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Introduction

We will now discuss how perturbations evolve in the low-z universe, assuming for simplicity that the background evolution is due to dark matter plus a cosmological constant, this is known as "ΛCDM". From the point of view of the low-z universe, the problem can be thought as a particular type of perturbation spectrum evolving according to the equations of motion for scales below the Hubble radius. The primordial perturbation spectrum (i.e. as generated by inflation) gets processed through the matter-radiation transition giving a processed spectrum which is the primordial times the square of the so-called transfer function, as we discussed before. Thus, we shall consider the evolution of perturbations with an "initial spectrum" given by this processed spectrum. For the most part we shall assume Gaussian initial conditions.

Equations of Motion (Eulerian)

Stress-Energy conservation for subhorizon perturbations leads to the following equations of motion for density and velocity fields (neglecting metric perturbations compared to $\vec{v}$ and $\mathcal{H}$)

$$
\begin{cases}
\dfrac{\partial \delta}{\partial \tau} + \vec{\nabla}\left[(1+\delta)\,\vec{v}\right] = 0 \\[2ex]
\dfrac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\Phi \\[2ex]
\nabla^2 \Phi = +4\pi G a^2 \bar{\rho}\,\delta
\end{cases}
$$

Together with the Friedmann equations,

$$
\begin{cases}
H^2 = \dfrac{8\pi G}{3}(\bar{\rho} + \rho_\Lambda) \\[2ex]
\dfrac{\ddot{a}}{a} = \dot{H} + H^2 = -\dfrac{4\pi G}{3}\underbrace{(\bar{\rho} - 2\rho_\Lambda)}_{\sum_i (\rho+3p)_i}
\end{cases}
$$

We can rewrite the source of Poisson equation by introducing $\Omega_m(a)$:

$$\Omega_m(a) \equiv \frac{\bar{\rho}}{\rho_c} = \frac{\bar{\rho}\, 8\pi G}{3H^2}$$

$$\Rightarrow \quad 4\pi G\, a^2 \bar{\rho} = \frac{3}{2} \mathcal{H}^2 \Omega_m(a)$$

$$\Rightarrow \quad \nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \Omega_m(a)\, \delta$$

In what follows we will assume the velocity field to be potential, and thus fully characterized by its divergence. From the equation of momentum conservation this is self consistent if the primordial vorticity vanishes, as the equations above guarantee this stays the same for non-linear evolution. However, at small scales the Euler equation does not hold, there are corrections due to multistreaming and these can generate vorticity.

We can rewrite the equations of motion as,

$$\begin{cases} \dfrac{\partial \delta}{\partial \tau} + \theta = -\vec{\nabla}(\delta \vec{v}) \\[2ex] \dfrac{\partial \theta}{\partial \tau} + \mathcal{H}\theta + \dfrac{3}{2}\mathcal{H}^2 \Omega_m \delta = -\vec{\nabla}\cdot\left[(\vec{v}\cdot\vec{\nabla})\vec{v}\right] \end{cases}$$

where we have put the non linear terms in the RHS.

Linear perturbation theory

In the linear approximation, we set to zero the non-linear terms. We have

$$\begin{cases} \dfrac{\partial \delta}{\partial \tau} + \theta = 0 \\[2ex] \dfrac{\partial \theta}{\partial \tau} + \mathcal{H}\theta + \dfrac{3}{2}\mathcal{H}^2 \Omega_m \delta = 0 \end{cases}$$

$$\Rightarrow \quad \frac{\partial^2 \delta}{\partial \tau^2} + \frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \delta}{\partial \tau^2} - \mathcal{H}\theta - \frac{3}{2}\mathcal{H}^2 \Omega_m \delta = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 \delta}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta}{\partial \tau} = \frac{3}{2}\mathcal{H}^2 \Omega_m \delta}$$

that's the evolution of the density field in linear PT ; for the velocity divergence we have $\theta = -\frac{\partial \delta}{\partial \tau}$

Since this is a second-order ODE, there are 2 independent solutions. Let's first consider the ideal case $\Omega_m = 1$, $\Omega_\Lambda = 0$. ~~Take~~ change variable from τ to $\ln a$

$$\mathcal{H}\frac{\partial \delta}{\partial \tau} = \mathcal{H}^2 \frac{\partial \delta}{\partial \ln a}$$

$$\frac{\partial^2 \delta}{\partial \tau^2} = \frac{\partial}{\partial \tau}\left(\mathcal{H}\frac{\partial \delta}{\partial \ln a}\right) = \frac{\partial \mathcal{H}}{\partial \tau}\frac{\partial \delta}{\partial \ln a} + \mathcal{H}^2 \frac{\partial^2 \delta}{\partial \ln a^2} = \mathcal{H}^2 \frac{\partial^2 \delta}{\partial \ln a^2} + \mathcal{H}\frac{\partial \mathcal{H}}{\partial \ln a}\frac{\partial \delta}{\partial \ln a}$$

$$\Rightarrow \quad \frac{\partial^2 \delta}{\partial \ln a^2} + \left(\frac{\partial \ln \mathcal{H}}{\partial \ln a} + 1\right)\frac{\partial \delta}{\partial \ln a} = \frac{3}{2}\Omega_m \delta$$

For $\Omega_m = 1$, $\quad H^2 = H_0^2 / a^3 \Rightarrow \mathcal{H}^2 = H_0^2 / a \Rightarrow \mathcal{H} \propto a^{-1/2} \Rightarrow \frac{\partial \ln \mathcal{H}}{\partial \ln a} = -1/2$

$$\Rightarrow \quad \frac{\partial^2 \delta}{\partial \ln a^2} + \frac{1}{2}\frac{\partial \delta}{\partial \ln a} = \frac{3}{2}\delta$$

This admits power-law solutions : $\delta \propto a^p \Rightarrow p^2 + p/2 = 3/2$

$\Rightarrow \quad p = 1, -3/2$

$$\Rightarrow \quad \begin{cases} \delta_+ \propto a & \text{growing mode} \\ \delta_- \propto a^{-3/2} & \text{decaying mode} \end{cases}$$

Now, let's look at the general case. Note that $\Omega_m = 1$ the decaying mode is the Hubble constant $\delta_- \propto H \propto a^{-3/2}$. This continues to be true in the more general case (in fact as long as sources are matter, cosmological constant and curvature), as you can check. Therefore

We can look for the growing mode by reducing the order of the ODE, i.e. $D_+ \equiv H\, y$ and find y by quadrature. The result of this algebra is

$$\begin{cases} D^-_{(a)} = H(a) \\ D^+_{(a)} = a^3 H^3_{(a)} \frac{5}{2} \Omega_m(a) \int_0^a \frac{da'}{a'^3 H^3_{(a')}} \end{cases}$$

and in general $\delta(\bar{x}, \tau) = A(\bar{x})\, D_-(a) + B(\bar{x})\, D_+(a)$

For other energy content, e.g. dark energy fluid with some w, or quintessence one must go back to the 2nd-order ODE to find the growing and decaying modes.

The evolution of velocity divergence field follows from δ. Defining

$$f \equiv \frac{d \ln D_+}{d \ln a} \qquad \text{and} \qquad g \equiv \frac{d \ln D_-}{d \ln a}$$

we have

$$\theta(\bar{x}, \tau) = -\mathcal{H}\left[A(\bar{x})\, D_-(a)\, g + B(\bar{x})\, D_+\, f\right]$$

For flat models with a cosmological constant, one can in fact find an ODE for $f(\Omega_m)$

$$f^2 + f'\, 3\Omega_m(\Omega_m - 1) + (2 - 3/2\, \Omega_m)\, f = \frac{3}{2} \Omega_m \qquad \left(\frac{d}{d\Omega_m} \equiv {}'\right)$$

with approximate behavior $f(\Omega_m) \propto \Omega_m^{5/9}$

2nd-order PT

In order to solve things to 2nd or higher order it is convenient to Fourier transform the equations of motion, we use convention:

$$A(\bar{k}) = \int \frac{d^3x}{(2\pi)^3}\, e^{-i k \cdot \bar{x}}\, A(\bar{x})$$

Then we have,

$$\begin{cases} \frac{\partial \delta(\vec{k})}{\partial \tau} + \theta(\vec{k}) = -\int d^3k_1\, d^3k_2\, \delta_D(\vec{k}-\vec{k}_{12})\, \alpha(\vec{k}_1,\vec{k}_2)\, \theta(\vec{k}_1)\, \delta(\vec{k}_2) \\ \frac{\partial \theta(\vec{k})}{\partial \tau} + \mathcal{H}\,\theta(\vec{k}) + \frac{3}{2}\Omega_m \mathcal{H}^2 \delta(\vec{k}) = -\int d^3k_1\, d^3k_2\, \delta_D(\vec{k}-\vec{k}_{12})\, \beta(\vec{k}_1,\vec{k}_2)\, \theta(\vec{k}_1)\, \theta(\vec{k}_2) \end{cases}$$

where $\alpha(\vec{k}_1,\vec{k}_2) \equiv \frac{\vec{k}_{12}\cdot\vec{k}_1}{k_1^2}$ $\qquad \beta(\vec{k}_1,\vec{k}_2) \equiv \frac{k_{12}^2\, \vec{k}_1\cdot\vec{k}_2}{2k_1^2 k_2^2}$

and $\vec{k}_{ij} \equiv \vec{k}_i + \vec{k}_j$

Consider first, again, $\Omega_m = 1$ - Then one can look for solutions for the fastest growing mode obeying

$$\delta(\vec{k},a) = \sum_{n=1}^{\infty} a^n\, \delta_n(\vec{k}) \qquad \theta(\vec{k},a) = -\mathcal{H} \sum_{n=1}^{\infty} a^n\, \theta_n(\vec{k})$$

where for $n=1$ $\delta_1(\vec{k}) = \theta_1(\vec{k})$ - Since $a \propto t^{2/3} \propto \tau^2$, $\mathcal{H} = \frac{2}{\tau}$ and $\frac{\partial\mathcal{H}}{\partial\tau} = -\frac{\mathcal{H}^2}{2}$

Then we have

$$\frac{\partial\delta}{\partial\tau} = \mathcal{H} \sum n\, a^n\, \delta_n \qquad \frac{\partial\theta}{\partial\tau} = -\mathcal{H}^2 \sum_n \left(n-\frac{1}{2}\right) a^n\, \theta_n$$

First equation then gives (look for a^n term)

$$n\,\delta_n - \theta_n = \sum_{m=1}^{n-1} \alpha\, \theta_m\, \delta_{n-m} \qquad (1)$$

where RHS means actually $\int d^3k_1\, d^3k_2\, \delta_D(\vec{k}-\vec{k}_{12})\, \alpha(\vec{k}_1,\vec{k}_2)\, \delta_{n-m}(\vec{k}_2)\, \theta_m(\vec{k}_1)$

Similarly, 2nd equation gives

$$-\left(n-\frac{1}{2}\right)\theta_n - \theta_n + \frac{3}{2}\delta_n = -\sum_{m=1}^{n-1} \beta\, \theta_{n-m}\, \theta_m$$

$$\Rightarrow \quad (2n+1)\,\theta_n - 3\,\delta_n = \sum_{m=1}^{n-1} 2\beta\, \theta_{n-m}\, \theta_m \qquad (2)$$

We can combine (1) and (2) by using matrix notation,

$$\begin{bmatrix} n & -1 \\ -3 & 2n+1 \end{bmatrix} \begin{pmatrix} \delta_n \\ \theta_n \end{pmatrix} = \sum_{m=1}^{n-1} \begin{pmatrix} \alpha\, \delta_{n-m} \\ \beta\, \theta_{n-m} \end{pmatrix} \theta_m$$ (6)

inverting matrix $\Rightarrow$ $$\begin{pmatrix} \delta_n \\ \theta_n \end{pmatrix} = \frac{1}{(2n+3)(n-1)} \begin{bmatrix} 2n+1 & 1 \\ 3 & n \end{bmatrix} \sum_{m=1}^{n-1} \begin{pmatrix} \alpha\, \delta_{n-m} \\ \beta\, \theta_{n-m} \end{pmatrix} \theta_m$$

What does this mean? This is a recursion relation, that gives n^{th} order solution as a product of 2 solutions of lower order such that the sum of orders is n. By writing

$$\delta_n(\vec{k}) = \int d^3q_1 \cdots d^3q_n\, [\delta_D]\, F_n(\vec{q}_1, \ldots, \vec{q}_n)\, \delta_1(\vec{q}_1) \cdots \delta_1(\vec{q}_n)$$

$$\theta_n(\vec{k}) = \int d^3q_1 \cdots d^3q_n\, [\delta_D]\, G_n(\vec{q}_1, \ldots, \vec{q}_n)\, \delta_1(\vec{q}_1) \cdots \delta_1(\vec{q}_n)$$

where $[\delta_D] \equiv \delta_D(\vec{k} - \vec{q}_1 - \cdots - \vec{q}_n)$, we can find the kernels F_n and G_n (by definition $F_1 = G_1 = 1$):

$$F_n = \sum_{m=1}^{n-1} \frac{G_m}{(2n+3)(n-1)} \Big[(2n+1)\, \alpha(\vec{k}_1, \vec{k}_2)\, F_{n-m} + 2\beta(\vec{k}_1, \vec{k}_2)\, G_{n-m} \Big]$$

$$G_n = \sum_{m=1}^{n-1} \frac{G_m}{(2n+3)(n-1)} \Big[3\alpha\, F_{n-m} + 2n\beta\, G_{n-m} \Big]$$

where F_n and G_n depend on $\vec{q}_1, \ldots, \vec{q}_n$, G_m on $\vec{q}_1, \ldots, \vec{q}_m$, and F_{n-m} and G_{n-m} on $\vec{q}_{m+1}, \ldots, \vec{q}_n$, and $\vec{k}_1 = \vec{q}_1 + \cdots + \vec{q}_m$ and $\vec{k}_2 = \vec{q}_{m+1} + \cdots + \vec{q}_n$.

For $n=2$ we have (after symmetrizing over $\vec{q}_1$ and $\vec{q}_2$)

$$F_2(\vec{q}_1, \vec{q}_2) = \frac{5}{7} + \frac{1}{2} \cos \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7} \cos^2$$

$$G_2(\vec{q}_1, \vec{q}_2) = \frac{3}{7} + \frac{1}{2} \cos \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{4}{7} \cos^2$$

where $\cos \equiv \dfrac{\vec{q}_1 \cdot \vec{q}_2}{q_1\, q_2}$

Before we look into applications of this, let's consider the case $\Omega_m \neq 1$. Now, to linear order we have

$$\begin{cases} \delta(\mathbf{k},\tau) = D\,\delta(\mathbf{k}) \\ \theta(\mathbf{k},\tau) = -\mathcal{H} f\, D\, \delta(\mathbf{k}) \end{cases}$$

where only the growing mode has been kept. As a generalization of the $\Omega_m=1$ ansatz, we look for separable solutions of the form:

$$\delta(\mathbf{k},\tau) = \sum_{n=1}^{\infty} D_n(\tau)\, \delta_n(\mathbf{k})$$

$$\theta(\mathbf{k},\tau) = -\mathcal{H} f \sum_{n=1}^{\infty} E_n(\tau)\, \theta_n(\mathbf{k})$$

where $D_1 = E_1 = D$. It is convenient to use the time variable $\ln D$, then

$$\frac{\partial}{\partial \tau} = \frac{d\ln D}{d\tau} \frac{\partial}{\partial \ln D} = \frac{d\ln D}{d\ln a} \frac{d\ln a}{d\tau} \frac{\partial}{\partial \ln D} = \mathcal{H} f \frac{\partial}{\partial \ln D}$$

Then, the ~~n~~ nth order term in continuity equation looks like

$$\frac{dD_n}{d\ln D}\delta_n - E_n \theta_n = \sum_{m=1}^{n-1} \alpha\, D_{n-m} E_m\, \theta_m\, \delta_{n-m}$$

To look at Euler we need

$$\frac{\partial}{\partial\tau}\theta \longrightarrow -\frac{\partial}{\partial\tau}(\mathcal{H} f E_n) = -\mathcal{H}^2 f^2 \frac{dE_n}{d\ln D} - \frac{\partial}{\partial\tau}(\mathcal{H} f)\, E_n$$

but since $\mathcal{H} f = \dfrac{d\ln D}{d\tau} \Rightarrow \dfrac{d}{d\tau}(\mathcal{H} f) = \dfrac{d}{d\tau}\dfrac{d\ln D}{d\tau} = \dfrac{1}{D}\dfrac{d^2 D}{d\tau^2} - \left(\dfrac{d\ln D}{d\tau}\right)^2$

From ODE for the growth factor: $\dfrac{d^2 D}{d\tau^2} = -\mathcal{H}\dfrac{dD}{d\tau} + \dfrac{3}{2}\Omega_m \mathcal{H}^2 D$

$$\Rightarrow \frac{d}{d\tau}(\mathcal{H} f) = -\mathcal{H} \underbrace{\frac{d\ln D}{d\tau}}_{\mathcal{H} f} + \frac{3}{2}\Omega_m \mathcal{H}^2 - \mathcal{H}^2 f^2 = -\mathcal{H}^2 f(f+1) + \frac{3}{2}\Omega_m \mathcal{H}^2$$

Then Euler gives

$$-\mathcal{H}^2 f^2 \frac{dE_n}{d\ln D}\theta_n + \mathcal{H}^2\left[f(f\cancel{+1}) - \frac{3}{2}\Omega_m\right] E_n\theta_n \cancel{- \mathcal{H}^2 f E_n \theta_n} + \frac{3}{2}\Omega_m \mathcal{H}^2 D_n \delta_n =$$

$$= -H^2 f^2 \sum_{m=1}^{n-1} \beta \; \theta_m \theta_{n-m} \; E_m E_{n-m}$$

Dividing by $-H^2 f^2$ get

$$\frac{dE_n}{d\ln D} \delta_n + \left(\frac{3}{2}\frac{\Omega_m}{f^2} - 1\right) E_n \theta_n + \frac{3}{2}\frac{\Omega_m}{f^2} D_n \delta_n = \sum_{m=1}^{n-1} \beta \; \theta_m \theta_{n-m} \; E_m E_{n-m}$$

Now, take a hard look at this and continuity equation above. You see that if $\frac{\Omega_m}{f^2} = 1$ then $D_n = E_n = D^n$ becomes a self-consistent (separable) solution to any order, thus giving the same recursion relations as in $\Omega_m = 1$ case provided we use D^n instead of a^n. Since for e.g. a ΛCDM model $f \sim \Omega_m^{5/9}$, then $\frac{\Omega_m}{f^2} \approx \Omega_m^{1-10/9} \sim \Omega_m^{-1/9}$ which is very close to unity, e.g. for $\Omega_m = 0.3$ $\Omega_m^{-1/9} \simeq 1.14$; and at higher redshift this approximation improves substantially. Therefore, to very good approximation the F_n, G_n kernels do not depend on cosmological parameters, it is all encoded in the linear growth factor.

In fact, the actual dependence can be calculated and for $n=2$ gives

$$F_2 = \frac{1}{2}(1+\epsilon) + \frac{1}{2}\cos\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \left(\frac{1}{2} - \frac{\epsilon}{2}\right)\cos^2$$

$$G_2 = \epsilon + \frac{1}{2}\cos\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + (1-\epsilon)\cos^2$$

where $\epsilon \simeq \frac{3}{7}\Omega_m^{-2/63}$ (for $\Omega_m > 0.1$) $\left(0.3^{-2/63} \simeq 1.038\right)$

Thus, within a few percent corrections to monopole and quadrupole terms the kernels are independent of cosmological parameters.

Finally, a simple application of the recursion relations is to consider the spherical average of the kernels, which defines the vertices,

$$\nu_n \equiv n! \int \frac{d\Omega_1}{4\pi} \cdots \frac{d\Omega_n}{4\pi} \, F_n(\mathbf{k}_1, \ldots, \mathbf{k}_n) \qquad \nu_n = \mu_n = 1$$

$$\mu_n \equiv n! \int \frac{d\Omega_1}{4\pi} \cdots \frac{d\Omega_n}{4\pi} \, G_n(\mathbf{k}_1, \ldots, \mathbf{k}_n)$$

We can derive a recursion relation for these vertices by taking the angle average of the recursion relations. The structure of these allows for the angle average to be done recursively; first average over angle between $\mathbf{k}_1$ and $\mathbf{k}_2$, thus replacing α and β by their angular averages, 1 and 1/3 respectively. Then we have

$$\nu_n = \sum_{m=1}^{n-1} \binom{n}{m} \frac{\mu_m}{(2n+3)(n-1)} \left[(2n+1)\, \nu_{n-m} + \frac{2}{3} \mu_{n-m} \right]$$

$$\mu_n = \sum_{m=1}^{n-1} \binom{n}{m} \frac{\mu_m}{(2n+3)(n-1)} \left[3\, \nu_{n-m} + \frac{2}{3}\, n\, \mu_{n-m} \right]$$

leading to $\nu_2 = \frac{34}{21}$ $\quad \nu_3 = \frac{682}{189}$ $\quad \mu_2 = +\frac{26}{21}$ $\quad \mu_3 = \frac{142}{63}$

This vertices are related to the spherical collapse dynamics. Indeed, if the initial density field is symmetric about $\mathbf{x}=0$ then $\delta_1(\mathbf{k}) = \delta_1(|\mathbf{k}|)$ then under spherical collapse this symmetry will be maintained, and the central density will be given by

$$\delta_{sc}(a) \underset{\substack{\Omega_m = 1 \\ \text{for simplicity}}}{=} \sum_n a^n \int d^3q_1 \ldots d^3q_n \, F_n(\mathbf{q}_1, \ldots, \mathbf{q}_n) \, \delta_1(|\mathbf{q}_1|) \cdots \delta_1(|\mathbf{q}_n|)$$

$$= \sum_n a^n \epsilon^n \frac{\nu_n}{n!} \qquad \text{where} \quad \epsilon \equiv \int d^3q \, \delta_1(|\mathbf{q}|)$$

and similarly for $\theta_{sc}(a)$ interms of the μ_n vertices.

We now go back to F_2 and G_2 for a physical interpretation of the different terms. For simplicity we drop the small cosmological parameter dependence. let's look at F_2 first. We can rewrite it as

$$F_2(k_1, k_2) = \frac{5}{14}\left[\alpha(k_1,k_2) + \alpha(k_2,k_1)\right] + \frac{2}{7}\beta(k_1,k_2)$$

$$F_2(k_1, k_2) = \frac{\nu_2}{2!} + \frac{1}{2}\underbrace{\frac{k_1\cdot k_2}{k_1 k_2}}_{\hat{k}_1\cdot\hat{k}_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{2}{7}\left[(\hat{k}_1\cdot\hat{k}_2)^2 - \frac{1}{3}\right]$$

or

$$F_2(k_1,k_2) = \frac{\nu_2}{2!} + \frac{1}{2}\hat{k}_1\cdot\hat{k}_2\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{2}{7}\left(\hat{k}_1^i\hat{k}_1^j - \frac{1}{3}\delta_{ij}\right)\left(\hat{k}_2^i\hat{k}_2^j - \frac{1}{3}\delta_{ij}\right)$$

The first term corresponds to the spherical averaged evolution, leading to a term $\frac{\nu_2}{2}\delta_1^2(\bar{x})$ in $\delta(\bar{x})$ [$\delta_1(\bar{x})$ being the linear density contrast]

The last term is also easy to understand, each factor corresponds to

$$\left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij}\right)\delta_1(k) \longrightarrow \left(\nabla_i\nabla_j\phi_1 - \frac{1}{3}\nabla^2\phi_1\,\delta_{ij}\right)$$

where ϕ_1 is the linear gravitational potential; such term is the tidal force effect of gravity. What about the middle term? To explain this is a bit long, but here is the idea: it comes from a transformation from following fluid elements (Lagrangian space) to computing things at fixed $\bar{x}$ (Eulerian) as we have done so far. In a sense, this is a kinematic term (that's why, for example, it does not get affected by cosmological parameters). What do we mean by this?

Consider taking time derivatives following the fluid rather than at fixed $\bar{x}$:

$$\frac{d}{dt} = \frac{\partial}{\partial\tau} + \bar{v}\cdot\bar{\nabla}$$

Then the equations of motion read (try it!)

$$\frac{d\delta}{d\tau} + \theta = - \int d^3k_1\, d^3k_2\, \delta_D(\mathbf{k}-\mathbf{k}_{12})\, \theta(\mathbf{k}_1)\, \delta(\mathbf{k}_2)$$

$$\frac{d\theta}{d\tau} + \mathcal{H}\theta + \frac{3}{2}\Omega_m \mathcal{H}^2 \delta = - \int d^3k_1\, d^3k_2\, \delta_D(\mathbf{k}-\mathbf{k}_{12}) \left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1 k_2}\right)^2 \theta(\mathbf{k}_1)\, \theta(\mathbf{k}_2)$$

We see now that the structure of the non-linear terms is different in the Lagrangian picture. We can define Lagrangian coupling coefficients α^L and β^L as

$$\begin{cases} \alpha^L(\mathbf{k}_1,\mathbf{k}_2) = 1 \\ \beta^L(\mathbf{k}_1,\mathbf{k}_2) = \left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1 k_2}\right)^2 \end{cases}$$

We see that these are related to the Eulerian ones by

$$\alpha = \alpha^L + \frac{\mathbf{k}_2\cdot\mathbf{k}_1}{k_1^2} \xrightarrow{\text{symmetrize}} \alpha^L + \frac{1}{2}\hat{k}_1\cdot\hat{k}_2\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)$$

$$\beta = \beta^L + \frac{1}{2}\hat{k}_1\cdot\hat{k}_2\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)$$

This is the "middle term" appearing in F_2 and G_2, F_2^L and G_2^L have no such term; therefore it corresponds to going from Lagrangian to Eulerian picture.

Notice that G_2 can be written in exactly the same form,

$$G_2 = \frac{\mu_2}{21} + \frac{1}{2}\hat{k}_1\cdot\hat{k}_2\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{4}{7}\left(\hat{k}_1^i \hat{k}_1^j - \frac{1}{3}\delta_{ij}\right)\left(\hat{k}_2^i \hat{k}_2^j - \frac{1}{3}\delta_{ij}\right)$$

Notice that velocities are much more sensitive to tidal fields, the tidal term is a factor of 2 larger.

Finally, note that $F_2(\mathbf{k},-\mathbf{k}) = G_2(\mathbf{k},-\mathbf{k}) = 0$, this is due to $\langle\delta\rangle = 0$, i.e. $\delta = \delta_1 + \delta_2 + \ldots$ and each term is a fluctuation, so $\langle\delta_2(\mathbf{k})\rangle = 0$. Indeed

$$\delta_2(\mathbf{k}) = \int \delta_D(\mathbf{k}-\mathbf{k}_{12})\, F_2(\mathbf{k}_1,\mathbf{k}_2)\, \overbrace{\langle \delta_1(\mathbf{k}_1)\, \delta_1(\mathbf{k}_2)\rangle}^{P(k_1)\,\delta_D(\mathbf{k}_1,\mathbf{k}_2)} = 0$$