measured by comoving observers with r, θ, ϕ = const.

"scale factor" overall time dependence due to the expansion of the Universe, $a(t)$ is solution of Einstein's eqs.

by isotropy this is a function of r, k denotes the spatial curvature

$$k = \begin{cases} 0 & \text{flat} \quad \text{no spatial curv.} \\ +1 & \text{closed} \quad <0 \;\text{"}\;\text{"} \\ -1 & \text{open} \quad >0 \;\text{"}\;\text{"} \end{cases}$$

This is the RW metric.

Most of the time we will work with units such that $c \equiv 1$.

[The 3D Ricci scalar is $\frac{6k}{a^2}$]

Another convenient choice of time variable that is often used is conformal time τ defined by:

$$dt^2 \equiv a^2(t)\, d\tau^2 \Rightarrow d\tau = dt/a$$

so that the metric reads :

$$g_{\mu\nu} = a^2(t)\, \tilde{g}_{\mu\nu}$$

where $\tilde{g}_{\mu\nu}$ is now independent of τ. This defines a conformal transformation (preserves angles).

There are many ways of writing the spatial part of the metric. Another common choice is $\left(\chi \equiv \int dr/\sqrt{1-kr^2} \right)$

$$- a^2(t) \left[d\chi^2 + \left\{ \begin{matrix} \sin\chi \\ \sinh\chi \\ \chi \end{matrix} \right\}^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right] \qquad k = \left\{ \begin{matrix} 1 \\ -1 \\ 0 \end{matrix} \right\}$$

(note that for the flat case, $k=0$, this agrees with the previous parametrization)

Kinematics of RW metric

We can understand some effects due to the expansion of the universe before we actually solve Einstein's equations for $a(t)$. This is what we call kinematics, as opposed to dynamics.

The first concept is redshift. Consider the propagation of light $\Rightarrow$ $ds^2 = 0$

and let's take a radial trajectory $d\theta = d\phi = 0$

$$\Rightarrow dt^2 = \frac{a^2(t)\, dr^2}{1-kr^2}$$

let's take an observer at the origin $r=0$, and a light source at fixed comoving coordinate r_s. If the source emits a light pulse @ $t=t_e$ and this is received by the observer @ $t=t_0$, we have :

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^{r_s} \frac{dr}{\sqrt{1-kr^2}}$$

~~as~~ the RHS does not depend on time if r_s is fixed -
for the next pulse, emitted @ $t = t_e + \delta t_e$ we have as well

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_0^{r_s} \frac{dr}{\sqrt{1-kr^2}} \Rightarrow \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)}$$

$$\Rightarrow \int_{t_e}^{t_e+\delta t_e} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)} \Rightarrow \frac{dt_e}{a(t_e)} = \frac{dt_0}{a(t_0)}$$

Now, if we think of these "pulses" as the peaks of the wave trains, we have that the light wavelength @ emission $\lambda_e \propto dt_e$, whereas the wavelength @ observer is $\lambda_0 \propto dt_0$, then

$$\frac{\lambda_{obs}}{\lambda_{emitted}} = \frac{dt_0}{dt_e} = \frac{a(t_0)}{a(t_e)} \equiv 1+z$$

redshift

we see that if $a(t_0) > a(t_e)$, that is, if the universe expands ($t_0 > t_e$) then the wavelenght of light is stretched by the expansion of the universe (shifting to the red part of the spectrum) and thus $z>0$ -

Now, let's consider that the source at $r=r_1$ has an intrinsic luminosity (energy/time) of L - If the observer @ $r=0$ has a detector of physical area A, the power @ the detector ~~[illegible]~~ is

$$P = L \underbrace{\left(\frac{a_1}{a_0}\right)^2} \frac{A}{4\pi a_0^2 r_1^2}$$

$A \rightarrow$ phys. area of detector

$4\pi a_0^2 r_1^2 \rightarrow$ phys. area over which L is spread @ detection time

one suppression factor is due to energy of photons being smaller
another " " " " time between photon arrivals is also stretched

Then, the flux at the observer is $F = \frac{P}{A} = \frac{L}{4\pi a_0^2 r_1^2 (1+z)^2}$

We define the <u>luminosity distance</u> d_L as

$$F = \frac{L}{4\pi\, d_L^2}$$

which in the absence of expansion will give the distance to the source, given L and F. We see that:

$$\boxed{d_L = r_1\, a_0\, (1+z)}$$

This expression is not very useful as it stands, since it depends on comoving distance to the source r_1. In order to obtain $r_1(z)$ we use that

$$\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \begin{cases} \arcsin r_1 \simeq r_1 + r_1^3/6 + \dots & k=+1 \\ r_1 & k=0 \\ \operatorname{arcsinh} r_1 = r_1 - r_1^3/6 & k=-1 \end{cases}$$

and expanding about times close to the present $(H_0(t-t_0) \ll 1)$ we have

$$a(t) \simeq a_0 + \dot{a}_0 (t-t_0) + \frac{\ddot{a}_0}{2}(t-t_0)^2$$

$$\Rightarrow \quad \frac{a(t)}{a_0} = 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots$$

where $H_0 = \dfrac{\dot{a}_0}{a_0}$ is the present value of the Hubble constant

$q_0 = -\dfrac{\ddot{a}_0}{a_0 H_0^2}$ is the deceleration parameter

see $\Rightarrow$ Homework

$$\boxed{H_0 d_L = z + \frac{1}{2}(1-q_0) z^2 + \dots}$$

This relationship is very important as it leads to a classical test of cosmological parameters. We will come back to this shortly. What one needs in this case is knowledge of some kind of "standard candle", objects of known L, measure F, get d_L and do this as a function of object redshift to get a handle on, say, q_0 and thus the acceleration

Dynamics

The evolution of the scale factor is given by solving Einstein's equations:

$$\underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} \equiv \underbrace{\underbrace{R_{\mu\nu}}_{\text{Ricci tensor}} - \frac{1}{2} g_{\mu\nu} \underbrace{R}_{\text{Ricci scalar}}}_{\text{second derivatives of the metric}} = 8\pi G \underbrace{T_{\mu\nu}}_{\text{stress-energy tensor of all matter, radiation, etc}}$$

These field equations are analogous to things you have seen before:

i) Newtonian gravity $\quad \underbrace{\nabla^2 \phi}_{\text{2nd derivatives of grav. potential}} = 4\pi G \underbrace{\rho}_{\text{mass density}}$

ii) Electromagnetism $\quad \Box^2 A_\mu = \frac{4\pi}{c} J_\mu$

the analogy is pretty clear
$$\begin{cases} A_\mu \to g_{\mu\nu} \\ \Box^2 A_\mu \to G_{\mu\nu} \\ T_{\mu\nu} \to J_\mu = (\rho, \vec{J}) \end{cases}$$

Similarly to E&M where charge conservation reads $\partial_\mu J^\mu = 0$, in GR we have stress-energy conservation $\nabla_\nu (T_{\mu\nu}) = 0$

Given the symmetries of the metric due to H & I, this also requires $T_{\mu\nu}$ to have a special form. Indeed $T_{\mu\nu}$ must be diagonal (as the metric) and by isotropy all space components must be the same, i.e.

$$T^\mu{}_\nu = \begin{bmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{bmatrix}$$

In the case of a perfect fluid, which we assume here, the energy density ρ is the 00 component, and p is the pressure. The form of $T^\mu{}_\nu$ comes fact from a more general result of the stress-energy of a perfect fluid in GR:

$$T_{\mu\nu} = -p\, g_{\mu\nu} + (\rho + p)\, u_\mu u_\nu$$

with 4-velocity u_μ. Since we are in comoving frame, where the fluid is locally at rest $u_0 = 1$ and $u_i = 0$, then we get back the previous result.

Conservation of stress-energy leads to a very simple result:

$$\nabla_\nu (T_{\mu\nu}) = 0 \quad \underset{\substack{\text{FRW metric} \\ + T^\mu_\nu \text{ above}}}{\Rightarrow} \quad \underbrace{d(\rho a^3)}_{\substack{\text{Change in} \\ \text{energy of a} \\ \text{comoving volume}}} = \underbrace{-p\, d(a^3)}_{-p\, dV \text{ work}}$$

1^{st} law of thermodynamics!
(a perfect fluid conserves entropy $\Rightarrow dE = T dS + p dV = -p dV$)

This simple conservation law leads to a straightforward derivation of $\rho(t)$ once the equation of state $p(\rho)$ is specified.

Many relevant cases can be covered by the simple equation of state:

$$p_{(t)} = w\, \rho_{(t)} \qquad \text{where } w \text{ is constant, indep. of time}$$

$$\Rightarrow \quad \rho(t) \sim a^{-3(1+w)}$$

Let's consider some special cases:

i) Matter: $p \simeq 0 \Rightarrow w = 0 \Rightarrow \rho_M(t) \sim a^{-3}$ (a comoving element conserves mass as it expands)

To see $p \simeq 0$, remember if we put c's back $\frac{p}{c^2} = w \rho$, for nonrelativistic matter $p \sim v^2 \rho \Rightarrow \frac{p}{c^2} \sim \frac{v^2}{c^2} \rho \simeq 0$. Keep in mind w of order unity means (due to c^2 factor) relativistic-size pressure!

ii) Radiation: $p = \frac{1}{3} \rho \Rightarrow w = 1/3 \Rightarrow \rho_R \sim a^{-4}$ (additional factor of a^{-1} due to photon redshift as universe expands)

iii) Vacuum energy: $p = -\rho \Rightarrow w = -1 \Rightarrow \rho_\Lambda = \text{const}!$

This is the cosmological constant case, in which

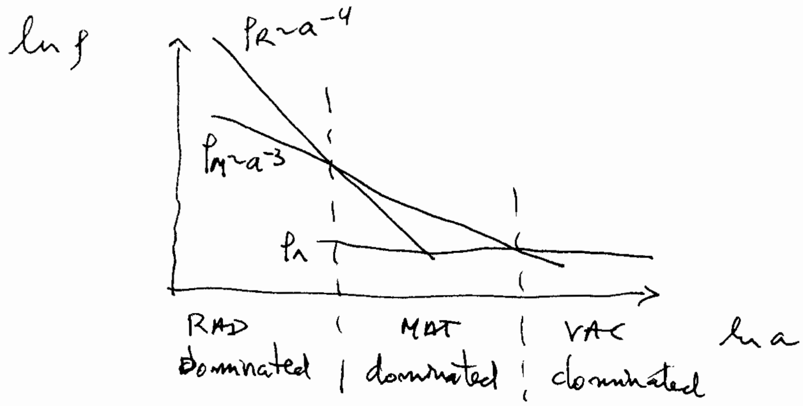
$$T_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \qquad \Rightarrow \qquad p = -\frac{\Lambda}{8\pi G} = -\rho$$

and adds a term to the RHS of Einstein eqs. that is $\Lambda g_{\mu\nu}$

How is it possible to expand and maintain the energy density? Well, if we had this stuff inside a cylinder and use a piston to expand it, what happens is that the negative pressure means we must do work to expand, and this work goes into the increase in energy inside:

$E_2 > E_1$

As a consequence of these scalings of $\rho(t)$, it means that if these 3 components are present, the evolution of the universe can be dominated by different components at different times (depending on how much there is of each of them), e.g.



Now, let's look at Einstein equations. By isotropy and homogeneity, only the 00 and ii components are non-trivial:

$k = -1, 0, +1$

$$00: \quad \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

$$ii: \quad 2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p$$

Actually, only one of these equations is necessary, the other can be derived ~~from~~ by adding stress-energy conservation! So, things are very simple.

In fact the 00 equation can be thought as the Newtonian analog of conservation of energy for a test particle ~~an~~ in an expanding spherical region of mass M and radius r:

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r} = \frac{1}{2} m \dot{r}^2 - \frac{GMm}{r} = \text{const.}$$

$$\Rightarrow \quad \frac{\dot{r}^2}{r^2} - \frac{2GM}{r^3} = \frac{\text{const.}}{r^2} \quad \Rightarrow \quad \left(\frac{\dot{r}}{r}\right)^2 - \frac{\text{const}}{r^2} = \frac{8\pi G}{3} \rho$$

the constant being identified in GR with the curvature!

It is customary to write the two independent equations as follows (known as Friedmann equations)

$$\begin{cases} H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) \end{cases}$$

The second equation is very important as it gives how the universe accelerates (the first one gives the expansion rate, or speed, measured by the Hubble constant H). We see that acceleration is given by $(\rho + 3p)$, and that in GR pressure gravitates (as it is part of $T_{\mu\nu}$), with negative pressure creating repulsive behavior ($\ddot{a} > 0$).

We can rewrite the first Friedmann eq as:

$$\frac{k}{a^2 H^2} = \frac{8\pi G}{3H^2} \rho - 1$$

this defines a characteristic density, for which the universe is flat ($k=0$) the <u>critical density</u> ρ_{crit}:

$$\rho_{crit} \equiv \frac{3H^2}{8\pi G} \quad \Rightarrow \quad \frac{k}{a^2 H^2} = \frac{\rho}{\rho_{crit}} - 1$$

Today, this corresponds to:

$$\rho_{crit}]_0 = \frac{3H_0^2}{8\pi G} = 1.8788 \times 10^{-29}\, h^2 \, \frac{g}{cm^3}$$

Therefore, the sign of the curvature is determined by the fraction:

$\Omega_{TOT} = \frac{\rho}{\rho_{crit}}$ (TOT stands for all ρ's : RAD, MAT, etc)

$$\begin{cases} \Omega_{TOT} = 1 \Rightarrow k=0 \quad \text{(flat)} \\ \Omega_{TOT} < 1 \Rightarrow k=-1 \quad \text{(open)} \\ \Omega_{TOT} > 1 \Rightarrow k=+1 \quad \text{(closed)} \end{cases}$$

Since $\rho = \sum_i \rho_i$ for different components it is useful to define Ω_i's as well : $\frac{\rho}{\rho_{crit}} = \Omega_{TOT} = \sum_i \Omega_i$ (e.g. Ω_M, Ω_Λ, Ω_{RAD}, …)

Another way of writing Friedmann equation is by thinking of curvature as an energy density ($\propto a^{-2}$), then

$$\rho_k = -\frac{3k}{8\pi G\, a^2} \quad \text{or} \quad \Omega_k = -\frac{k}{a^2H^2}$$

(Ω_i's are in general functions of time!)

$$\Rightarrow -\Omega_k \equiv \frac{k}{a^2H^2} = \Omega_{TOT} - 1 = \Omega_M + \Omega_{RAD} + \Omega_\Lambda + \dots - 1$$

(or $\Omega_k + \Omega_{TOT} = 1$, always!)

Notice : $\Omega_\Lambda = \frac{\Lambda}{3H^2}$ $\left(= \frac{\rho_\Lambda}{\rho_{crit}}\right)$

You can check that the deceleration parameter is given by

$$q = \frac{\Omega_M}{2} + \underbrace{\Omega_{RAD}}_{\simeq 0} - \Omega_\Lambda \underset{\text{if } \Omega_{TOT}=1}{=} \frac{3}{2}\Omega_M + \underbrace{2\Omega_{RAD}}_{\simeq 0} - 1$$

Or, for one component with equation of state $p = w\rho$:

$$q = \Omega_{TOT}\frac{(1+3w)}{2}$$

Since $q \equiv -\ddot{a}/(a_0H_0^2)$, the universe accelerates (as oppose to decelerating) if the equation of state is such that $\underline{w < -1/3}$.

couple of points worth noticing from the Friedmann equation,

i) In order to avoid having the universe expand (i.e. static) we need

$$H=0 \Rightarrow \frac{8\pi G}{3}\rho = \frac{k}{a^2} \qquad \ddot{a}=0 \Rightarrow \rho+3p=0$$

Which requires a ~~___~~ positive curvature and $w=-1/3$ - In order to achive this Einstein introduced the cosmological constant:

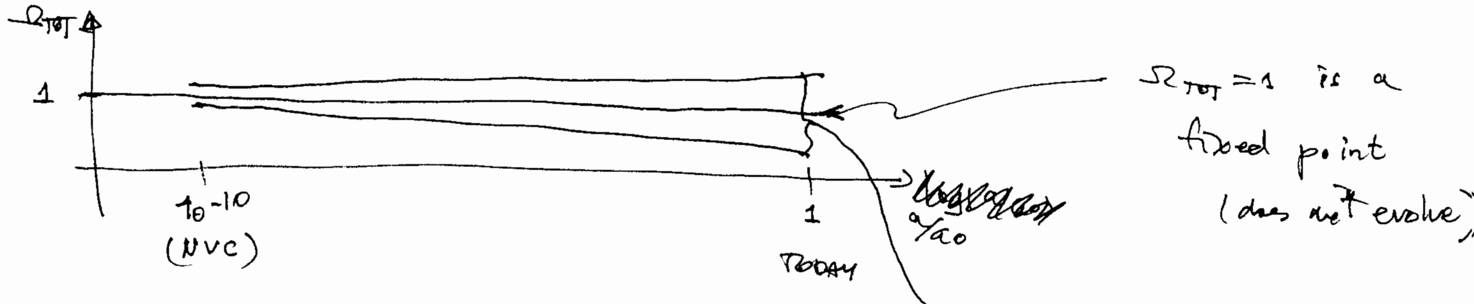
$$\rho = \rho_m + \rho_\Lambda = \frac{3k}{8\pi G a^2} = -3p \underset{\substack{\uparrow \\ \text{only } \Lambda \\ \text{has pressure}}}{=} 3\rho_\Lambda = \frac{3\Lambda}{8\pi G} \Rightarrow \Lambda = \frac{k}{a^2}$$

However, this doesn't work because it is unstable to perturbations! Try it!

ii) From $-\Omega_k = \frac{k}{a^2H^2} = \Omega_{TOT}-1$ we see that at early times curvature is negligible - Indeed, for $H^2 \sim a^{-3}$ (MAT), a^{-4} (RAD)

$$\Rightarrow \Omega_k \sim \begin{cases} a & \text{MAT} \\ a^2 & \text{RAD} \end{cases} \quad \text{as } a\to 0 \quad \Rightarrow \quad \Omega_{TOT} \xrightarrow[a\to 0]{} 1$$

Now, today Ω_{TOT} is of order one (~~act~~ actually $=1$ to very good accuracy!) Since deviations away from $\Omega_{TOT}=1$ grow fast (as $\sim a$ or a^2 early in RAD era), unless Ω_{TOT} is very close to 1 early (say @ Nucleosynthesis when $\frac{a}{a_0} \simeq 10^{-10}$) we would today have clearly Ω_{TOT} very different from 1



Unless $\Omega_{TOT}=1$ exactly, there is no convincing explanation for such a fine tunning of initial conditions : <u>FLATNESS PROBLEM</u>

any tiny difference $\Omega_{TOT}\neq 1$ gets very quickly amplified

Some particular solutions of the Friedman equation which you should know right away are

i) Flat, MATTER dominated universe : $k=0$, $\Omega_M = 1$

$$\Rightarrow \quad H^2 = \frac{8\pi G}{3}\rho_M \sim a^{-3} \quad \text{or} \quad \dot{a}^2 \sim \frac{1}{a} \quad \text{or} \quad \sqrt{a}\,da \sim dt, \quad a^{3/2} \sim t$$

$$\Rightarrow \boxed{a(t) \sim t^{2/3}}$$

ii) Flat, RAD dom.

$$H^2 \sim a^{-4} \quad \Rightarrow \quad \dot{a} \sim \frac{1}{a} \qquad a\,da \sim dt \qquad a^2 \sim t \Rightarrow$$

$$\boxed{a(t) \sim t^{1/2}}$$

iii) in general for an equation of state $p = w\rho$, $k=0$ flat case

gives $\boxed{a \sim t^{\frac{2}{3}(1+w)^{-1}}}$

In the special case of $w = -1$ $\quad H^2 = \text{const.} \equiv H_0^2 \Rightarrow \quad \dot{a} = a H_0 \quad \frac{da}{a} = dt\, H_0$

$$\Rightarrow \boxed{a(t) \propto e^{H_0 t}}$$

exponential expansion

In the general case that $\Omega_{TOT} \neq 1$, there are some parametric solutions but they are not very illuminating (you can check them up in books).