

Random (Gaussian) fields

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Before we enter the discussion on cosmological perturbations we need to review a few key concepts on statistics. The reason is that in cosmology we do not predict the evolution of single systems but rather the statistical properties of a large set of them.

In our case the "set" we are after are basically density perturbations. In cosmology there is no way of predicting which particular point in the universe should be ^{over}dense or underdense from first principles, it rather we can predict what's the probability that a point in space should be dense or underdense, or what's the probability that if point \vec{x}_1 is ^{over}dense, then point \vec{x}_2 is overdense, etc. This is particularly so as the source of density perturbations is thought to be quantum fluctuations from Inflation, all we can make are statistical predictions. Therefore, we will think of the observable universe as being a particular realization out of a statistical ensemble of possibilities.

The way we model this is by using the concept of a random field. This is the natural generalization of a random variable, say r , which has some probability distribution function (PDF) $P(r)$. The statistical properties of r are fully determined by $P(r)$. For example,

$$\langle r \rangle = \int r P(r) dr \quad : \text{average} \quad \left(\int P(r) dr = 1 \right)$$

$$\langle r^2 \rangle = \int r^2 P(r) dr \quad : \text{second moment}$$

$$\sigma_r^2 \equiv \langle r^2 \rangle - \langle r \rangle^2 \quad : \text{variance}$$

$$\langle r^3 \rangle = \int r^3 P(r) dr \quad : \text{third moment}$$

Under sufficiently nice conditions, knowing all the moments allows one to reconstruct the PDF - We get to measure moments, as we'll discuss.

Then from this we can say something about $P(\sigma)$, for example. (2)

we said, a random field is a generalization of random variable, for particular a random field $\phi(\vec{x})$ means that we have a random variable at each point in space \vec{x} , so we have an infinite number of random variables - We write any quantity that fluctuates in time as its mean value plus fluctuations away from this mean:

$$\phi(\vec{x}, t) = \underbrace{\langle \phi(\vec{x}, t) \rangle}_{\text{mean}} + \underbrace{\delta\phi(\vec{x}, t)}_{\text{fluctuation}} \equiv \bar{\phi}(t) + \delta\phi(\vec{x}, t)$$

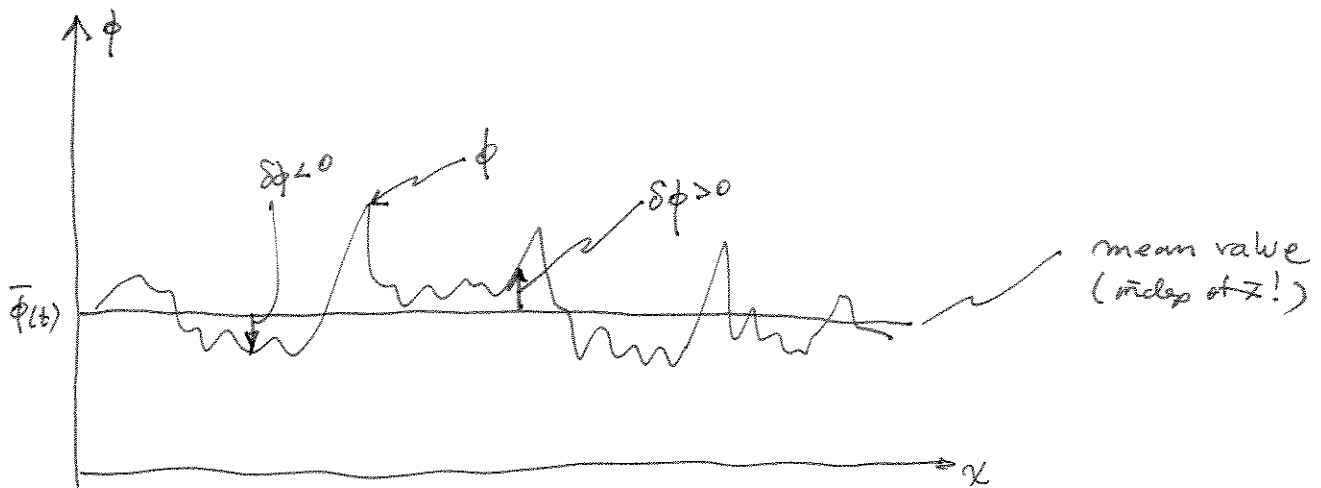
to that we have made the assumption that the mean depends only on time, this is so ~~under~~ under conditions we discuss below. First note that typically one defines a dimensionless fluctuation amplitude:

$$\delta_\phi = \frac{\phi - \bar{\phi}}{\bar{\phi}} = \frac{\delta\phi}{\bar{\phi}}$$

and sometimes is easier to discuss the statistical properties of δ_ϕ , which

is zero mean: $\langle \delta_\phi \rangle = \langle \frac{\phi - \bar{\phi}}{\bar{\phi}} \rangle = \frac{1}{\bar{\phi}} \langle \phi - \bar{\phi} \rangle = \frac{1}{\bar{\phi}} [\bar{\phi} - \bar{\phi}] = 0$

the picture is -



now, a few key points - First, in cosmology we are only capable of observing a single universe (unfortunately!), therefore, unlike usual experimental

sciences, we cannot "run the universe" again from a different (3)
of initial conditions (consistent with the statistical properties) and
~~the~~ observe different realizations of the ensemble, and then
average to construct the expectation values.

Therefore we assume the ergodic hypothesis, where cosmic random
fields are assumed to be ergodic, which means that averages
of the ensemble are equal to spatial averages:

$$\int \Phi P(\Phi) d\Phi = \langle \Phi(x,t) \rangle \stackrel{\text{ergodicity}}{=} \frac{1}{V} \int_V d^3x \Phi(x,t) \equiv \bar{\Phi}(t)$$

is a key assumption, as otherwise it would be hard to do much in
cosmology. Note that by definition we get that the average of Φ is
a function of time $\bar{\Phi}(t)$.

What constraints does ergodicity impose? When is it valid? Mathematically,
ergodicity holds for Gaussian (we'll define this shortly) random fields with
continuous power spectra which are statistically homogeneous

As we have an infinite number of random variables in a random field, we
take averages of "different random variables", i.e. of products
of the field at different points - this defines the correlation functions

$\langle \delta_\Phi(x_1) \delta_\Phi(x_2) \rangle$: two-point correlation function

$\langle \delta_\Phi(x_1) \delta_\Phi(x_2) \delta_\Phi(x_3) \rangle$: 3-pt " "

⋮

$\langle \delta_\Phi(x_1) \dots \delta_\Phi(x_N) \rangle$: N-pt " "

When points coincide, they reduce to moments, i.e. $\langle \delta_\Phi(x)^2 \rangle$, $\langle \delta_\Phi(x)^3 \rangle$, ..

Now in our case we are thinking about physics, these correlation
functions will go to zero as you take some point to ∞ compared to the

thers, these correlation functions will vanish as some inverse
er of separation (more accurate statement on this later, when we
cross connected correlation functions) -

If the fluctuations were independent from point to point, then
se correlators vanish (this is the case for Poisson fluctuations) -
gain, we assume ergodicity so we'll calculate this using spatial
verages, instead of the ensemble average which would be:

$$\langle \delta_\phi(x_1) \dots \delta_\phi(x_N) \rangle = \int \underbrace{P[\delta_\phi(x_1), \dots, \delta_\phi(x_N)]}_{\text{joint PDF for } N \text{ "random variables" } \delta_\phi(x_1) \dots \delta_\phi(x_N)} \delta_\phi(x_1) \dots \delta_\phi(x_N) D\delta_\phi(x_1) \dots D\delta_\phi(x_N)$$

we turn ergodicity into a tool for calculating corr. function well be discussed
rthly, we first need to introduce another pair of key properties.

Random Fields in cosmology are typically statistically homogeneous
& isotropic, a natural extension of the ideas of homogeneity
& isotropy to correlation functions. It means that correlation
notions are invariant under translations (stat. homogeneity)
& rotations (stat. isotropy).

stat. homog. \downarrow

stat homog: $\langle \delta_\phi(x_1) \dots \delta_\phi(x_N) \rangle \stackrel{\text{stat. homog.}}{=} f(x_{21}, x_{31}, \dots, x_{N1})$

that is, f is only a function of the relative separation between points

$x_{21} \equiv x_2 - x_1$
 $x_{31} \equiv x_3 - x_1$
etc

then, if we do a translation $\vec{x}_i \rightarrow \vec{x}_i + \delta\vec{x}$ $i=1, \dots, N$
although the field themselves are not homogeneous: $\delta_\phi(x_i) \neq \delta_\phi(x_i + \delta x)$
that's because precisely they are fluctuating!)

The correlation function is invariant.

at. isotropy: correlation functions are invariant under rotations!

is means that the physics that created and eroded these fluctuations is not distinguish a particular point from another (stat. homog) or particular direction (stat. isotropy).

Note this simplifies correlation functions in a dramatic way:

- two-point function in 3D

$$\langle \delta_\phi(\vec{x}_1) \delta_\phi(\vec{x}_2) \rangle = \xi_\phi(\vec{x}_1, \vec{x}_2) \xrightarrow{\text{STAT HOMO.}} \xi_\phi(\vec{x}_1 - \vec{x}_2) \xrightarrow{\text{STAT ISOTROPY}} \xi_\phi(|\vec{x}_1 - \vec{x}_2|)$$

a function of 6 variables in principle
function of 3 var.
function of 1 var.

= function

$$\langle \delta_\phi(\vec{x}_1) \delta_\phi(\vec{x}_2) \delta_\phi(\vec{x}_3) \rangle = \xi_\phi(\vec{x}_{31}, \vec{x}_{21}) = \xi_\phi(x_{21}, x_{31}, x_{12})$$

9 vars.
STAT HOMO. 6 vars
STAT ISOTROPY 3 vars.

we are ready to say how one would take advantage of ergodicity, st. homogeneity and isotropy in observations to measure correlation notions - for the two-point function:

$$\xi_\phi(r) = \int \frac{d\Omega_r}{4\pi} \int_V \frac{d^3r'}{V} \delta_\phi(\vec{r}') \delta_\phi(\vec{r}' + \vec{r})$$

and then from right to left: take a point \vec{r}' , evaluate δ_ϕ there, then take other point at $\vec{r}' + \vec{r}$ (where \vec{r} has magnitude r , the distance at which we want to calculate the two-point function) evaluate δ_ϕ , multiply together to correlate, then move to another \vec{r}' and keep going

until you use all your V - Note that so far we just used translation invariance. We can further average over all different directions of \vec{r} , that's the outer integral over direction space

direction space

It is natural to deal with fluctuations in Fourier space rather than "real space" as we have been doing so far - the reason for this is that small amplitude fluctuations can be linearized and that means each Fourier mode evolves independently - Statistically, translation invariance singles out Fourier modes as they are two-point-wise correlated, as we shall see.

The Fourier transform is defined as (beware of different conventions!)

$$A(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}} A(\vec{x}) \frac{d^3x}{(2\pi)^3}$$

and the inverse: $A(\vec{x}) = \int e^{i\vec{k}\cdot\vec{x}} A(\vec{k}) d^3k$

from the first of this we see that if $A(\vec{x})$ is real $\Rightarrow A(-\vec{k}) = A(\vec{k})^*$,

which means that $A(-\vec{k})$ has the same information as $A(\vec{k})$ [it is just the complex conjugate], it is the same "Fourier mode".

Also note when dealing with inverses, it is useful to remember the delta function $\delta_D(\vec{x})$, which can be written as

$$(2\pi)^3 \delta_D(\vec{x}) = \int e^{-i\vec{k}\cdot\vec{x}} d^3k$$

here $\int \delta_D(\vec{x}) d^3x = 1$ $\delta_D(\vec{x}) = 0$ unless $\vec{x} = 0$, $\int \delta_D(\vec{x}) F(\vec{x}) d^3x = F(\vec{0})$

or more generally $\int F(\vec{x}) \delta_D(\vec{x} - \vec{x}_0) d^3x = F(\vec{x}_0)$

o, when we decompose any random field $\Phi(x)$ in Fourier

does

$$\Phi(x) = \int e^{i\vec{k}\cdot\vec{x}} d^3k \Phi(\vec{k})$$

\uparrow Fourier modes (waves) \uparrow Fourier coefficients

can now think of the Fourier coefficients $\Phi(\vec{k})$ as a random field, or a collection of random variables (two per \vec{k}^2 , since Φ is plax!). We can then take correlators in Fourier space:

$$\langle \delta_\phi(\vec{k}) \rangle = \left\langle \int e^{-i\vec{k}\cdot\vec{x}} \frac{d^3x}{(2\pi)^3} \delta_\phi(\vec{x}) \right\rangle = \underbrace{\int e^{-i\vec{k}\cdot\vec{x}} \frac{d^3x}{(2\pi)^3}}_{\text{go outside average over PDF because they are not random variables}} \underbrace{\langle \delta_\phi(\vec{x}) \rangle}_{=0} = 0$$

and, similarly:

$$\begin{aligned} \langle \delta_\phi(\vec{k}_1) \delta_\phi(\vec{k}_2) \rangle &= \int \underbrace{e^{-i\vec{k}_1\cdot\vec{x}_1} e^{-i\vec{k}_2\cdot\vec{x}_2}}_{e^{-i\vec{k}_1\cdot\vec{x}_2} e^{-i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}_2}} \underbrace{\frac{d^3x_1}{(2\pi)^3} \frac{d^3x_2}{(2\pi)^3}}_{\frac{d^3x_{12}}{(2\pi)^3} \frac{d^3x_{12}}{(2\pi)^3}} \underbrace{\langle \delta_\phi(\vec{x}_1) \delta_\phi(\vec{x}_2) \rangle}_{\delta^3(\vec{x}_1-\vec{x}_2)} \\ &= \delta^3(\vec{k}_1+\vec{k}_2) \underbrace{\int \frac{d^3r}{(2\pi)^3} \xi_\phi(r) e^{-i\vec{k}_1\cdot\vec{r}}}_{\equiv P_\phi(k_1)} : \text{power spectrum} = \text{Fourier transform of } \xi \end{aligned}$$

$$\langle \delta_\phi(\vec{k}_1) \delta_\phi(\vec{k}_2) \rangle = \delta^3(\vec{k}_1+\vec{k}_2) P_\phi(k_1)$$

st thing to note is that this correlator, unlike in real space, is only nonzero $\vec{k}_1 = -\vec{k}_2$, which means that Fourier modes are uncorrelated, (recall $\delta(k^2)$ $\delta(k^2)^*$). This can be understood just from translational invariance.

if we make a translation:

$$\begin{aligned} \vec{r} \rightarrow \vec{r} + \vec{\lambda} \quad \delta_\phi(\vec{k}) &\Rightarrow \int e^{i\vec{k}\cdot\vec{r}} \frac{d^3r}{(2\pi)^3} \delta_\phi(\vec{r}-\vec{\lambda}) = e^{i\vec{k}\cdot\vec{\lambda}} \delta_\phi(\vec{k}) \\ \Rightarrow \langle \delta_\phi(\vec{k}_1) \delta_\phi(\vec{k}_2) \rangle &\rightarrow e^{i\vec{\lambda}\cdot(\vec{k}_1+\vec{k}_2)} \langle \delta_\phi(\vec{k}_1) \delta_\phi(\vec{k}_2) \rangle \end{aligned}$$

in order to have translation invariance, then need $t_1 + t_2 = 0$, that's (8)
 $\delta_\phi(t_1 + t_2)$ appears. Note also $P(k)$ depends only on $|\vec{k}|$, due
 stat. isotropy -

similarly, if you calculate 3-pt correlator in Fourier space,

$$\langle \delta_\phi(k_1) \delta_\phi(k_2) \delta_\phi(k_3) \rangle = \underbrace{\delta_0(k_1 + k_2 + k_3)}_{\text{again, translation invariance}} \underbrace{B(k_1, k_2, k_3)}_{\text{Bispectrum: FT of 3-pt function by isotropy depends only on 3 vars.}}$$

well, let's go back to basics for a second, and ask the question whether
 N-pt correlations are independent - let's start with a field Φ with
 zero mean:

$$\langle \Phi \rangle = \overline{\Phi(t)} \quad (\text{average})$$

if we calculate $\langle \Phi_1 \Phi_2 \rangle$, in principle this could be non-zero just because
 Φ_1 and Φ_2 are equal to the mean, i.e. there is no "true" correlation - for
 this reason we define the connected correlation $\langle \Phi_1 \Phi_2 \rangle_c$ as:

$$\langle \Phi_1 \Phi_2 \rangle = \underbrace{\langle \Phi_1 \rangle \langle \Phi_2 \rangle}_{\text{unconnected piece}} + \langle \Phi_1 \Phi_2 \rangle_c$$

here we see the splitting that makes sense physically - as point 1 gets far

from 2 $\langle \Phi_1 \Phi_2 \rangle_c \rightarrow 0$ typically, but $\langle \Phi_1 \Phi_2 \rangle \rightarrow \langle \Phi \rangle^2$, so a
 non-zero $\langle \Phi_1 \Phi_2 \rangle$ in this limit is not independent of $\langle \Phi \rangle$ but $\langle \Phi_1 \Phi_2 \rangle \rightarrow \langle \Phi \rangle^2$

it makes more sense to work with connected correlations -

one way to avoid this at the 2-pt level is to work with $\delta\phi$ instead of ϕ :

$$\langle \delta\phi \rangle = 0$$

$$\langle \delta\phi(1) \delta\phi(2) \rangle = \langle \delta\phi(1) \delta\phi(2) \rangle_c$$

$$\text{and same: } \langle \delta\phi(1) \delta\phi(2) \delta\phi(3) \rangle = \langle \delta\phi(1) \delta\phi(2) \delta\phi(3) \rangle_c$$

$$\text{but: } \langle \delta\phi(1) \delta\phi(2) \delta\phi(3) \delta\phi(4) \rangle = \underbrace{\langle \delta\phi(1) \delta\phi(2) \rangle \langle \delta\phi(3) \delta\phi(4) \rangle}_{\text{unconnected pieces}} + \langle \delta\phi(1) \delta\phi(3) \rangle \langle \delta\phi(2) \delta\phi(4) \rangle + \langle \delta\phi(1) \delta\phi(4) \rangle \langle \delta\phi(2) \delta\phi(3) \rangle + \langle \delta\phi(1) \delta\phi(2) \delta\phi(3) \delta\phi(4) \rangle_c$$

which again, defines the part of the 4-pt function that is needed - Indeed if we take, say, point 1 & 2 to infinity, $\langle \phi(1) \dots \phi(4) \rangle_c$ will typically vanish, whereas $\langle \delta\phi(1) \dots \delta\phi(4) \rangle \simeq \langle \delta\phi(1)\delta\phi(2) \delta\phi(3)\delta\phi(4) \rangle$ clearly connected correlations contain all the information

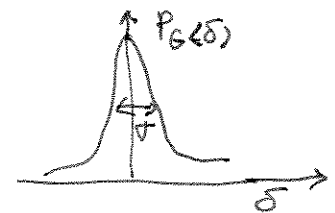
Gaussian Random Fields

Gaussian Random Fields are the simplest example of random fields, and are defined by saying that all connected correlations of order larger than 2 are zero.

Therefore a Gaussian field is completely characterized by its power spectrum 2-pt function (and also the mean, but typically we take this to be zero)

for a field @ a single point a Gaussian PDF is:

$$P_G(\delta) = \frac{1}{\sqrt{2\pi r^2}} \exp\left[-\frac{1}{2} \frac{\delta^2}{r^2}\right]$$



where $r^2 = \langle \delta^2(x) \rangle$ is the only correlation function (or moment, in this case) that enters, as it should for a Gaussian.

This generalizes trivially to N points, i.e. a multi-variate Gaussian is

$$P_G(\delta_1, \dots, \delta_N) = \frac{1}{2\pi \sqrt{\det C}} \exp\left[-\frac{1}{2} \delta_i C_{ij}^{-1} \delta_j\right]$$

where $C_{ij} \equiv \langle \delta_i \delta_j \rangle$

Now, in Fourier space things look even simpler, because the two-pt correlator is diagonal, then

$$P_G[\delta(k)] = \prod_{\vec{k}} \frac{1}{\sqrt{2\pi P(k)}} e^{-\frac{|\delta(k)|^2}{2P(k)}}$$

product of indep. Gaussians
variance in Fourier space

now, a few more definitions. Typically power spectra are

α -laws:

$$P(k) \sim k^\alpha$$

is then known as the spectral index. If $P(k)$ is more generic function α we can define a local, or effective, spectral index as

$$n_{\text{eff}}(k) \equiv \frac{d \ln P(k)}{d \ln k}$$

which reduces to $\alpha = \text{const.}$ if $P(k) \sim k^\alpha$

Whereas $P(k)$ measures the amplitude of Fourier coefficients @ k , in 3D we have a lot of modes at a given k , so a more physical measure the amplitude of fluctuations is a dimensionless quantity:

$$\Delta(k) = \underbrace{4\pi k^3}_{\substack{\# \text{ of modes} \\ \text{per } d\ln k \\ \text{in 3D}}} P(k)$$

this is dimensionless because $\langle \delta(\vec{x}) \delta(\vec{x}') \rangle = \frac{\int d^3k_1 \int d^3k_2 \delta(\vec{k}_1 + \vec{k}_2) P(k_1) P(k_2)}{V^2}$
so $P(k)$ has units of volume $k^3 \sim \frac{1}{V}$

we can write the variance in real space in terms of an integral over the power spectrum:

amplitude squared of fluctuations in real space : $\sigma^2 = \langle \delta^2 \rangle = \int d^3k_1 d^3k_2 e^{i\vec{k}_1 \cdot \vec{x}} e^{i\vec{k}_2 \cdot \vec{x}} \underbrace{\langle \delta(\vec{k}_1) \delta(\vec{k}_2) \rangle}_{P(k_1) \delta_{\vec{k}_1, -\vec{k}_2}}$

$$\Rightarrow \sigma^2 = \int d^3k P(k) = \int d\ln k \underbrace{4\pi k^3 P(k)}_{\Delta(k)}$$

$\Delta(k)$: amplitude squared of all modes in 3D