VI

Inflation : Basics

See eg. Liddle & Lyth
[perturb: Holland & Wald gr-qc/0205058]

Inflation was originally designed to solve a number of problems,

i) horizon problem: the CMB temperature is the same at scales larger than the Hubble radius during decoupling, how come causally disconnected regions have the same temperature?

ii) flatness problem: the universe is very close to flat, any small (tiny) deviation from flatness originally leads to strong deviations later, flatness today suggests some mechanism must have flatten the universe early on, or else requires very special conditions (fine tunning)

iii) relic abundance: Some grand unified theories predicted the production of lots of heavy monopoles that would have survived until the present: Why don't we see them? (also gravitinos and moduli)

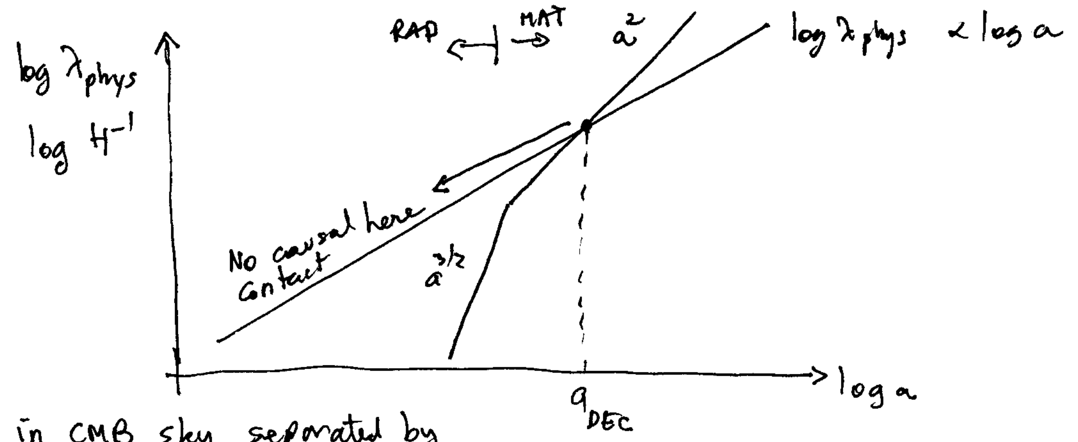
The most important aspect of inflation, however, is the creation of density perturbations with the correct power spectrum necessary to source the growth of structure in the Universe according to observations. We will discuss that next class.

The basic idea of inflation is a long-enough (can be quantified) period in the early universe during which the universe expanded exponentially, making the universe flat, diluting the abundance of any unwanted relic and solving the horizon problem. To see this recall the evolution of a given length scale in the standard FRW model is $\lambda \propto a$ with a the scale factor, and that

RAD era: $\rho \propto a^{-4} \Rightarrow H^2 \propto a^{-4} \Rightarrow a \propto t^{1/2} \Rightarrow H^{-1} \sim t \sim a^2$

MAT era: $\rho \propto a^{-3} \Rightarrow H^2 \propto a^{-3} \Rightarrow a \propto t^{2/3} \Rightarrow H^{-1} \sim t \sim a^{3/2}$

Then we have the following situation,



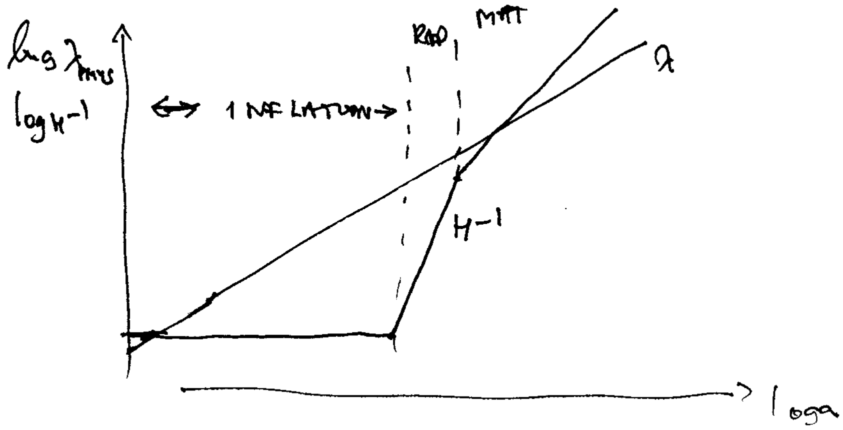
Points in CMB sky separated by scales larger than H^{-1} @ DEC have no reason to share the same temperature, which observationally they do to 1 part in 10^5. As you can see from the plot the problem arises because λ_{phys} grows slower than the Hubble radius, in other words, in standard cosmology

$$\frac{d}{dt}\left(\frac{\lambda_{phys}}{H^{-1}}\right) < 0$$

If we had a period in the early universe where such inequality is reversed, we can have scales enter back into H^{-1} at early times, for that we need

$$\frac{d}{dt}\left(\frac{\lambda_{phys}}{H^{-1}}\right) \propto \frac{d}{dt}(aH) = \ddot{a} > 0$$

that is, we need a long enough epoch of acceleration. As we discussed before one way of achieving this is having the energy density of the universe dominated by vacuum energy, in which case $\rho =$ const. and $H^2 =$ const $\Rightarrow a \propto e^{Ht}$, leading to exponential expansion. The diagram would look like this then:



Now, whatever may drive inflation is not standard stuff, since we need an equation of state with $\rho + 3p < 0$ to have acceleration. The standard models of inflation assume there is some scalar field, which for simplicity here we take as a minimally coupled one with lagrangian

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - V(\phi) \simeq \frac{1}{2} \dot{\phi}^2 - V(\phi)$$

No gradients to respect homogeneity & isotropy (which would set in pretty quickly after Inf. starts)

The stress energy tensor then reads as a perfect fluid with density and pressure

$$\begin{cases} \rho_\phi = \dfrac{\dot{\phi}^2}{2} + V(\phi) \\ p_\phi = \dfrac{\dot{\phi}^2}{2} - V(\phi) \end{cases} \qquad \left(T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \mathcal{L} g_{\mu\nu} \right)$$

From stress-energy conservation we find the equation of motion

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0$$

The simplest way to achieve inflation is the so-called slow-roll regime, in which $\dot{\phi}^2 \ll V(\phi)$, so that the effective equation of state reads $p_\phi = -\rho_\phi$. In such a case the scalar field is rolling down the potential with terminal velocity due to friction from the expansion of the universe and $\ddot{\phi} \approx 0$. In this regime,

$$\begin{cases} 3H\dot{\phi} + V'(\phi) \simeq 0 \\ H^2 \simeq \dfrac{8\pi G}{3} V(\phi) = \dfrac{V(\phi)}{3 M_{pl}^2} \end{cases}$$

This implies restrictions on the potential. The consistency condition to neglect $\ddot{\phi}$ is that

$$\frac{\ddot{\phi}}{3H\dot{\phi}} \ll 1$$

from $\frac{d}{dt}\left[3H\dot\phi + V'(\phi)\right] \approx 0 \quad \Rightarrow \quad 3\dot H\dot\phi + 3H\ddot\phi + V''\dot\phi \approx 0$

$$\Rightarrow \quad \frac{\ddot\phi}{3H\dot\phi} \sim -\frac{\dot H}{3H^2} - \frac{V''}{9H^2} \ll 1$$

From this we can derive two small quantities, the slow-roll parameters $\varepsilon(\phi)$ and $\eta(\phi)$. Let's see:

$$1 \gg \frac{V''}{9H^2} = \frac{V'' M_{pl}^2}{3V} \equiv \frac{1}{3}\eta(\phi) \quad \Rightarrow \quad \eta(\phi) \ll 1 .$$

Also, since $\quad 2H\dot H = \frac{V'\dot\phi}{3M_{pl}^2} = -\frac{V'^2}{3M_{pl}^2 H}$

$$\Rightarrow \quad \frac{\dot H}{-3H^2} = \frac{V'^2}{54 M_{pl}^2 H^4} = \frac{M_{pl}^2}{6}\frac{V'^2}{V^2} \equiv \frac{1}{3}\varepsilon(\phi) \ll 1$$

Before we jump into technical stuff about the generation of perturbations, let us discuss qualitatively how inflation generates density perturbations from quantum fluctuations. For this purpose let's consider a free field, and expand the perturbations in Fourier modes

$$\phi(\vec x, t) = \int d^3k \, e^{ikx} \, \tilde\phi(\vec k)$$

Then we have

$$\ddot{\tilde\phi} + 3H\dot{\tilde\phi} + \frac{k^2}{a^2}\tilde\phi = 0$$

At small scales we recover the standard flat-space result, as it should be, a simple harmonic oscillator,

$$k \gg aH \quad \Rightarrow \quad \ddot{\tilde\phi} + \frac{k^2}{a^2}\tilde\phi \simeq 0$$

At large scales, on the other hand, the amplitude of the oscillations is frozen:

$k \ll aH \quad \rightarrow \quad \ddot{\tilde{\phi}} + 3H\dot{\tilde{\phi}} \approx 0 \quad \Rightarrow \quad \tilde{\phi} = \text{const.}$

(the other indep solution is a decaying mode)

To calculate quantum fluctuations, we quantize this free field, noticing that each mode can be thought of as an independent harmonic oscillator with a Lagrangian

$$\mathcal{L}_k = \frac{a^3}{2}\left[\,|\dot{\tilde{\phi}}|^2 - \frac{k^2}{a^2}|\tilde{\phi}|^2\right] \rightarrow \frac{m}{2}\left[\dot{x}^2 - \omega^2 x^2\right]$$

where $m = a^3$ plays the role of a mass, and k^2/a^2 of frequency squared. You can check that Euler-Lagrange eqs. for this $\mathcal{L}_k$ gives EOM as above. Now remember from quantum mechanics the ground state of the oscillator has a variance

$$\langle x^2 \rangle = \frac{\hbar}{m\omega}$$

(characteristic length squared in HO)

which we can translate into a power spectrum in our case,

$$P_\phi(k) \equiv \langle |\tilde{\phi}|^2 \rangle \sim \frac{\hbar}{a^3\, k/a}$$

Now, small-scale modes don't see the expansion of the universe (as discussed above) and adjust adiabatically. On the other hand, once a mode crosses the Hubble radius, the fluctuations get frozen at a value

$$P_\phi(k) \sim \frac{\hbar}{a_*^3\, k/a_*} \quad \text{with} \quad k = a_* H \quad \Rightarrow \quad P_\phi(k) \sim \hbar\,\frac{H^2}{k^3}$$

with an amplitude characterized by the Hubble constant during inflation and a spectrum $P_\phi \sim k^{-3}$, which get imprinted as curvature on gravitational potential perturbations, leading to a density power spectrum through Poisson equation, $\delta \sim \phi k^2$

$P_\delta \sim k^4 P_\phi \sim k^1$ this is known as Harrison-Zeldovich spectrum