Classification of metric perturbations, ~~[illegible]~~

[I follow MTW Ch. 21
Liddle & Lyth Ch. 14
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ADM Formalism

In the absence of perturbations, the universe is modeled as homogeneous and isotropic, and there is a preferred coordinate system where such symmetries are explicitly obvious, that is, we slice spacetime into a 3+1 decomposition, choosing the time axis to be such that at constant time e.g. the energy density and pressure are constant as we move on a spatial hypersurface; in other words ρ and p are only functions of time. It is obvious that one could do a change of coordinates such that ρ and p will not be just functions of time, by mixing t and $\vec{x}$ in the definition of t', this ~~[illegible]~~ unnecessary complication shows that what we call a density perturbation for example, is a coordinate dependent concept and that ~~one~~ must be careful in ~~[illegible]~~ automatically identifying e.g. density perturbations on a spatial hypersurface with physical perturbations, as some component of it (or all in the above example) could be just due to a "gauge mode" (i.e. reflecting our choice of coordinates), not a true physical perturbation.

Nevertheless to talk about perturbations and their evolution we need to do a 3+1 split of spacetime into time and spatial directions, the general framework for doing such a split is known as ADM formalism (due to Arnowitt, Deser and Misner 1962).

Consider ~~[illegible]~~ a general spatial hypersurface in spacetime described by the "position" functions X^μ ~~of~~ for the coordinates x^i, $X^\mu(x^i)$. ~~[illegible]~~

At any point in this hypersurface we have a basis formed by the 3 4-vectors tangent to the hypersurface

$$X_i^{\ \mu} \equiv X^\mu{}_{,i}$$

and the unit vector normal to the hypersurface ~~is~~ n^μ such that

$$g_{\mu\nu} X_i^\mu n^\nu = 0 \qquad\qquad g_{\mu\nu} n^\mu n^\nu = -1$$

$(n \perp X_i)$ $\qquad\qquad$ (n is unit vector)

Now let's deform this hypersurface in a continuous fashion, obtaining a family of hypersurfaces $X^\mu = X^\mu(x^i, t)$. Define the deformation 4-vector N^μ:

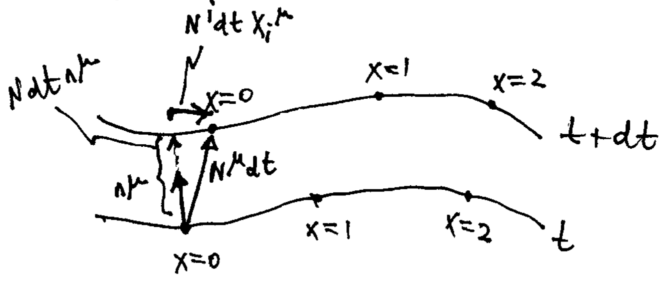
$$N^\mu \equiv \dot{X}^\mu = \frac{\partial X^\mu(x^i,t)}{\partial t}$$

By definition, this is the 4-vector that connects points with the same coordinates x^i in contiguous hypersurfaces. We can decompose this 4-vector in our basis:

$$N^\mu \equiv \underbrace{N}_{\text{"lapse function"}} n^\mu + \underbrace{N^i}_{\text{"shift function"}} X_i^\mu$$

This defines two objects whose geometric interpretation is as follows: $N\,dt$ measures the proper time between the two contiguous hypersurfaces, and $N^i\,dt$ measures the coordinate spatial distance ~~between~~ at the upper hypersurface ~~between the normal intersection~~ ~~of~~ to the point with same coordinates x^i (measured from the normal vector that passes through x^0 in the lower hypersurface).

Too many words! Easiest to draw it:



Now, to follow the dynamics of any field, we project it parallel and perpendicular to the hypersurface - the metric itself (what we are interested in) is no exception, but since we use it to define what we call normal direction, two of its projections are trivial

$$g_{\perp\perp} \equiv g_{\mu\nu} n^\mu n^\nu = -1 \qquad g_{\perp i} \equiv g_{\mu\nu} n^\mu X_i^\mu = 0$$

leaving only

$$g_{ij} \equiv \gamma_{ij} = g_{\mu\nu} X_i^\mu X_j^\nu$$

which defines the 3D metric γ_{ij} induced on the hypersurface -

The ADM decomposition of the metric follows from using the basis $\{N^\mu, X_i^\mu\}$ (since N^μ is in the "time" direction),

$$\begin{cases} g_{00} \equiv g_{\mu\nu} N^\mu N^\nu = \gamma_{ij} N^i N^j - N^2 \\ g_{0i} \equiv g_{\mu\nu} N^\mu X_i^\nu = N_i \\ g_{ij} \equiv g_{\mu\nu} X_i^\mu X_j^\mu = \gamma_{ij} \end{cases}$$

The interval then reads

$$ds^2 = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt)$$

and the volume element (check it!)

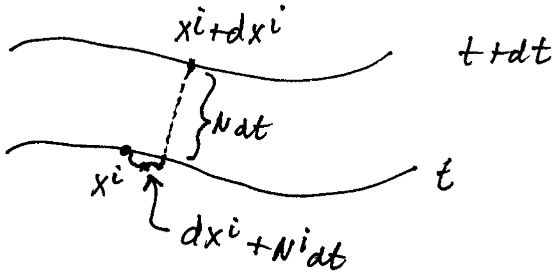
$$\sqrt{-g}\, d^4x = N \sqrt{\gamma}\, d^4x$$

The expression above for the interval is all we need, note that this procedure assumes nothing about Einstein equations, it is just a purely geometric procedure, thus it is valid for any metric (we will mostly work in linear perturbation theory though)

Another way of looking at the expression for the interval is that

$$ds^2 = \underbrace{-N^2\, dt^2}_{\substack{\text{proper time}\\ \text{from lower}\\ \text{to upper 3-geometry}}} + \underbrace{\delta_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt)}_{\substack{\text{proper distance in}\\ \text{lower 3-geometry}}}$$

pictorially:



Since $g_{\perp\perp} = -1$, $g_{\perp i} = 0$, and $g_{ij} = \delta_{ij}$, this picture gives interval above.

Scalar - Vector - Tensor (SVT) Decomposition

We will now consider a classification of perturbations to the metric away from flat Robertson Walker $(k=0)$ which reads

$$ds^2 = -dt^2 + a^2\,\delta_{ij}\, dx^i dx^j = a^2_{(\tau)}\left\{-d\tau^2 + \delta_{ij}\, dx^i dx^j\right\}$$ (unperturbed $k=0$ FRW)

where we introduced conformal time τ, such that $dt \equiv a(\tau)\, d\tau$. In the first form we see that in terms of ADM variables the unperturbed metric has

$$\begin{cases} N^{(0)} = 1 \\ N^{i\,(0)} = 0 \\ \gamma_{ij}^{(0)} = \delta^{ij} \end{cases}$$

For the perturbed metric we can write, without loss of generality,

$$ds^2 = a^2_{(\tau)}\left\{ -(1+2A)\, d\tau^2 + 2B_i\, d\tau\, dx^i + \left[(1+2D)\,\delta_{ij} + 2E_{ij}\right] dx^i dx^j \right.$$

where we have introduced two 3-scalar fields (A, D), one 3-vector

B_i and one symmetric traceless 3-tensor E_{ij}. Note that no generality has been lost by assuming E_{ij} to be traceless as any trace can be put into D. We kan see that this implies for the perturbed ADM components:

$$\left.\begin{aligned} \delta N &= A \\ \delta N^i &= -B^i a \end{aligned}\right\} \text{(to first order in metric perturbations)}$$

$$\delta\gamma_{ij} = 2D\,\delta_{ij} + 2E_{ij}$$

(I'm using t, not τ for this identification)

As we can see, the number of degrees of freedom still adds up to 10,

$$\left.\begin{array}{ll} \text{2 scalars} & (2) \\ \text{1 vector} & (3) \\ \text{1 traceless symmetric tensor} & (5) \end{array}\right\} \quad 2+3+5 = 10 \ \checkmark$$

An important property in cosmological settings is that even when density perturbations are large, metric perturbations remain small (i.e. we are never in the strong gravity field regime). Therefore we can linearize the LHS of Einstein equations, to obtain a set of linear PDE's. These can be decoupled by using normal modes under translation and rotation (because background is isotropic + homogeneous). The scalar, vector and tensor eigenmodes of the Laplacian operator form a complete set and correspond in the flat ($k=0$) case to the usual Fourier modes. The SVT decomposition is then a simple generalization of the standard decomposition in fluid mechanics of the velocity field into a divergence and vorticity component (the so-called Helmholtz theorem), that is

if $\vec\nabla\cdot\vec v = \theta$ and $\vec\nabla\times\vec v = \vec\omega$

$$\Rightarrow \quad \vec v = -\vec\nabla\phi + \vec\nabla\times\vec A$$

where
$$\begin{cases} \phi(\vec{r}) = \dfrac{1}{4\pi} \displaystyle\int \dfrac{\theta(\vec{r}\,')\, d^3r'}{|\vec{r}-\vec{r}\,'|} \\[2ex] \vec{A}(\vec{r}) = \dfrac{1}{4\pi} \displaystyle\int \dfrac{\vec{\omega}(\vec{r}\,')\, d^3r'}{|\vec{r}-\vec{r}\,'|} \end{cases}$$

Things are ~~much~~ easier in Fourier space as

$$\vec{\nabla}\cdot\vec{v} = -\nabla^2\phi = \theta$$

$$\vec{\nabla}\times\vec{v} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) \underset{\uparrow \atop \vec{\nabla}(\vec{\nabla}\cdot\vec{A})=0}{=} -\nabla^2\vec{A} = \vec{\omega}$$

Then in Fourier space

$$\vec{v}(\vec{k}) = \frac{i\vec{k}\,\theta(\vec{k})}{k^2} + i\,\frac{\vec{k}\times\vec{\omega}(\vec{k})}{k^2} \qquad \vec{k}\cdot\vec{\omega} = 0$$

Thus the 3 components ~~of~~ $\vec{v}$ have been split into a scalar piece (θ) and a transverse vector $(\vec{\omega})$. In ~~a~~ a slightly different notation (more relevant to cosmology) we have:

$$v_i = \underbrace{\frac{k_i}{k}\phi}_{\text{scalar}} + \underbrace{v_i^V}_{\text{vector}}$$

where v_i^V satisfies $k_i v_i^V = 0$

Similarly, we can do it for a tensor. We decompose into a scalar part, a vector and an intrinsic tensorial piece:

$$T_{ij} = \underbrace{\left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij}\right) T}_{T_{ij}^S \ (\text{scalar})} + \underbrace{\frac{k_i T_j^V + k_j T_i^V}{2k}}_{T_{ij}^V \ (\text{vector})} + T_{ij}^T$$

with the conditions:

$$\begin{cases} k_i T_i^V = 0 \\ k_i T_{ij}^T = 0 \end{cases}$$

Notice then that for a vector, 3 components are split into

1 scalar degree of freedom (DOF)

and 2 vector DOF

For a tensor, symmetric, its 6 components are split into

1 scalar (DOF)

2 vector DOF

3 tensor DOF

In the case we are interested in for the metric, one of the tensor DOF is removed by the condition that its trace vanishes.

Apart from the decoupling mentioned above (only true in linear theory), the SVT decomposition is useful beyond this, as different terms have different physical interpretations. The scalar piece is the familiar Newtonian-like gravity, and it's sourced by energy density fluctuations. The vector piece is something new, not present in Newtonian gravity, called gravitomagnetism, and it would be sourced by vorticity perturbations. Finally, the tensor piece is gravitational radiation also absent in Newtonian case. We will see next week that both scalar and tensor modes are generically produced during inflation.

To sumarize, in terms of SVT decomposition we have

2 scalars	(A, D)	$\Rightarrow$	2 S dof's
1 vector	$\vec{B}$	$\Rightarrow$	1 S + 2V dof's
1 tensor (traceless)	E_{ij}	$\Rightarrow$	1 S + 2V + 2T dof's
			4 S + 4V + 2T dof's

This totals 10 degrees of freedom as it should. However, remember ~~xxxxx~~ that due to general covariance ~~there~~ there are 4 out of these 10 which can be set by choosing coordinates, or choosing a "gauge".

We can apply the SVT decomposition to the gauge transformation itself, that is

$$x'^{\mu} = x^{\mu} + \xi^{\mu}$$

We have a 4-vector that we can split into 1 scalar ξ^0 and 1 3-vector which in turn can be split into a scalar and 2 vector dof's, i.e.

$$\xi^0 \Rightarrow 1S \quad \text{dof}$$

$$\xi^i \Rightarrow 1S + 2V \quad \text{dof}$$

In the linear approximation this means we can set 2S dof's by choosing a gauge, and similarly 2V dof's. In other words,

out of 4S dof's ⇒ only 2 are physical
" " 4V " ⇒ " " " "

And it follows that the 2T dof are physical, and in other words, gauge invariant. One can form 2S and 2V dof's which are gauge invariant as well by looking for the 2 linear combinations ~~[illegible]~~ that are unaffected by GT, this leads to the Gauge-invariant formalism (which we won't use).

The choice of the 2S and 2V dof's that sets the gauge will lead to different equations for the perturbations, as we shall see next class.