# Generation of Fluctuations from Inflation

Now let's discuss the generation of perturbations during inflation in detail, for a scalar field with some potential $V(\phi)$. We split into homogeneous part + perturbations,

$$\phi(x,t) = \phi_0(t) + \delta\phi(\vec{x},t)$$

Similarly, the stress-energy-momentum tensor

$$T^\mu{}_\nu = \nabla^\mu\phi\,\nabla_\nu\phi - \left[\frac{1}{2}\nabla^\alpha\phi\,\nabla_\alpha\phi - V(\phi)\right]\delta^\mu{}_\nu$$

can be decomposed into background and linear perturbations,

$$T^\mu{}_\nu = \langle T\rangle^\mu{}_\nu + \delta T^\mu{}_\nu$$

with

$$\begin{cases} \langle T\rangle^0{}_0 = \frac{1}{2}\frac{\dot\phi_0^2}{a^2} + V(\phi_0) \equiv \rho_\phi \\ \langle T\rangle^0{}_i = 0 \\ \langle T\rangle^i{}_j = \left[-\frac{1}{2}\frac{\dot\phi_0^2}{a^2} + V(\phi_0)\right]\delta^i{}_j \equiv -p_\phi\,\delta^i{}_j \end{cases}$$

$\dot{} \equiv \frac{d}{d\tau}$, $\tau$: conformal time

and

$$\begin{cases} \delta T^0{}_0 = \frac{1}{a^2}\left[-\dot\phi_0^2 A + \dot\phi_0\,\delta\dot\phi + V' a^2\,\delta\phi\right] \equiv \delta\rho_\phi \\ \delta T^0{}_i = \frac{1}{a^2}\dot\phi_0\,\delta\phi_{,i} \equiv (\rho_\phi + p_\phi)(v_i - B_i) \\ \delta T^i{}_j = \frac{1}{a^2}\left[\dot\phi_0^2 A - \dot\phi_0\,\delta\dot\phi + V' a^2\,\delta\phi\right]\delta^i{}_j \equiv -\delta p_\phi\,\delta^i{}_j \end{cases}$$

where we have done the mapping into the $T^\mu{}_\nu$ for a perfect fluid as discussed previously. We see in particular from $\delta T^i{}_j$ that ~~for~~ for a scalar field $\overleftrightarrow{\Sigma} = 0$, thus the tensor modes will have no sources. Also $\delta T^0{}_i$ is proportional to $\delta\phi_{,i}$ and thus it is purely a scalar mode, so there will be no sources for vector modes ~~[illegible]~~ (and thus vanish). So we won't discuss vector modes. Tensor modes are easiest, so we discuss them first.

## Generation of Tensor perturbations

We discussed last class the classical equations of motion for tensor modes:

$$(\partial_\tau^2 + 2\mathcal{H}\partial_\tau - \nabla^2)\, E_{ij}^T = 8\pi G a^2 \Sigma_{ij}^T \underset{\uparrow}{=} 0$$

$\Sigma_{ij} = 0$ for scalar fields

If we denote each of the two polarizations of $E_{ij}$ by $h_+$ and $h_\times$, each of these evolves then according to (in Fourier space)

$$\ddot{h} + 2\mathcal{H}\dot{h} + k^2 h = 0$$

(This is the same equation of motion we wrote earlier for a free scalar field
$$\ddot{\phi} + 3H\dot{\phi} + \left(\frac{k}{a}\right)^2\phi = 0$$
where here $\dot{} = d/dt$)

$(\dot{} = d/d\tau)$

It is convenient to define

$$\tilde{h} = \frac{M_{pl}}{\sqrt{2}}\, a h = \frac{1}{\sqrt{16\pi G}}\, a h$$

the factor $\frac{M_{pl}}{\sqrt{2}}$ is to convert $h$ into a canonically normalized scalar field, i.e. so that its kinetic term in the action has a $\frac{1}{2}$ prefactor as a scalar field. The additional factor of "a" is to get rid of the $\dot{h}$ term in the equation of motion, which makes easier the mapping into the harmonic oscillator problem from which quantization will follow. Then we have,

$$\ddot{\tilde{h}} + \left(k^2 - \frac{\ddot{a}}{a}\right)\tilde{h} = 0$$

which looks like a harmonic oscillator equation with frequency $\omega^2_{(\tau)} = k^2 - \ddot{a}/a$ that is time dependent (the time dependence coming from the $2\mathcal{H}\dot{h}$ term above).

The first step in quantizing these metric perturbations is to define the momentum conjugate to $\tilde{h}$

$$\pi = \dot{\tilde{h}}$$

and then promote $\tilde{h}$ and $\pi$ to operators which satisfy the commutation relations (all quantities depend on $\tau$)

$$\begin{cases} [\hat{\tilde{h}}(\vec{x}), \hat{\tilde{h}}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0 \\ [\hat{\tilde{h}}(\vec{x}), \underbrace{\hat{\pi}(\vec{y})}_{= \partial_\tau \hat{\tilde{h}}}] = i\hbar\, \delta_D(\vec{x}-\vec{y}) \end{cases}$$

The operator $\hat{\tilde{h}}$ satisfies the same equation of motion as the corresponding classical variable and thus its general solution can be written as

$$\hat{\tilde{h}}(\vec{x},\tau) = \int \left[ v(\vec{k},\tau)\, e^{i\vec{k}\cdot\vec{x}}\, \hat{a}_{\vec{k}} + v^*(\vec{k},\tau)\, \hat{a}^+_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}} \right] d^3k$$

(We work in Heisenberg picture, operators evolve, states do not)

where the mode functions $v$ obey the classical eqn. of motion

$$\ddot{v} + \left(k^2 - \frac{\ddot{a}}{a}\right) v = 0$$

and the operators $\hat{a}_{\vec{k}}$ & $\hat{a}^+_{\vec{k}}$ are the annihilation & creation operators that satisfy the commutation relations,

$$\begin{cases} [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = [\hat{a}^+_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = 0 \\ [\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = \hbar\, \delta_D(\vec{k}-\vec{k}') \end{cases}$$

These are consistent with the commutation relations above provided that the mode functions obey a normalization condition, namely

$$\underset{= i\hbar\delta_D}{[\hat{h}, \hat{\pi}]} \overset{[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = \hbar\delta_D}{\Rightarrow} \langle v_{(k)}, v_{(k)} \rangle \equiv \frac{i}{\hbar}\left[ v^*_{(k)} \dot{v}_{(k)} - \dot{v}^*_{(k)} v_{(k)} \right] = 1$$

since $\langle , \rangle$ is up to a constant the Wronskian of the equation of motion for $v$, it does not depend on time, so if satisfied initially it will continue to hold. This normalization is the first condition that the mode functions must satisfy. We need a second condition to fully fix the mode functions (as they obey 2nd order ODE) which comes from choosing the vacuum. Since

$$\hat{h}(\vec{k}) = v(\vec{k},\tau)\,\hat{a}_{\vec{k}} + v^*(-k,\tau)\,\hat{a}^+_{-\vec{k}} \qquad (v(-\vec{k}) = v(\vec{k}))$$

we have $\langle v(\vec{k}), \hat{h}(\vec{k})\rangle = \hat{a}_k$ and $-\langle v^*_{(\vec{k})}, \hat{h}(\vec{k})\rangle = \hat{a}^+_{-\vec{k}}$

so any change in $v(\vec{k})$ that keeps the same $\hat{h}$ leads to a change in $\hat{a}_k$ and thus the definition of the vacuum state $\hat{a}_{\vec{k}}|0\rangle = 0$. Other states are built them by operating with $\hat{a}^+_{\vec{k}}$.

One can check that ~~one~~ solution of the mode function is

$$v_{(\vec{k})} = \frac{\mathcal{H}}{\sqrt{2k^3}}\left(\frac{k}{\mathcal{H}}+i\right)e^{ik/\mathcal{H}} = \frac{1}{\sqrt{2k}}\left[1+\frac{i\mathcal{H}}{k}\right]e^{ik/\mathcal{H}}$$

and the other independent solution being $v^*_{(\vec{k})}$. Which one (or linear combination of them) should go in front of $\hat{a}_{\vec{k}}$?

For this we look into the $\frac{k}{aH} = \frac{k}{\mathcal{H}} \gg 1$ limit, i.e. for modes which are well below Hubble and argue that for such modes things should reduce to Minkowski (as in this limit the frequency of oscillator becomes time-independent). Then we should pick $v(\vec{k},\tau)$ to have <u>positive frequency</u>,

$$h(\vec{k}) \to \underbrace{(2\pi a)^{-3/2}}_{\text{usual } V^{-1/2} \text{ factor}} \frac{e^{-i\int E_k dt}}{\sqrt{2E_k}} \Rightarrow \hat{h} \sim a h \sim v \underset{E_k \sim k_{phys} \sim \frac{k}{a} \text{ for massless}}{\sim} \frac{1}{\sqrt{k}}\, e^{-i\int \frac{k}{a} dt}.$$

Noting that $\frac{d}{dt}\left(\frac{k}{\mathcal{H}}\right) = -\frac{k}{\mathcal{H}}\frac{da/dt}{a^2} = -\frac{k}{a}$

then $e^{ik/\mathcal{H}}$ corresponds to positive frequency mode $e^{-i\int\frac{k}{a}dt}$

Then we have,

$$\hat{h}(\vec{k}) = \frac{\mathcal{H}}{\sqrt{2k^3}}\left[\left(\frac{k}{\mathcal{H}}+i\right)e^{ik/\mathcal{H}}\,\hat{a}_{\vec{k}} + \left(\frac{k}{\mathcal{H}}-i\right)e^{-ik/\mathcal{H}}\,\hat{a}^{+}_{-\vec{k}}\right]$$

We now assume that the relevant state during inflation is the so-defined vacuum state (known as Bunch-Davies vacuum), and thus calculate the expectation value of the fluctuations as

$$\langle 0|\,\hat{h}(\vec{k})\,\hat{h}(\vec{k}')|0\rangle = v(k)\,v^*(k')\,\langle 0|\,\hat{a}_{\vec{k}}\,\hat{a}^{+}_{-\vec{k}'}\,|0\rangle \underset{\uparrow}{=} \hbar\,|v(\vec{k})|^2\,\delta_D(\vec{k}+\vec{k}')$$

$$[\hat{a}_{\vec{k}},\hat{a}^{+}_{\vec{k}'}] = \hbar\,\delta_D(\vec{k}+\vec{k}')$$

Then we have for the power spectrum of $\tilde{h}$: $\quad P_{\tilde{h}}(k) = \hbar\,|v(\vec{k})|^2$

and mapping back to $h = \frac{\sqrt{2}}{aM_{pl}}\tilde{h}$ we have

$$P_h(k) = \hbar\,\frac{2}{a^2M_{pl}^2}\,|v(\vec{k})|^2$$

and remembering that there are 2 independent tensor modes, we have

$$P_{tensor}(k) = \hbar\,\frac{4}{a^2M_{pl}^2}|v(\vec{k})|^2 = \hbar\,\frac{4}{a^2M_{pl}^2}\,\frac{\mathcal{H}^2}{2k^3}\left(1+\frac{k^2}{\mathcal{H}^2}\right)$$

This should look familiar!

$$\begin{cases} \text{For } \frac{k}{\mathcal{H}} = \frac{k}{aH} \gg 1 \Rightarrow P_t \sim \frac{1}{a^2k} \\ \text{For } \frac{k}{\mathcal{H}} \ll 1 \Rightarrow P_t \sim \frac{H^2}{k^3} \end{cases}$$

these are the results we derived before with our simple toy model.

So, after modes cross Hubble we can write ($k \ll \mathcal{H}$) ⑥

$$\boxed{P_{tensor}(k) \simeq (\hbar)\, \frac{2}{M_{pl}^2}\, \frac{H^2}{k^3} = (\hbar)\ 16\pi G\, \frac{H^2}{k^3}}$$

(we'll set $\hbar = 1$ from now on)

- Note that since the amplitude of the power is proportional to $H^2 \propto V$ during inflation, a measurement of this amplitude will tell us the energy scale during inflation, one of the most interesting numbers we would like to know. In fact $\frac{2}{M_{pl}^2} H^2 = \frac{2}{3} \frac{V}{M_{pl}^4}$ ~ the energy density during inflation in Planck units!

- Also it is interesting to note that long-wavelength modes ($k \ll \mathcal{H}$) act as a sort of classical stochastic variable, i.e.

since $v \sim \frac{\mathcal{H} i}{\sqrt{2k^3}}$ and $v^* - \frac{\mathcal{H} i}{\sqrt{2k^3}}$

we have for the long-wavelength perturbations,

$$\hat{h}_{long}(\bar{x}) = \int_0^{\mathcal{H}} d^3k\, \frac{\mathcal{H} i}{\sqrt{2k^3}} \left[ a_{\bar{k}}\, e^{i\bar{k}\cdot\bar{x}} - a^+_{\bar{k}}\, e^{-i\bar{k}\cdot\bar{x}} \right] = \int_0^{\mathcal{H}} d^3k\, \frac{\mathcal{H} i}{\sqrt{2k^3}}\, e^{i\bar{k}\cdot\bar{x}} \left[ a_{\bar{k}} - a^+_{-\bar{k}} \right]$$

but the operator $a_{\bar{k}} + a^+_{-\bar{k}}$ behaves as a classical stochastic variable,

$$[\, a_{\bar{k}} - a^+_{-\bar{k}}\, ,\, a_{\bar{k}'} - a^+_{-\bar{k}'}\, ] = -\underbrace{[\, a_{\bar{k}}, a^+_{-\bar{k}'}]}_{\hbar\, \delta_D(\bar{k}+\bar{k}')} - \underbrace{[\, a^+_{-\bar{k}}, a_{\bar{k}'}]}_{-\hbar\, \delta_D(\bar{k}+\bar{k}')} = 0$$

So, since the modes commute and the vacuum state obeys Gaussian fluctuations, for practical purposes we have a classical Gaussian random field of tensor modes with power spectrum $P_{tensor}$ -

## Generation of Scalar Perturbations

We now consider the generation of scalar modes, which is ~~in~~ fact more complicated than for tensor modes because one has to deal with the scalar fluctuations from $T_{\mu\nu}$ (i.e. $\delta\phi$ for the inflaton) and the metric simultaneously. This can be done in (at least) two ways: i) by working with the so-called "Mukhanov-Sasaki" variable, or ii) by choosing a gauge in which the two decouple. We shall follow (ii)

Let's consider the spatially flat gauge,

$$ds^2\Big|_{scalar} = a^2\left[-(1+2A)\,d\tau^2 - 2\,B_i^S\,dx^i d\tau + \delta_{ij}\,dx^i dx^j\right]$$

i.e we have chosen the slicing so that the spatial part of the metric is unperturbed (i.e. $D=E=0$ in this gauge). The interesting thing is that then, to leading order in slow-roll parameters the evolution of inflaton perturbations is the same as a free scalar field (show it!)

$$\ddot{\delta\phi} + 2\mathcal{H}\,\dot{\delta\phi} + k^2\,\delta\phi = 0 \qquad \text{(spatially flat gauge)}$$

which is the same as for a tensor mode polarization (canonically normalised).

Then we have for its power spectrum

$$P_{\delta\phi}(k) = \frac{H^2}{2k^3} \qquad \text{(spatially flat gauge)}$$

which gives the spectrum after Hubble crossing. However this is not that useful yet because i) it is a gauge-dependent result ii) the inflaton decays, so $\phi$ is gone by now, so this does not yet connect to things we can measure.

To overcome these issues we now map this result into the power spectrum of "curvature fluctuations", which is easily related to observables, and it is gauge invariant.

Consider the following variable, known as the "primordial curvature perturbation"

$$\zeta \equiv \frac{\delta}{3(1+w)} + D - \frac{\nabla^2 E}{3}$$

i) you can check that since, under a gauge transformation

$$\begin{cases} D \to D - \frac{1}{3}\nabla^2\beta - \mathcal{H}\alpha \\ E \to E - \beta \\ \delta \to \delta + 3\mathcal{H}\alpha(1+w) \end{cases}$$

$\zeta$ is gauge invariant.

ii) in slices of uniform energy density ($\delta = 0$) and threads that have no shear ($E = 0 \Rightarrow E_{ij}^S = 0$) the spatial part of the metric can be written as

$$g_{ij} = a^2(1+2D)\,\delta_{ij} \simeq a^2 e^{2\zeta}\,\delta_{ij} \equiv a^2_{(\bar{x},\tau)}\,\delta_{ij}$$

where the local scale factor

$$a(\bar{x},\tau) \equiv a(\tau)\,e^{\zeta(\bar{x},\tau)}$$

that is, $\zeta$ defines the local scale factor in uniform density slices (with no shear). This can be taken as the non-perturbative (i.e. beyond linear fluctuations) definition of $\zeta$.

iii) in the spatially flat gauge we have then

$$\zeta = \frac{\delta}{3(1+w)} = \frac{1}{3}\frac{\delta\rho}{\rho+p} = -\mathcal{H}a\frac{\delta\rho}{\dot{\rho}} \qquad \text{(spatially flat gauge)}$$

$$\dot{} = \frac{d}{d\tau}$$

so it measures the dimensionless density perturbations.

iv) it is conserved on super-Hubble scales for adiabatic fluctuations — You can show this formally from the definition of $\zeta$ plus Einstein and stress-energy conservation, or from thinking of a super-Hubble patch evolving as a "separate universe" (i.e causally disconnected) in which then the continuity equation holds as in homogeneous case

$$\dot{\rho}_{(\tau)} = -3 \frac{\partial \ln a(\vec{x},\tau)}{\partial \tau} \left[ \rho(\tau) + p(\vec{x},\tau) \right]$$

where we work in the uniform density gauge $\hookrightarrow \rho = \rho(\tau)$ - Then, since

$$\frac{\partial \ln a(\vec{x},\tau)}{\partial \tau} = \mathcal{H} + \dot{\zeta}$$

$$\Rightarrow \quad \dot{\rho}_{(\tau)} = -3 \left[ \mathcal{H}_{(\tau)} + \dot{\zeta}_{(\vec{x},\tau)} \right] \left[ \rho(\tau) + p(\vec{x},\tau) \right]$$

For adiabatic perturbations where $p = p(\rho)$ $\Rightarrow$ $p$ does not have $\vec{x}$ dependence in uniform density gauge $\Rightarrow$ $\dot{\zeta}(\vec{x},\tau) = 0$ this is true in any gauge. Since $\zeta$ is gauge-invariant

~~[illegible]~~

v) Finally, we can relate $\zeta$ to $\Phi$ & $\Psi$, the potentials in the conformal Newtonian gauge as follows - In this gauge we have

$$\zeta = \frac{\delta}{3(1+w)} + D = \frac{\delta}{3(1+w)} - \Psi \qquad (D \equiv -\Psi \quad A \equiv \Phi)$$

Now the 00 Einstein Equation gives (see last class)

$$-\nabla^2 \Psi + 3\mathcal{H}(\dot{\Psi} + \mathcal{H}\Phi) = -\frac{3}{2}\mathcal{H}^2 \delta$$

then at super-Hubble scales (can drop gradients) $\Rightarrow$ $\delta = -2\left(\frac{\dot{\Psi}}{\mathcal{H}} + \Phi\right)$

and then
$$\boxed{-\zeta = \Psi + \frac{2}{3}\,\frac{\Phi + \dot{\Psi}/\mathcal{H}}{1+w}}$$
(super-Hubble, Conformal Newtonian Gauge)

Now let's consider an era when $w \approx$ const. (e.g. RAD or MAT dominated) and consider adiabatic fluctuations (so $\zeta$ = const.) – During such periods when there is no anisotropic stress (e.g. neglect neutrinos) $\Phi = \Psi$ (this is a consequence of $i \neq j$ Einstein eqs. $\mathcal{D}_{ij}(\Phi - \Psi) = -8\pi G a^2 \Sigma^S_{ij}$) – Then we have that $\dot{\zeta} = 0$ implies we can neglect $\dot{\Phi}$ in the equation above (will induce a decaying mode on top of what we are about to write) and then

$$\boxed{-\zeta = \frac{5+3w}{3+3w}\,\Phi}$$
(super-Hubble, conf. Newt. gauge, no aniso stress, $w \approx$ const.)

which gives for RAD ($w = 1/3$)
$$\Phi = \Psi = -\frac{2}{3}\zeta$$

and for MAT ($w=0$)
$$\Phi = \Psi = -\frac{3}{5}\zeta$$

This will allow us to map the gauge invariant perturbations $\zeta$ created by inflation, conserved outside Hubble, to the conformal newtonian scalar modes as they re-enter the Hubble radius during RAD and MAT eras, leading to the initial conditions for CMB and LSS evolution – Note that under the assumptions above (adiabatic fluctuations, no anisotropic stress) we only have to deal with a single scalar mode encoded by the gauge invariant variable $\zeta$.

Let us now go back to our discussion of the inflationary perturbation $\delta\phi$ and finalize its relation to $\zeta$. Since $P_{\delta\phi}(k) = H^2/2k^3$ in the spatially flat gauge, we see that in this gauge $(D=E=0)$ we have

$$\boxed{\zeta = -Ha\frac{\delta\rho}{\dot\rho} = -Ha\frac{\cancel{V'}\delta\phi}{\cancel{V'}\dot\phi_0} = -Ha\frac{\delta\phi}{\dot\phi_0} = -\mathcal{H}\frac{\delta\phi}{\dot\phi_0}}$$

This has a simple physical interpretation. One can calculate the curvature perturbation $\zeta$ using the fact that it corresponds to the local scale factor in slices of uniform energy density, then

$$\zeta \sim -\frac{\delta a}{a} = -H\,\delta t = -\mathcal{H}\,\delta\tau = -\frac{\mathcal{H}}{d\phi_0/d\tau}\delta\phi = -\mathcal{H}\frac{\delta\phi}{\dot\phi_0} \quad \checkmark$$

the longer inflation lasts in a patch the less is the energy density due to additional expansion

inflaton ends @ different scale factors in different regions of the universe due to $\delta\phi$

Then we have for the curvature perturbation scalar power spectrum,

$$P_\zeta(k) = \left(\frac{\mathcal{H}}{\dot\phi_0}\right)^2\frac{H^2}{2k^3} = \frac{H^4}{\left(\frac{d\phi_0}{dt}\right)^2}\frac{1}{2k^3}$$

recall that during slow-roll $3H\frac{d\phi_0}{dt} \sim -V'$

and using $H^2 = \frac{V}{3M_{pl}^2}$ we have

$$\frac{H^4}{(d\phi_0/dt)^2} = \left(\frac{V}{3M_{pl}^2}\right)^2\left(\frac{3H}{V'}\right)^2 = \frac{H^2}{M_{pl}^2}\left(\frac{V}{V'M_{pl}}\right)^2$$

$$= \frac{H^2}{2M_{pl}^2}\frac{1}{\epsilon} \qquad \epsilon \equiv \frac{1}{2}\left(\frac{V'}{V}\right)^2 M_{pl}^2$$

$$\Rightarrow \boxed{P_\zeta(k) = \frac{H^2}{4M_{pl}^2k^3}\frac{1}{\epsilon}}$$

Notice that compared to the tensor modes power we here,

$$\frac{P_\zeta(k)}{P_{tensor}(k)} = \frac{\cancel{H^2}}{4\cancel{M_{pl}^2}\cancel{k^3}}\frac{1}{\epsilon}\frac{\cancel{M_{pl}^2}\cancel{k^3}}{2\cancel{H^2}} = \frac{1}{8\epsilon} \quad \text{or} \left[r \equiv \frac{P_{tensor}}{P_\zeta} = 8\epsilon\right]$$

"tensor-to-scalar ratio"

So the scalar modes fluctuations are <u>enhanced</u> compared to tensor fluctuations by $(8\epsilon)^{-1}$ (recall $\epsilon \ll 1$).

Now we write both scalar and tensor fluctuations using the dimensionless power spectrum $\Delta(k) \equiv 4\pi k^3 P(k)$

$$\Delta_\zeta(k) = \frac{\pi}{\epsilon}\frac{H^2}{M_{pl}^2} \underset{H^2 = \frac{V}{3M_{pl}^2}}{\Rightarrow} \boxed{\Delta_\zeta(k) = \frac{\pi}{3}\frac{V}{M_{pl}^4}\frac{1}{\epsilon}} \quad @\ k = aH$$

where the k dependence (small) is coming from evaluating $\frac{V}{\epsilon}$ @ Hubble radius crossing (when fluctuations in $\zeta$ freeze). The above formula is easy to remember: $\Delta_\zeta \sim \frac{V}{M_{pl}^4}\frac{1}{\epsilon}\Big|_{k=aH}$, the amplitude is energy density in units of Planck enhanced by slow-roll parameter evaluated at Hubble crossing!

For tensors we have similarly,

$$\Delta_{tensor}(k) = 8\pi\frac{H^2}{M_{pl}^2} \Rightarrow \boxed{\Delta_{tensor}(k) = 8\frac{\pi}{3}\frac{V}{M_{pl}^4}} \quad @\ k = aH$$

Let us now study the (small) k-dependence of these expressions. They are not truly scale invariant because different k-modes exit Hubble at different times and the inflaton is slowly rolling, so V is changing slightly (and $\epsilon$ too). So we need to map a change in k to a change in $\phi$ when $\frac{k}{aH} = 1$

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So let us define the spectral indices $n_s$ & $n_T$ of scalar and tensor fluctuations as

$$\begin{cases} n_s - 1 \equiv \dfrac{d\ln \Delta_\zeta}{d\ln k} = \dfrac{d\ln V}{d\ln k} - \dfrac{d\ln \epsilon}{d\ln k} & @\, k \simeq aH \\ n_T \equiv \dfrac{d\ln \Delta_{tensor}}{d\ln k} = \dfrac{d\ln V}{d\ln k} & @\, k \simeq aH \end{cases}$$

(the somewhat different convention for $n_s$ vs $n_T$ is because for scalars this definition leads to $\Delta_\zeta \sim k^{n_s-1} \sim \Delta_\Phi \underset{\substack{\uparrow \\ \text{Poisson inside Hubble}}}{\sim} \Delta_\delta k^{-4} \sim P_\delta k^{-1}$
$\Rightarrow P_\delta \propto k^{n_s}$, so $n_s$ corresponds to the spectral index of density perturbations)

So we have $\begin{cases} \Delta_\zeta(k) = A_s(k_*) \left(\dfrac{k}{k_*}\right)^{n_s(k)-1} \\ \Delta_{tensor}(k) = A_T(k_*) \left(\dfrac{k}{k_*}\right)^{n_T(k)} \end{cases}$ with $k_*$ some arbitrary "pivot scale".

To figure out $n_s$ & $n_T$ we transform the derivative with respect to $\ln k$ to $\phi$ close to Hubble crossing,

$$dk \underset{\substack{\uparrow \\ \text{zeroth order} \\ \text{in slow roll}}}{\simeq} H\,da = H \frac{da}{dt} \underbrace{\frac{dt}{d\phi}}_{= -V'/3H} d\phi = \frac{H^2 a}{-V'/3H}\, d\phi$$

$$\Rightarrow d\ln k = -\underbrace{\left(\frac{aH}{k}\right)}_{=1} \frac{3H^2}{V'}\, d\phi = -\frac{V}{V'}\frac{1}{M_{pl}^2}\, d\phi$$

$$\Rightarrow \left.\frac{d}{d\ln k}\right|_{\frac{k}{aH}=1} \simeq -M_{pl}^2 \frac{V'}{V}\frac{d}{d\phi}$$

$$\Rightarrow \left.\frac{d\ln V}{d\ln k}\right|_{\frac{k}{aH}=1} \simeq -M_{pl}^2 \frac{V'}{V}\frac{d\ln V}{d\phi} \simeq -2\epsilon$$

and
$$\left.\frac{d\ln\epsilon}{d\ln k}\right|_{\frac{k}{aH}=1} = -M_{Pl}^2\frac{V'}{V}\frac{d\ln\epsilon}{d\phi} = -M_{Pl}^2\frac{V'}{V}\,2\frac{V}{V'}\left[\frac{V''}{V}-\left(\frac{V'}{V}\right)^2\right]$$
$$\epsilon = \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2$$

using that $\eta = M_{Pl}^2\frac{V''}{V}$ we get then

$$\left.\frac{d\ln\epsilon}{d\ln k}\right|_{k=aH} = 4\epsilon - 2\eta$$

then we have:

"scalar tilt": $$\boxed{\begin{aligned} n_S - 1 &= 2\eta - 6\epsilon \\ n_T &= -2\epsilon \end{aligned}}$$

If we ever detect the stochastic background of tensor modes and furthermore (!) measure its spectral index $n_T$ accurately enough we would be able to test the following consistency relation for single field slow-roll inflation

$$\boxed{r = \frac{\Delta_{\text{tensor}}}{\Delta_\zeta} = 8\epsilon = -4n_T}$$