Linear (Newtonian) Evolution of Fluctuations

We now discuss how perturbations evolve since inflation until the present. We showed that inflation generates a spectrum of curvature perturbations that is nearly scale invariant. As modes ~~become~~ larger than the Hubble radius during inflation, their scale is larger than the typical time scale and thus retardation effects become important, and thus GR is needed to figure out their evolution. However, the end result of such a calculation is extremely simple: curvature perturbations are conserved for $\frac{k}{aH} < 1$ until they come back inside the Hubble radius during RAD or MAT.

After perturbations become sub-Hubble again their evolution can be described by simple Newtonian analysis when in MAT era, and with simple change of $\bar\rho \to \bar\rho + \bar p$ one can also describe their evolution in RAD era, without the need of going through the details of GR perturbations. This is the strategy we are going to follow here.

The contents of the universe we are going to include are

1) baryons : ρ_B (typically, we'll assume BBN-type values $\Omega_B h^2 \simeq 0.02$)

the equation of state in this case is given by collisional pressure:

$$p_B = \frac{\rho_B T_B}{m_B} \ll \rho_B \quad \text{since baryons are non-relativistic (NR)}$$

therefore for homogeneous case $p_B \simeq 0$ is a good approximation. However for perturbations we also need to know the sound speed c_s:

$$c_s^2 = \frac{\delta p_B}{\delta \rho_B}$$

2) dark matter : ρ_M For example made of WIMPS of mass m, and could be cold or hot (in principle, though most of the time we will use CDM)

We assume they are NR when the universe becomes matter dominated.

CDM has negligible pressure, $p_M + \delta p_M \sim 0$ (ie. $c_s^2{}_{DM} \simeq 0$)

3) radiation : ρ_R with usual equation of state for photons $p_R = \frac{1}{3}\rho_R$

4) cosmological constant (or dark energy) : ρ_Λ or ρ_w – we will often mention how things change when this is included.

Newtonian Equations of motion

As we mentioned, when $\frac{k}{aH} \gg 1$ during MAT era the relativistic equations $T_{\mu\nu;}{}^{\nu} = 0$ and the GR field equations reduce to the Newtonian analysis we now derive.

It is very convenient to use comoving coordinates $\vec{x}(t)$, ie. physical coordinates $\vec{r}(t)$ are

$$\vec{r}(t) = a(t)\ \vec{x}(t)$$

$\Rightarrow$ physical velocity is

$$\frac{d\vec{r}}{dt} = \vec{V}(t) = \dot{a}\,\vec{x} + a\,\frac{d\vec{x}}{dt} = \underbrace{H\,\vec{r}}_{\text{Hubble expansion}} + \underbrace{\vec{v}}_{\text{"Peculiar velocity"}}$$

or, written in comoving variables

$$\vec{V}(t) = \mathcal{H}\,\vec{x} + \vec{v}$$

where $\mathcal{H} = aH = \dot{a}$ is the conformal expansion rate. That is, if we introduce conformal time τ by $a\,d\tau = dt$ we have

$$\mathcal{H} = \frac{d\ln a}{d\tau} = a\,\frac{d\ln a}{dt} = aH$$

$\mathcal{H}^{-1}$ can be thought of as the comoving Hubble radius, i.e.

$$\text{comoving Hubble radius} = \text{physical Hubble radius}/a = \frac{H^{-1}}{a} = \frac{1}{aH} = \frac{1}{\mathcal{H}} = \mathcal{H}^{-1}$$

and the peculiar velocity $\vec{v} = a \frac{d\vec{x}}{dt} = \frac{d\vec{x}}{d\tau}$

Newtonian dynamics is given by the set of equations,

$$\begin{cases} \frac{\partial \rho}{\partial t} + \bar{\nabla}_r \cdot [\rho \vec{V}] = 0 & \text{continuity equation (conservation of mass)} \\ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \bar{\nabla}_r)\vec{V} = -\frac{\bar{\nabla} P}{\rho} - \bar{\nabla}_r \Phi_{tot} & \text{Euler eqn. (conservation of momentum)} \\ \nabla_r^2 \Phi_{tot} = 4\pi G \rho & \text{Poisson eqn.} \end{cases}$$

Note that all quantities are physical, $\bar{\nabla}_r$ means $\frac{\partial}{\partial \vec{r}}$ with r physical, $\vec{V}$ is physical velocity, Φ_{TOT} is the total gravitational potential (due to homogeneous or background + fluctuations).

We can rewrite the first 2 equations in a more familiar form by defining the time derivative following a fluid element

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V} \cdot \bar{\nabla}_r$$

$$\Rightarrow \quad \frac{d\rho(\vec{x},t)}{dt} = -\rho \, \bar{\nabla}_r \cdot \vec{V} = -3H(\vec{x},t)\, \rho(\vec{x},t)$$

where $H(\vec{x},t) \equiv \frac{1}{3} \bar{\nabla}_r \cdot \vec{V} = \frac{1}{3} \bar{\nabla}_r \cdot [H\vec{r} + \vec{v}] = H(t) + \frac{\bar{\nabla}_r \cdot \vec{v}}{3}$ is the locally defined Hubble parameter (equal to $H(t)$ in the absence of perturbations)

Similarly,
$$\frac{d\vec{V}}{dt} = -\frac{\bar{\nabla}_r P}{\rho} - \bar{\nabla}_r \Phi_{tot}$$

Now, let's write $\Phi_{tot} = \Phi_b + \Phi$, where Φ is due to perturbations, Φ_b due to background. Solving Poisson equation for the background:

$$\nabla_r^2 \Phi_b = 4\pi G \rho(t) \quad \rightarrow \quad \Phi_{b(\vec{x},t)} = \frac{2\pi G}{3} r^2 \rho(t)$$

For this background solution,

$$\frac{d\vec{V}}{dt} = -\nabla_r \phi_b \Rightarrow \quad \ddot{a}\ \vec{x} = \frac{\ddot{a}}{a}\vec{r} = -\frac{4\pi G}{3}\bar{\rho}(t)\ \vec{r}$$

$$\Rightarrow \quad \frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G\bar{\rho}}{3}$$

, the usual acceleration equation in the absence of pressure

[to include relativistic pressure, must use GR]

The equations of motion for perturbations follow by writing:

$$\rho(x,t) = \bar{\rho} + \delta\rho(\bar{x},t) \equiv \bar{\rho}(t)\left[1+\delta(\bar{x},t)\right]$$

$$p(x,t) = \bar{p} + \delta p(\bar{x},t)$$

$$H(x,t) = \bar{H} + \delta H(\bar{x},t)$$

$$\phi_{TOT} = \frac{2\pi G}{3}\bar{\rho}\, r^2 + \Phi(x,t)$$

$$\vec{V} = H\vec{r} + \vec{v} \quad \Rightarrow \quad \delta H = \frac{1}{3}\nabla_r\cdot\vec{v}$$

Now, we need to convert from partial derivatives in $(\bar{r},t)$ to $(\bar{x},\tau)$, where $\bar{x}$ are <u>comoving</u> positions and τ is conformal time – Then

$$\left.\frac{\partial}{\partial t}\right)_{\bar{r}} = \underbrace{\left.\frac{\partial\tau}{\partial t}\right)_{\bar{r}}}_{\frac{1}{a}}\frac{\partial}{\partial\tau} + \underbrace{\left.\frac{\partial\bar{x}}{\partial t}\right)_{\bar{r}}}_{-\frac{\mathcal{H}}{a}\bar{x}}\frac{\partial}{\partial\bar{x}} = \frac{1}{a}\frac{\partial}{\partial\tau} - \frac{\mathcal{H}}{a}\,\bar{x}\cdot\bar{\nabla}$$

at fixed $\bar{r}$

$$\nabla_r = \left.\frac{\partial}{\partial\bar{r}}\right)_t = \underbrace{\left.\frac{\partial\tau}{\partial\bar{r}}\right)_t}_{0}\frac{\partial}{\partial\tau} + \underbrace{\left.\frac{\partial\bar{x}}{\partial\bar{r}}\right)_t}_{1/a}\frac{\partial}{\partial\bar{x}} = \frac{1}{a}\bar{\nabla}$$

Then continuity equation reads:

$$\frac{\partial\rho}{\partial t} = (1+\delta)\frac{\partial\bar{\rho}}{\partial t} + \bar{\rho}\frac{\partial\delta}{\partial t} \overset{\bar{\rho}\sim a^{-3}}{=} -3(1+\delta)H\bar{\rho} + \bar{\rho}\frac{\partial\delta}{\partial t} = -3H\rho + \bar{\rho}\frac{\partial\delta}{\partial t}$$

$$\nabla_r\cdot(\rho\vec{V}) = \rho\underbrace{\nabla_r\cdot\vec{V}}_{3H+\nabla_r\cdot\vec{v}} + \vec{V}\cdot\nabla_r\left[\bar{\rho}(1+\delta)\right] = 3H\rho + \rho\,\nabla_r\cdot\vec{v} + \bar{\rho}\,(\vec{V}\cdot\vec{\nabla})(1+\delta)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot (\rho \vec{V}) = \bar{\rho} \frac{\partial \delta}{\partial t} + \rho\, \vec{\nabla}_r \cdot \vec{V} + \bar{\rho}\, (\vec{V} \cdot \vec{\nabla}_r)(1+\delta)$$

$$\frac{\partial \delta}{\partial t} = \frac{1}{a}\frac{\partial \delta}{\partial \tau} - \frac{\mathcal{H}}{a}(\vec{x} \cdot \vec{\nabla})\delta \qquad (\vec{V} \cdot \vec{\nabla}_r)(1+\delta) = \vec{V} \cdot \vec{\nabla}_r \delta = \frac{\mathcal{H}}{a}(\vec{x} \cdot \vec{\nabla})\delta + \frac{1}{a}(\vec{v} \cdot \vec{\nabla})\delta$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \vec{\nabla}_r \cdot (\rho \vec{V}) = \frac{\bar{\rho}}{a}\left[\frac{\partial \delta}{\partial \tau} - \cancel{\mathcal{H}(\vec{x} \cdot \vec{\nabla})\delta} + (1+\delta)\, \vec{\nabla} \cdot \vec{v} + \cancel{\mathcal{H}(\vec{x} \cdot \vec{\nabla})\delta} + (\vec{v} \cdot \vec{\nabla})\delta\right]$$

$$\Rightarrow \boxed{\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1+\delta)\vec{v}] = 0}$$

Euler:

$$\frac{\partial \vec{V}}{\partial t} = \frac{1}{a}\frac{\partial \vec{V}}{\partial \tau} - \frac{\mathcal{H}}{a}(\vec{x} \cdot \vec{\nabla})\vec{V} = \frac{1}{a}\frac{\partial \mathcal{H}}{\partial \tau}\vec{x} + \frac{1}{a}\frac{\partial \vec{v}}{\partial t} - \frac{\mathcal{H}^2}{a}\vec{x} - \frac{\mathcal{H}}{a}(\vec{x} \cdot \vec{\nabla})\vec{v}$$

$$(\vec{V} \cdot \vec{\nabla}_r)\vec{V} = \frac{1}{a}[\mathcal{H}\vec{x} + \vec{v}] \cdot \vec{\nabla}(\mathcal{H}\vec{x} + \vec{v}) = \frac{\mathcal{H}^2}{a}\vec{x} + \frac{\mathcal{H}}{a}(\vec{x} \cdot \vec{\nabla})\vec{v} + \frac{\mathcal{H}}{a}\vec{v} + \frac{1}{a}(\vec{v} \cdot \vec{\nabla})\vec{v}$$

$$\Rightarrow \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}_r)\vec{V} = \frac{1}{a}\left[\frac{\partial \mathcal{H}}{\partial \tau}\vec{x} + \frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v}\right] = -\frac{1}{a}\vec{\nabla}\phi_{TOT} - \frac{1}{a}\frac{\vec{\nabla} p}{\rho}$$

$$\Rightarrow \frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}\left[\Phi_{TOT} + \frac{1}{2}\frac{\partial \mathcal{H}}{\partial \tau}x^2\right] - \frac{\vec{\nabla} p}{\rho}$$

Let $\Phi \equiv \phi_{TOT} + \frac{1}{2}\frac{\partial \mathcal{H}}{\partial \tau}x^2$ as expected, $\frac{1}{2}\frac{\partial \mathcal{H}}{\partial \tau}x^2 = -\Phi_b$

$$\Rightarrow \boxed{\frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}\Phi - \frac{\vec{\nabla} p}{\rho}}$$

Poisson

$$\nabla_r^2 \Phi_{TOT} = \frac{1}{a^2}\nabla^2\phi_{TOT} = \frac{1}{a^2}\nabla^2\left[\Phi - \frac{1}{2}\frac{\partial \mathcal{H}}{\partial \tau}x^2\right] \overset{\nabla^2 x^2 = 6}{=} \frac{1}{a^2}\nabla^2\Phi - \frac{3}{a^2}\frac{\partial \mathcal{H}}{\partial \tau} = 4\pi G\rho$$

From Friedmann $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}$ (for $\Lambda = 0$) and $\frac{\partial \mathcal{H}}{\partial \tau} = a\frac{\partial \dot{a}}{\partial t} = a\ddot{a}$

$$\Rightarrow 3\frac{\partial \mathcal{H}}{\partial \tau} = -4\pi G\bar{\rho}a^2$$

$$\Rightarrow \nabla^2\Phi = 4\pi G(\rho - \bar{\rho})a^2 = 4\pi G\bar{\rho}a^2\delta = \frac{3}{2}\Omega_m\mathcal{H}^2\delta$$

$$\Rightarrow \boxed{\nabla^2\Phi = \frac{3}{2}\mathcal{H}^2\Omega_m\delta}$$

(BTW, these equations are also valid in the presence of Λ, since Λ is only homog. thus only have to modify background)

Linear Solutions : Growing & Decaying Modes

We now solve the equations above in linear perturbation theory, by assuming that $\delta \ll 1$ and $\vec{\nabla}\cdot\vec{v}/\mathcal{H} \ll 1$.

For simplicity in illustrating the results, we will often take $\Omega_m = 1$, in which case:

$$a(t) \sim t^{2/3} \qquad a\, d\tau = dt \Rightarrow d\tau \sim \frac{dt}{t^{2/3}} \Rightarrow \tau \sim t^{1/3} \Rightarrow a(\tau) \sim \tau^2$$

$$\Rightarrow H = \frac{2}{3t} \qquad \mathcal{H} = \frac{2}{\tau} \qquad \frac{dH}{dt} \equiv \dot{H} = -\frac{2}{3t^2} = -\frac{3}{2}H^2 \,, \quad \frac{d\mathcal{H}}{d\tau} \equiv \mathcal{H}' = -\frac{2}{\tau^2} = -\frac{\mathcal{H}^2}{2}$$

Linearizing the continuity equation we have:

$$\frac{\partial \delta}{\partial \tau} + \vec{\nabla}\cdot\vec{v} \equiv \boxed{\frac{\partial \delta}{\partial \tau} + \theta = 0} \qquad \theta \equiv \vec{\nabla}\cdot\vec{v} \quad \text{velocity divergence}$$

For the Euler equation we have

$$\frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} = -\nabla\Phi - \frac{\nabla p}{\bar{\rho}}$$

(where note we did $\dfrac{\nabla p}{\bar{\rho}(1+\delta)} \simeq \underbrace{\dfrac{\nabla p}{\bar{\rho}}}_{\text{1st order}} \underbrace{(1-\delta)}_{\text{1st order}} = \simeq \dfrac{\nabla p}{\bar{\rho}}$ to 1st order)

taking divergence of this we have,

$$\boxed{\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta} = -\nabla^2\Phi - \frac{\nabla^2 p}{\bar{\rho}} \underset{\text{Poisson}}{=} \boxed{-\frac{3}{2}\mathcal{H}^2\Omega_m\delta - \frac{\nabla^2 p}{\bar{\rho}}}$$

Let's look at these in Fourier space

$$A(\vec{k}) = \int \frac{d^3x}{(2\pi)^3} A(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}} \qquad A(\vec{x}) = \int d^3k\, A(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \vec{\nabla} \longrightarrow i\vec{k} \quad \text{and} \quad \nabla^2 \longrightarrow -k^2$$

$$\Rightarrow \frac{\partial \delta_k}{\partial \tau} + \theta_k = 0 \qquad \& \qquad \frac{\partial \theta_k}{\partial \tau} + \mathcal{H}\theta_k = -\frac{3}{2}\Omega_m\mathcal{H}^2\delta_k + \frac{k^2 p_k}{\bar{\rho}}$$

From this we can find a 2nd order differential equation for δ_k:

$$\frac{\partial^2 \delta_k}{\partial \tau^2} + \frac{\partial \theta_k}{\partial \tau} = 0 \Rightarrow \boxed{\frac{\partial^2 \delta_k}{\partial \tau^2} + \underbrace{\mathcal{H}\frac{\partial \delta_k}{\partial \tau}}_{\text{"friction" due to expansion}} = \underbrace{\frac{3}{2}\Omega_m\mathcal{H}^2\delta_k}_{\text{Gravity}} - \underbrace{\frac{k^2 p_k}{\bar{\rho}}}_{\text{Pressure (suppresses grow}}}$$

Before we can solve this we need to specify the equation of state for perturbations:

Pressureless case : $p_k = 0$

This is a very important case, as it describes very well the behavior of CDM (cold dark matter):

$$\frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_k}{\partial \tau} = \frac{3}{2} \Omega_m \mathcal{H}^2 \delta_k$$

Since nothing depends explicitly on k except δ_k (but $\Omega_m(\tau)$, $\mathcal{H}(\tau)$) this accepts factorizable solutions:

$$\delta_k(\tau) = D(\tau) \, A_k$$ where A_k depends only on k.

$$\Rightarrow \quad \boxed{\frac{d^2 D}{d\tau^2} + \mathcal{H} \frac{dD}{d\tau} = \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2(\tau) \, D(\tau)}$$

this ODE describes the evolution of the growth factor $D(\tau)$.

In order to solve it we need $\mathcal{H}(\tau)$ and $\Omega_m(\tau)$. These are given by Friedmann equations, which in these variables are

$$\begin{cases} \mathcal{H}^2 \left(1 - \Omega_m(\tau)\right) = k \\ \mathcal{H}' = -\frac{\mathcal{H}^2}{2} \Omega_m(\tau) \end{cases}$$

i) Consider the simplest case where $\Omega_m = 1$, $k = 0$, $\mathcal{H}' = -\frac{\mathcal{H}^2}{2}$, $a \sim \tau^2$

$$\Rightarrow \quad \frac{d^2 D}{d\tau^2} + \frac{2}{\tau} \frac{dD}{d\tau} = \frac{3}{2} \frac{4}{\tau^2} D(\tau)$$

Clearly this has power-law solutions in τ, and therefore in a.

Proposing $D \sim a^p$ get $p = 1, -3/2$

the 2 indep. solutions are $\Rightarrow$ $\begin{cases} D_+ \sim a & \text{"growing mode"} \\ D_- \sim a^{-3/2} & \text{"decaying mode"} \end{cases}$

and in general we have:

$$\delta_k(\tau) = A_k\, a + B_k\, a^{-3/2}$$

where A_k and B_k will be determined by initial conditions, and these will determine whether the growing or decaying mode (or what linear combination) is excited.

What about velocity perturbations? Since in linear theory we have $\theta_k = -\frac{\partial \delta_k}{\partial \tau}$,

$$\theta_k(\tau) = -\mathcal{H}\left(A_k\, a - \frac{3}{2} B_k\, a^{-3/2}\right)$$

Since this is proportional to $-\mathcal{H}$ it is convenient to define

$$\Theta_k \equiv \frac{-\theta_k(\tau)}{\mathcal{H}} = A_k a - \frac{3}{2} B_k\, a^{-3/2}$$

let's now look at initial conditions. Since the overall value of a is arbitrary, let's say initial conditions are set @ $a=1$, and then we evolve forward. Then:

$$\Theta_k^{(0)} = A_k - \frac{3}{2} B_k \qquad \delta_k^{(0)} = A_k + B_k$$

$$\Rightarrow \begin{cases} A_k = \left(\delta_k^{(0)} + \frac{2}{3}\Theta_k^{(0)}\right)\frac{3}{5} = \frac{3}{5}\delta_k^{(0)} + \frac{2}{5}\Theta_k^{(0)} \\ B_k = \left(\delta_k^{(0)} - \Theta_k^{(0)}\right)\frac{2}{5} = \frac{2}{5}\delta_k^{(0)} - \frac{2}{5}\Theta_k^{(0)} \end{cases}$$

$$\Rightarrow \begin{cases} \delta_k(\tau) = \left[\frac{3}{5}\delta_k^{(0)} + \frac{2}{5}\Theta_k^{(0)}\right] a + \left[\frac{2}{5}\delta_k^{(0)} - \frac{2}{5}\Theta_k^{(0)}\right] a^{-3/2} \\ \Theta_k(\tau) = \left[\quad " \qquad " \quad\right] a - \frac{3}{2}\left[\quad " \qquad " \quad\right] a^{-3/2} \end{cases}$$

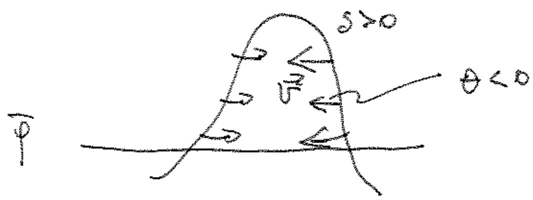
or, in matrix form:

$$\begin{pmatrix} \delta_k(\tau) \\ \Theta_k(\tau) \end{pmatrix} = \left\{ \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} a - \frac{1}{5}\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} a^{-3/2} \right\} \begin{pmatrix} \delta_k^{(0)} \\ \Theta_k^{(0)} \end{pmatrix}$$

We then see that the initial condition where:

$$\begin{pmatrix}\delta_k^{(0)}\\ \Theta_k^{(0)}\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix} \delta_k^{(0)}$$

excites only the growing mode. Physically this corresponds to $\delta_k^{(0)} = \Theta_k^{(0)} = -\theta^{(0)}/\mathcal{H}$, so the velocity divergence is opposite to the ~~divergen~~ density, therefore it corresponds to initial velocities pointing towards a high-density peak:



as it should be for a growing perturbation (mass keeps accumulating later so δ grows). On the contrary ~~an~~ ~~mac~~ initial condition $\propto \begin{pmatrix}1\\ -3/2\end{pmatrix}$ corresponds to velocities flowing <u>out</u> of high densities and therefore this is a purely decaying mode.

So far we discussed everything in terms of velocity divergence, but since velocity is a vector field, that's not the whole story.

As any vector field, we can decompose it into a divergence ("scalar mode") and a vorticity ("vector mode"),

$$\vec{v} = \vec{\nabla}\psi_v + \vec{\nabla}\times\vec{A}_v$$

So that
$$\begin{cases} \vec{\nabla}\cdot\vec{v} = \nabla^2\psi_v \equiv \theta & (\text{since } \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}_v) = 0) \quad \underline{\text{divergence}} \\ \vec{\nabla}\times\vec{v} = -\nabla^2\vec{A}_v \equiv \vec{\omega} & (\vec{\nabla}\times\vec{\nabla}\psi_v = 0\,;\ \vec{\nabla}\cdot\vec{A}_v = 0) \quad \underline{\text{vorticity}} \end{cases}$$

In Fourier space these relations take a very simple form:

$$\vec{v}_k = \underbrace{\frac{-i\vec{k}}{k^2}\,\theta_k}_{\substack{\text{longitudinal} \\ \text{(parallel to } \vec{k})}} + i\,\underbrace{\frac{\vec{k}\times\vec{\omega}_k}{k^2}}_{\substack{\text{transverse} \\ \text{(perpendicular to } \vec{k})}}$$

Indeed, since $\vec{\nabla} \to i\vec{k}$:

$$\begin{cases} i\vec{k}\cdot\vec{v}_k = \theta_k \\ i\vec{k}\times\vec{w}_k = -\frac{1}{k^2}\,\vec{k}\times(\vec{k}\times\vec{w}_k) = \vec{w}_k \end{cases} \qquad (\vec{k}\cdot\vec{w}_k = 0)$$

A velocity field with $\theta\neq 0$ and $\vec{w}=0$ is said to be potential or irrotational. If $\theta=0$ and $\vec{w}\neq 0$ it is said to be solenoidal (like the magnetic field).

How do vorticity perturbations evolve?

Since $\frac{\partial\vec{v}}{\partial\tau} + \mathcal{H}\vec{v} = -\nabla\Phi - \frac{\nabla p}{\bar{\rho}}$ in linear theory

$$\Rightarrow \quad \frac{\partial\vec{w}}{\partial\tau} + \mathcal{H}\vec{w} = 0$$

Since vorticity is not sourced by density perturbations, it decays due to the expansion of the universe:

$$\frac{1}{\mathcal{H}}\frac{\partial\vec{w}}{\partial\tau} = \frac{\partial\vec{w}}{\partial\ln a} = -\vec{w} \quad \Rightarrow \quad \boxed{\vec{w}\propto a^{-1}}$$

This can be understood from conservation of angular momentum:

$$\vec{L} = \vec{p}\times\vec{r} \quad \Rightarrow \quad \vec{L} = \rho\, d^3r\; \vec{v}\times\vec{r} \qquad \text{(all physical)}$$

$$\rho = \bar{\rho}(1+\delta) \underset{\delta\ll 1}{\simeq} \bar{\rho} \sim a^{-3} \qquad d^3r \sim a^3 \qquad \vec{v} = \underbrace{\vec{v}_{\parallel}}_{\text{from }\theta} + \underbrace{\vec{v}_{\perp}}_{\text{from }\vec{w}} \qquad \vec{v}_{\parallel}\times\vec{r} = 0$$

$$\Rightarrow \quad \vec{L} \propto \vec{v}_{\perp}\times\vec{r} \sim \frac{1}{a}\,a = \text{const.}$$

Therefore, we will deal with a <u>potential flow</u>, and the description is only through θ.

ii) Now, let's consider the case when $\Omega_m \neq 1$, observationally we are interested in the case when $\Omega_m < 1$. Consider the case where no extra energy densities are around (i.e. the universe is open with $\Omega_{TOT} = \Omega_m < 1$). A solution can be obtained by a change of variables to

$$x = \frac{1}{\Omega_m} - 1 \geq 0$$

At early times $\Omega_m \to 1$ and $x \to 0$, at late times $\Omega_m \to 0$ and $x \to \infty$ - So we can think of x as a new time variable - Solving the equations in this case is possible and it follows that

$$\begin{cases} D_+ = 1 + \dfrac{3}{x} + 3\sqrt{\dfrac{1+x}{x^3}} \ln\left[\sqrt{1+x} - \sqrt{x}\right] \\ D_- = \sqrt{\dfrac{1+x}{x^3}} \end{cases}$$

For small x, $\Omega_m \simeq 1$ and we recover $D_+ \simeq \frac{2}{5}x$, $D_- \simeq x^{-3/2}$ and $x \simeq a$ - For large $x \gg 1$, we see that $D_+ \simeq 1$, $D_- \simeq \frac{1}{x}$

Therefore fluctuations ~~stop~~ growing when $\Omega_m \to 0$, as expected since there is "not enough gravity" - The same is ~~true~~ when the universe goes from MAT to being dominated by vacuum energy (or dark energy), fluctuations stop growing as the universe accelerates -

In general, when $\Omega_m \neq 1$, we have

$$\delta_k = A_k D_+(a) + B_k D_-(a) \underset{\text{after long enough time}}{\simeq} A_k D_+(a)$$

$\Rightarrow$ since $\vec{v}_k = -\frac{i\vec{k}}{k^2}\theta_k$

we have $\theta = -\frac{\partial \delta}{\partial \tau} = -\mathcal{H}\frac{\partial \delta}{\partial \ln a}$ so: $\vec{v}_k = \frac{i\vec{k}}{k^2} A_k \mathcal{H} \frac{d D_+(a)}{d \ln a}$

defining $f \equiv \frac{d \ln D_+}{d \ln a}$ ($f \to 1$ as $\Omega_m \to 1$), we have

$$\boxed{\theta_k = -\mathcal{H} f \delta_k} \quad \text{and} \quad \boxed{\vec{v}_k = \frac{i\vec{k}}{k^2}\mathcal{H} f \delta_k}$$

In fact a differential equation can be obtained for f as a function of Ω_m - A useful approximation of a solution to this is

$f(\Omega_m) \sim \Omega_m^{0.6}$ for open universes $\qquad f(\Omega_m) \sim \Omega_m^{5/9}$ for flat univ. with cosmological constant

Solutions with Pressure

We now go back to the equations of motion for δ_k and add pressure:

$$\frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_k}{\partial \tau} = \frac{3}{2}\Omega_B \mathcal{H}^2 \delta_k - k^2 \frac{p_k}{\bar\rho}$$

where we considered the case of baryons. ~~...~~ In pple this can also be used when baryons are coupled to photon's at scales below the Hubble radius as well, in that case the extra term in Poisson equation due to radiation is small because δ_k^{RAD} is small. We'll discuss this in more detail below.

In order to solve this we must first ~~...~~ determine the equation of state for pressure perturbations. In ~~...~~ real space we can use

$$\nabla^2 \frac{\delta p}{\bar\rho} \underset{\substack{\uparrow \\ \text{assume} \\ \text{pressure is} \\ \text{a function of} \\ \text{density alone}}}{=} c_s^2 \nabla^2 \frac{\delta\rho}{\bar\rho} \longrightarrow -k^2 c_s^2 \delta_k \qquad \text{where } c_s^2 = \left(\frac{\delta p}{\delta \rho}\right)_{\underset{\substack{\uparrow \\ \text{at constant} \\ \text{entropy}}}{S}}$$

c_s is the adiabatic sound speed. For radiation $c_s^2 = 1/3$. For baryons, after recombination we can use a monoatomic ideal gas (hydrogen) with $\gamma = 5/3$, i.e. since $\rho_B \simeq n m_B + \frac{3}{2} nkT$ $\quad p = nkT \quad$ $\left(s = \frac{\rho + p}{T}\right)$

$$\Rightarrow \quad c_s^2 = \frac{5}{3}\frac{kT}{m_B} \qquad (c_s \text{ will be in general a function of time})$$

Going back to the eqs. of motion

$$\frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_k}{\partial \tau} = \left(\frac{3}{2}\Omega_B \mathcal{H}^2 - k^2 c_s^2\right)\delta_k$$

the RHS of this equation defines the <u>Jeans wavenumber</u> k_J

$$k_J^2 c_s^2 = \frac{3}{2}\Omega_B \mathcal{H}^2 = 4\pi G \bar\rho_B a^2$$

Remember k's are comoving, so $k^{phys} = k/a$ and

$$\boxed{k_J^{phys} = \frac{\sqrt{4\pi G \bar\rho_B}}{c_s} \quad \Big| \quad \lambda_J = \frac{2\pi}{k_J^{phys}} = \frac{2\pi c_s}{\sqrt{4\pi G \bar\rho_B}}}$$

Jeans Scale

then we can write

$$\frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_k}{\partial \tau} = c_s^2 \quad (k_J^2 - k^2)\,\delta_k$$

Thus for $k \ll k_J$ ($\lambda \gg \lambda_J$) perturbations evolve like in the pressureless case. However, for small scales $k > k_J$ and pressure takes over, perturbations cannot grow, pressure wins over gravity and prevents growth.

Physically, recall that $H^2 \sim G\bar{\rho} \Rightarrow (G\bar{\rho})^{-1/2} \sim H^{-1}$ = Hubble time

$\Rightarrow \lambda_J \sim c_s \times$ Hubble time $\sim$ sound horizon

$\Rightarrow \lambda > \lambda_J$ means pressure cannot react over such large scales to counteract gravity, so perturbations grow as in pressureless case.

Note that in the absence of expansion, $\tau = t$, $\ddot{\delta}_k = -c_s^2 k^2 \delta_k$ at $\lambda \ll \lambda_J$, and if c_s is independent of time $\delta_k \sim e^{\pm i k c_s t}$, just usual sound waves. When add expansion, details will depend on $c_s(a)$.

It is useful the Jeans mass, the mass in a radius equal to $\lambda_J/2$:

$$M_J = \frac{4\pi}{3}\rho_0 \left(\frac{\lambda_J}{2}\right)^3 = \frac{\pi^{5/2}\, c_s^3}{6\, G^{3/2} \bar{\rho}^{1/2}}$$

One can think of this mass scale as the minimum mass object that can collapse and form in the universe. The evolution of λ_J and M_J depend on $c_s(a)$:

$$\lambda_J \sim c_s\, a^{3/2} \qquad M_J \sim c_s^3\, a^{3/2}$$

During the RAD era, since baryons are tightly coupled to photons, baryons pressure is provided by photons, therefore $c_s^2 = 1/3$ and the Jeans mass in baryons is

$\rho_i = 27.75\, \Omega_i\, 10^{10} M_\odot/h\, /\, (\mathrm{Mpc}/h)^3 \qquad M_\odot = 2 \times 10^{33}\,\mathrm{g} \simeq 1.1 \times 10^{57}\,\mathrm{GeV}$

$$M_J^B = \frac{4\pi}{3}\rho_B \left(\frac{\lambda_J}{2}\right)^3 \simeq 5.4 \times 10^{18}\, \Omega_B h^2 \left(\frac{T}{1\,\mathrm{eV}}\right)^{-3/2} M_\odot$$

this is a ~~large~~ mass scale

Let's compare this with baryon mass within H^{-1},

$$M_H^B = \frac{4\pi}{3}\,\bar\rho_B\,(H^{-1})^3$$

$$\frac{M_J^B}{M_H^B} = \left(\frac{\lambda_J}{2H^{-1}}\right)^3 \underset{\uparrow}{=} \left[\frac{2\pi c_s}{\sqrt{4\pi G\rho_B}}\sqrt{\frac{8\pi G\rho_R}{3}}\right]^3 = (\pi c_s)^3\left(\frac{2\rho_R}{3\rho_B}\right)^{3/2} \gg 1$$

i.e. $\lambda_J \gg H^{-1}$

$\rho \simeq \rho_R + \rho_B \simeq \rho_R$ in RAD era

Thus, during RAD era, baryons cannot grow for scales $\lambda \lesssim H^{-1}$, due to photon pressure. After recombination, when $\bar e$ and p recombine and matter decouples from radiation we have:

$$c_s^2 = \frac{5}{3}\frac{T_B}{m} \underset{\uparrow}{=} \frac{5}{3}\frac{T^2}{m T_{REC}}$$

this is a huge drop in c_s^2!

$T_B \sim a^{-2}$ (NR, decoupled)
$T \sim a^{-1}$ for photons $\Big\} \Rightarrow T_B = \frac{T_{REC}}{a^2} = T_{REC}\left(\frac{T}{T_{REC}}\right)^2 = \frac{T^2}{T_{REC}}$

$$\Rightarrow M_J^B \underset{\substack{\uparrow\\ \rho\simeq\rho_B}}{=} \frac{\pi^{5/2}}{6}\frac{c_s^3}{G^{3/2}\rho_B^{1/2}} \underset{\substack{\uparrow\\ c_s^2\sim T^2\\ \rho_B\sim a^{-3}\sim T^3}}{\sim} T^{3/2} \sim (1+z)^{3/2}$$

putting numbers: $\boxed{M_J^B \simeq 1\text{-}3\times10^5\,(\Omega_B h^2)^{-1/2}\left(\frac{z}{1100}\right)^{3/2} M_\odot}$

Note the substantial drop in the Jeans mass; after REC/DEC baryon perturbations with $M > M_J^B$ can grow and collapse to form objects. Only after this transition can sub-Hubble baryon perturbations grow, before REC/DEC baryons cannot move freely through plasma due to photon pressure to collapse.

Free-streaming or Collisionless Landau damping

So far we have assumed an ideal fluid description, we now discuss free streaming which is a mechanism that erases fluctuations @ small scales. This operates in collisionless dark matter, the idea is that collisionless particles can stream out of overdense regions into underdense regions and thus smooth out density perturbations. Note that this effect is just due to their free motion, not the perturbed velocity field that responds to density perturbations. It is most important for relativistic particles, and cannot be modeled by a perfect fluid.

We will just estimate the length scale at which this becomes important.

The physical scale l_{FS} can be estimated by just

$$l_{FS}(t) = a(t) \underbrace{\int_0^t \frac{v(t)\,dt}{a(t)}}_{\text{comoving length}}$$

let's assume that for $t < t_{NR}$ (when dark matter is REL) the universe is RAD dominated, usually a good approximation. Then

$$\boxed{l_{FS}(t) = a(t) \int_0^t \frac{dt}{a(t)} = 2t \propto a^2} \qquad t < t_{NR}$$

If $t_{NR} < t < t_{EQ}$ (note once it becomes NR, $v \sim 1/a \Rightarrow v = a_{NR}/a$)

$$\boxed{l_{FS}(t)} = \frac{a}{a_{NR}} 2t_{NR} + a(t)\int_{t_{NR}}^t \frac{dt}{a}\frac{a_{NR}}{a} \underset{a = a_{NR}\left(\frac{t}{t_{NR}}\right)^{1/2}}{\overset{\uparrow}{=}} \frac{a}{a_{NR}} 2t_{NR} + a(t)\frac{\cancel{a_{NR}}}{\cancel{a_{NR}}}\frac{t_{NR}}{a_{NR}}\int_{t_{NR}}^t \frac{dt}{t}$$

$$= \frac{a}{a_{NR}} 2t_{NR} + \frac{a\, 2t_{NR}}{a_{NR}} \frac{1}{2} \ln t/t_{NR} = \boxed{\frac{a}{a_{NR}} 2t_{NR}\left[1 + \ln\frac{a}{a_{NR}}\right]} \qquad t_{NR} < t < t_{EQ}$$

For $t > t_{EQ}$ ($a \sim t^{2/3}$) then

$$l_{FS}(t) = \frac{a}{a_{EQ}} l_{FS}(t_{EQ}) + a(t)\int_{t_{EQ}}^t \frac{a_{NR}}{a_{EQ}^2}\left(\frac{t_{EQ}}{t}\right)^{4/3} dt$$

$$\Rightarrow \boxed{l_{FS}(t) = 2t_{NR}\frac{a}{a_{NR}}\left(1 + \ln\frac{a_{EQ}}{a_{NR}}\right) + 3t_{NR}\frac{a}{a_{NR}}\left(1 - \sqrt{\frac{a_{EQ}}{a}}\right)} \qquad t_{EQ} < t$$

For $a \gg a_{EQ}$ $\quad l_{FS} \simeq \frac{2t_{NR}}{a_{NR}}\, a\left[\frac{5}{2} + \ln\frac{a_{EQ}}{a_{NR}}\right]$

To compare l_{FS} with λ it is most illuminating to do it in comoving lengths since $\lambda_{comov} = \text{const}$. Then, the comoving free streaming length is:

$$\frac{l_{FS}(t)}{a(t)} \simeq \begin{cases} \frac{2t}{a} = \frac{2t_{NR}}{a}\left(\frac{t}{t_{NR}}\right) = \frac{2t_{NR}}{a_{NR}^2}\, a & t < t_{NR} \\ \frac{2t_{NR}}{a_{NR}}\left[1 + \ln(a/a_{NR})\right] & t_{NR} < t < t_{EQ} \\ \frac{2t_{NR}}{a_{NR}}\left[5/2 + \ln\frac{a_{EQ}}{a_{NR}}\right] & t \gg t_{EQ} \end{cases}$$

So, for $t < t_{NR}$ increases like a, then it only grows logarithmically and in MAT era saturates. This is the largest value it takes, and today:

$$l_{FS}(t_0) = \frac{a_0}{a_{NR}} \, 2 t_{NR} \left[\frac{5}{2} + \ln \frac{a_{EQ}}{a_{NR}} \right]$$

to evaluate this we need to get NR time as a function of particle mass

$$\left(\frac{T_{NR}}{T}\right)^3 = \frac{n}{n_\gamma} \text{ is conserved during expansion} \qquad \Omega_{DM} = \left(\frac{mn}{\rho_{crit}}\right)_{Today} = \frac{m n_\gamma}{\rho_{crit}} \frac{n}{n_\gamma} \simeq 30 \left(\frac{m}{1\,\mathrm{keV}}\right)\left(\frac{n}{n_\gamma}\right) h^{-2}$$

putting numbers $\quad \frac{a_{NR}}{a_0} = 7 \times 10^{-7} \left(\frac{m}{1\,\mathrm{keV}}\right)^{-1} \frac{T_{NR}}{T} \qquad t_{NR} = 1.2 \times 10^{7} \left(\frac{m}{1\,\mathrm{keV}}\right)^{-2} \left(\frac{T_{NR}}{T}\right)^2$

$$\frac{a_{EQ}}{a_{NR}} \simeq \frac{m}{1\,\mathrm{keV}} \left(\Omega h^2\right)^{-1} \left(\frac{T}{T_{NR}}\right)$$

$$\Rightarrow \quad l_{FS}(t_0) \simeq 40\ \mathrm{Mpc}\ \left(\Omega_{DM} h^2\right)^{-1} \left(\frac{T_{NR}}{T}\right)^4 = 0.5\ \mathrm{Mpc}\ \left(\Omega_{DM} h^2\right)^{1/3} \left(\frac{m}{1\,\mathrm{keV}}\right)^{-4/3}$$

For a massive neutrino: $\quad m \simeq 30\,\mathrm{eV} \qquad \frac{T_{NR}}{T} \sim 0.7 \qquad \Omega_\nu h^2 = \frac{m_\nu}{100\,\mathrm{eV}}$

$$\Rightarrow \quad l_{FS}(t_0) \simeq 30\ \mathrm{Mpc} \left(\frac{m_\nu}{30\,\mathrm{eV}}\right)^{-1}$$

For a heavy cold relic: $\quad m \simeq 1\,\mathrm{keV} \qquad \Omega_{DM} h^2 \simeq 1 \quad \Rightarrow \quad l_{FS}(t_0) \simeq 0.5\ \mathrm{Mpc} \left(\frac{m}{1\,\mathrm{keV}}\right)^{-4/3}$

This modifies the spectrum of perturbations @ small scales erasing perturbations of scales $< l_{FS}$.