Relativistic Perturbation Theory, Gauges

[I follow Bertschinger astro-ph/9503125 4.3 to 4.6]

Stress-Energy Tensor perturbations and SVT decomposition

Last class we discussed the metric perturbations and how they can be decomposed into their scalar (S), vector (V) and tensor (T) components. In order to complete the picture, we need to do the same for the stress tensor and the Einstein tensor, and then write the Einstein equations for perturbations.

Let us consider a fluid (next class we will extend this to a scalar field), for which

$$T^{\mu\nu} = (\rho+p)\, u^\mu u^\nu + p\, g^{\mu\nu} + \Sigma^{\mu\nu} \qquad \left(\begin{array}{l} \Sigma^\mu{}_\mu = 0 \\ \Sigma^\mu{}_\nu u^\nu = 0 \end{array} \right)$$

In order to write down perturbations of $T^{\mu\nu}$, therefore, we need to consider perturbations of ρ, p, u^μ and $\Sigma^{\mu\nu}$, apart from metric perturbations that we already considered. The velocity perturbations is the only non trivial piece because it involves metric perturbations due to the normalization condition $u^\mu u_\mu = -1$.

We have by definition,

$$u^\mu = \underset{\substack{\uparrow \\ \text{proper time}}}{\frac{dx^\mu}{ds}} = \underset{\substack{\uparrow \\ \text{conformal time}}}{\frac{dx^\mu}{d\tau}} \frac{d\tau}{ds}$$

In our coordinate system $x^0 = \tau$, thus we have

$$\frac{dx^\mu}{d\tau} = \left(1, \frac{dx^i}{d\tau}\right)$$

Now, you may recall from previous course one defines a peculiar velocity $\vec{v}$ perturbation by looking at deviations from Hubble flow: ②

total physical velocity: $\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(a\vec{x}) = \dot{a}\vec{x} + a\frac{d\vec{x}}{dt} = \mathcal{H}\vec{x} + \frac{d\vec{x}}{d\tau} =$

$$= \underbrace{\mathcal{H}\vec{x}}_{H\vec{r}} + \vec{v}$$

Hubble flow

peculiar velocity (perturbation)

where we introduced the conformal expansion rate $\mathcal{H} = \frac{d\ln a}{d\tau} = Ha$, with $H = \frac{d\ln a}{dt}$ the Hubble constant. Then we have

$$\frac{dx^\mu}{d\tau} = (1, \vec{v})$$

with $\vec{v}$ the "usual" 3-velocity. Now we also have

$$\frac{ds}{d\tau} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = \sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^iv^j}$$

which due to the form of our metric,

$$ds^2 = a^2(\tau)\left\{-(1+2A)d\tau^2 - 2B_i\, d\tau\, dx^i + \left[(1+2D)\delta_{ij} + 2E_{ij}\right]dx^i dx^j\right\}$$

leads to

$$\frac{ds}{d\tau} = a(\tau)\sqrt{(1+2A) + 2B_i v^i - (1+2D)v^2 - 2E_{ij}v^iv^j}$$

which gives

$$u^\mu = \frac{dx^\mu}{ds} = \frac{(1, \vec{v})}{a\sqrt{(1+2A) + 2B_i v^i - (1+2D)v^2 - 2E_{ij}v^iv^j}}$$

We see from here that in the absence of metric perturbations,

$$u^\mu \to \frac{(1, \vec{v})}{a\sqrt{1-v^2}}$$

which is the standard 4-velocity in special relativity, apart from

the fact that we are using conformal time to take care of ③
the expansion of the universe (thus the a factor extra). In the presence of metric perturbations the components get changed as the measure of proper distances and times get affected by perturbations.

Now, we are going to work in the limit of small metric perturbations and we will also assume velocities are much smaller than the speed of light, $v \ll 1$, thus we can linearize in v and metric perturbations. Linearizing, we have

$$u^\mu \simeq \frac{1}{a}(1, \vec{v})(1-A) \simeq \frac{1}{a}(1-A, \vec{v})$$

we will also need the components

$$\boxed{u_0 = g_{00}u^0 + g_{0i}u^i \simeq -a(1+A)}$$

$$\uparrow$$

$$-a^2(1+2A)\frac{(1-A)}{a}$$

$$-a^2 B_i v^i/a$$

$$\boxed{u_i = g_{i0}u^0 + g_{ij}u^j} =$$

$$= -a^2 B_i \frac{(1-A)}{a} + a^2 \frac{v^j}{a}\left[(1+2D)\delta_{ij} + 2E_{ij}\right]$$

$$\simeq \boxed{a(v_i - B_i)}$$

Now we are ready to write down the full stress energy with the only assumption of $v/c \ll 1$ and small metric perturbations, we have

$$T^0{}_0 = -\rho$$

$$T^i{}_0 = -(\rho+p)v^i$$

$$T^0{}_i = (\rho+p)(v_i - B_i)$$

$$T^i{}_j = p\,\delta^i{}_j + \Sigma^i{}_j$$

and we have used mixed components for simplicity (thus avoiding extra factors due to metric perturbations). Note that we have not yet decomposed ρ and p into background + perturbations. Both v^i and Σ^i_j are perturbations already. The only assumptions are $v/c \ll 1$ and small metric perturbations (also v, Σ times metric perturbations are ignored). Using these approximations we can write down conservation of stress energy (∇ are gradients using 3D metric)

$$\begin{cases} \dfrac{\partial \rho}{\partial \tau} + 3(\mathcal{H}+\dot{D})(\rho+p) + \vec{\nabla}\cdot\left[(\rho+p)\vec{v}\right] = 0 \\[2ex] \dfrac{\partial}{\partial \tau}\left[(\rho+p)(\vec{v}-\vec{B})\right] + 4\mathcal{H}(\rho+p)(\vec{v}-\vec{B}) = -\vec{\nabla}p - \vec{\nabla}\cdot\overleftrightarrow{\Sigma} - (\rho+p)\vec{\nabla}A \end{cases}$$

Note that ρ is energy density here, and p in $(\rho+p)$ appears because the change in energy density is also affected by ~~expansion~~ $p\,dV$ work due to expansion for a relativistic fluid in which p is comparable to ρ. Also notice in expansion term $\mathcal{H}+\dot{D}$ appear, that's because the perturbed scale factor is $\simeq a(1+D)$ in the spatial part (E_{ij} has no trace), thus the effective conformal Hubble is $\frac{\partial \ln a(1+D)}{\partial \tau} \simeq \mathcal{H} + \frac{\partial D}{\partial \tau}$

The appearence of $\vec{v}-\vec{B}$ in the momentum conservation equation is due to the fact this is the velocity in a frame where the shift vector is zero, in other words, $\vec{v}-\vec{B}$ takes care of the fact that some $\frac{d\vec{x}}{d\tau}$ is due to the shift vector. The rest of the terms are standard from Newtonian conservation, note $\vec{\nabla}A$ is the gradient of the gravitational potential.

The standard procedure would be now to linearize $T^{\mu\nu}$, i.e assume density and pressure ~~have~~ small perturbations compared to their background values, and similarly for $\vec{v}$ and $\overleftrightarrow{\Sigma}$. One then does the SVT decomposition for $T^{\mu\nu}$ which is equivalent to decompose $\vec{v}$ into S and V, and $\overleftrightarrow{\Sigma}$ into S, V and T as we did last class. Before we say more about this, let us consider again Gauge transformations.

Gauge transformations (again)

Let's consider how perturbations transform under a gauge transformation,

$$\tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}(x^{\nu})$$

Consider the metric first. We want $\tilde{g}_{\mu\nu}(x)$:

$$\tilde{g}_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(\tilde{x} - \xi) = \tilde{g}_{\mu\nu}(\tilde{x}) - \tilde{g}_{\mu\nu,\tilde{\lambda}}\, \xi^{\lambda} + \mathcal{O}(\xi^2)$$

where $_{,\tilde{\lambda}} \equiv \frac{\partial}{\partial \tilde{x}^{\lambda}}$, which we can set to $\frac{\partial}{\partial x^{\lambda}}$ working to 1st order in PT.

But: $$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} g_{\alpha\beta}(x) = (\delta^{\alpha}_{\mu} - \xi^{\alpha}_{,\tilde{\mu}})(\delta^{\beta}_{\nu} - \xi^{\beta}_{,\tilde{\nu}})\, g_{\alpha\beta}(x)$$

$$\Rightarrow \tilde{g}_{\mu\nu}(x) \underset{\uparrow}{\simeq} g_{\mu\nu}(x) - \xi^{\alpha}_{,\mu} g_{\alpha\nu} - \xi^{\alpha}_{,\nu} g_{\mu\alpha} - g_{\mu\nu,\lambda}\, \xi^{\lambda}$$

to first order in PT

Now, we decompose our gauge transformation into SVT:

$$\begin{cases} \xi^0 \equiv \alpha(x) & (1S) \\ \xi^i = \nabla^i \beta + \epsilon^i \quad \text{with } \vec{\nabla}\cdot\vec{\epsilon} = 0, & (1S + 2V) \end{cases}$$

Then we have

here $\dot{} \equiv \frac{d}{d\tau}$

$$\begin{cases} \tilde{A} = A - \dot{\alpha} - \mathcal{H}\alpha \\ \tilde{B}_i = B_i - \nabla_i(\alpha - \dot{\beta}) - \dot{\epsilon}_i \\ \tilde{D} = D - \frac{1}{3}\nabla^2\beta - \mathcal{H}\alpha \\ \tilde{E}_{ij} = E_{ij} - \mathcal{D}_{ij}\beta - \frac{1}{2}(\nabla_i\epsilon_j + \nabla_j\epsilon_i) \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{B}^S_i = B^S_i - \nabla_i(\alpha - \dot{\beta}) \\ \tilde{B}^V_i = B^V_i - \dot{\epsilon}_i \end{cases}$$

$$\mathcal{D}_{ij} \equiv \nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2$$

$$\Downarrow$$

$$\tilde{E}^S_{ij} = E^S_{ij} - \mathcal{D}_{ij}\beta$$

$$\tilde{E}^V_{ij} = E^V_{ij} - \frac{1}{2}(\nabla_i\epsilon_j + \nabla_j\epsilon_i)$$

and $\tilde{E}^T_{ij} = E^T_{ij}$ (gauge invariant)

One can similarly do this for $T_{\mu\nu}$ rather than $g_{\mu\nu}$, and get the transformation laws for $\delta\rho, v, \Sigma$, etc.
let me assume $p=0=\Sigma$ for simplicity - Then:

$$\tilde{\rho}(x) = \rho(x) + \xi^\lambda \rho_{,\lambda} + \mathcal{O}(\xi^2)$$

$$\Rightarrow \quad \tilde{\delta} = \delta + 3\mathcal{H}\alpha \qquad \left[\begin{array}{c} \text{if we add pressure this just becomes} \\ \tilde{\delta} = \delta + 3\mathcal{H}\alpha\,(1+w) \end{array} \right]$$

and for the velocity, $\nabla^i \beta_{,0}$

$$\tilde{v}^i = v^i + \xi^i{}_{,0} \quad \Rightarrow \quad \begin{cases} \tilde{v}^{i\,S} = v^{i\,S} + \beta^i{}_{,0} \\ \tilde{v}^{i\,V} = v^{i\,V} + \epsilon^i{}_{,0} \end{cases}$$

Let us now consider some gauges. First notice we have 2S and 2V dof at our disposal to play with. In the <u>synchronous gauge</u>, one sets

$$\begin{cases} g_{00} \equiv -1 & 1S \\ g_{0i} = 0 & 1S + 2V \end{cases}$$

then $A=0$, $B_i^S = B_i^V = 0$

The synchronous gauge has the property that there is a set of freely falling observers that have constant spatial coordinates, that is, $u^i = 0$. This follows from the geodesic equation

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0$$

for $d\lambda = \sqrt{-ds^2}$ and $u^\mu = \frac{dx^\mu}{d\lambda}$. when $A=0$, $B_i=0$, it follows that $\Gamma^i_{00} = 0 \Rightarrow u^i = 0$ is a geodesic. In other words, each observer carries a clock reading conformal time τ and a fixed spatial coordinate label x^i. Obviously the slicing is perpendicular to the geodesics so the shift is zero ($B_i = 0$)

The synchronous gauge has some peculiarities. First, it is not just a gauge but a family of gauges: one can do coordinate transformations with $\beta = \beta_0(\vec{x}) \int \frac{dt}{a(t)}$ and $\epsilon = \epsilon(\vec{x})$ and still satisfy the gauge conditions. This arises from the freedom to adjust the initial setting of clocks and coordinate labels of the observers following geodesics. Also, because it is a set of Lagrangian coordinates (with x^i fixed to observers), when obs. trajectories cross coordinates become singular. This only happens when density fluctuations are "large", i.e. $\delta \to 1$, so it is not a concern when dealing with early universe.

The synchronous gauge is good for numerical solutions because of its stability (eg. CMBfast uses it), though the metric perturbations are a bit hard to interpret since there is no analog of Newtonian potential. One can e.g. solve in this gauge and then transform to another gauge for easier interpretation if desired.

The Poisson gauge corresponds to imposing the gauge conditions,

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 & (1S) \\ \vec{\nabla} \cdot \overleftrightarrow{E} = 0 & (1S + 2V) \end{cases}$$

Thus in this case one eliminates the scalar part of $\vec{B}$, and the scalar and vector parts of E_{ij}, again, for a total of 2S and 2V conditions. Thus the only degrees of freedom left are

$$\begin{cases} A, D & (2S) \\ B_i^V & (2V) \\ E_{ij}^T & (2T) \end{cases}$$

In the linear approximation S, V and T perturbations evolve independently, and the scalar part of the metric is thus gauge is often called the conformal Newtonian gauge.

The Einstein equations in the Poisson gauge are

$$G^0_0 : \quad \nabla^2 D \pm 3\mathcal{H} (\dot{D} \pm \mathcal{H} A) = -4\pi G a^2 \bar{\rho}\, \delta$$

$$(G^0_i)^S : \quad \nabla_i (\dot{D} - \mathcal{H} A) = 4\pi G a^2 \left[(\rho + p)(v_i - B^V_i) \right]^S$$

$$(G^0_i)^V : \quad \nabla^2 B^V_i = -16\pi G a^2 \left[(\rho + p)(v_i - B^V_i) \right]^V$$

$$G^i_i : \quad -\ddot{D} + \mathcal{H}(\dot{A} - 2\dot{D}) + (2\dot{\mathcal{H}} + \mathcal{H}^2) A + \frac{1}{3} \nabla^2 (A + D) = 4\pi G a^2 (p - \bar{p})$$

$$(G^i_{j\neq i})^S : \quad \mathcal{D}_{ij} (D + A) = -8\pi G a^2 \Sigma^S_{ij}$$

$$(G^i_j)^V : \quad -(\partial_\tau + 2\mathcal{H}) \nabla_i B^V_j = 8\pi G a^2 \Sigma^V_{ij}$$

$$(G^i_j)^T : \quad (\partial^2_\tau + 2\mathcal{H}\partial_\tau - \nabla^2) E^T_{ij} = 8\pi G a^2 \Sigma^T_{ij}$$

Of course, not all of these equations are independent, they are related by energy-momentum conservation. One can take either the first 3 or the second 3 (or some linear combination) plus the last one (for tensor modes).

What is interesting about the Poisson gauge is that one can obtain the scalar and vector potentials directly from the instantaneous stress-energy distribution with no time integration required, i.e matter and metric perturbations are algebraically related (except for the tensor modes). We can take these 3 equations to be

$$\nabla^2 B^V_i = -16\pi G a^2 \left[(\rho + p)(v_i - B^V_i) \right]^V$$

$$\mathcal{D}_{ij} (D + A) = -8\pi G a^2 \Sigma^S_{ij}$$

and combining the first two above we have

$$\nabla^2 D = -4\pi G a^2 \left[\bar{\rho}\,\delta + 3\mathcal{H} \hat{\Phi} \right] \quad \text{with} \quad -\nabla \hat{\Phi} \equiv \left[(\rho + p)(v_i - B^V_i) \right]^S$$

We can now take a look at the Newtonian limit from here. First, if there is no expansion $\mathcal{H}=0$ and $\Sigma=0$ the equation for D can be written as Poisson equation

$$\nabla^2 A = 4\pi G\, a^2\, \delta\rho$$

When there is expansion, there is an additional source to energy density, that is $\hat{\Phi}$ which is the scalar part of momentum density. We can estimate the relativistic correction by

$$\mathcal{H}\hat{\Phi} \sim \frac{\mathcal{H}}{k}\bar{\rho}\, v$$

$$\Rightarrow \quad \frac{3\mathcal{H}\hat{\Phi}}{\bar{\rho}\,\delta} \sim 3\,\frac{\mathcal{H}}{k}\,\frac{1}{\delta}\, v \sim 3\left(\frac{k_H}{k}\right)\frac{1}{\delta}\,\frac{v}{c}$$

$\mathcal{O}(1)$

where we have defined the Horizon wavevector $k_H \equiv \frac{2\pi}{c(\mathcal{H}a)^{-1}} \overset{a=1\ \text{(today)}}{=} \frac{2\pi}{3000}\,\frac{h}{\text{Mpc}}$

Thus the correction only becomes important at distances comparable to the horizon. Similarly, stresses are typically small, then $\Sigma^{S} \sim \left(\frac{c_s}{c}\right)^2$ with c_s the characteristic thermal speed of particles, thus $A \sim -D$ to order $(v/c)^2$. Also, B_i^V is negligible due to no vorticity.

The physical interpretation in this gauge is simple since we have the Newtonian potential + small corrections, and $D \sim -A$ gives the perturbative (small) corrections to the local scale factor. One inconvenience of this gauge is that when relativistic corrections become important at horizon scales the extra term in the Poisson equation leads to numerical instabilities.