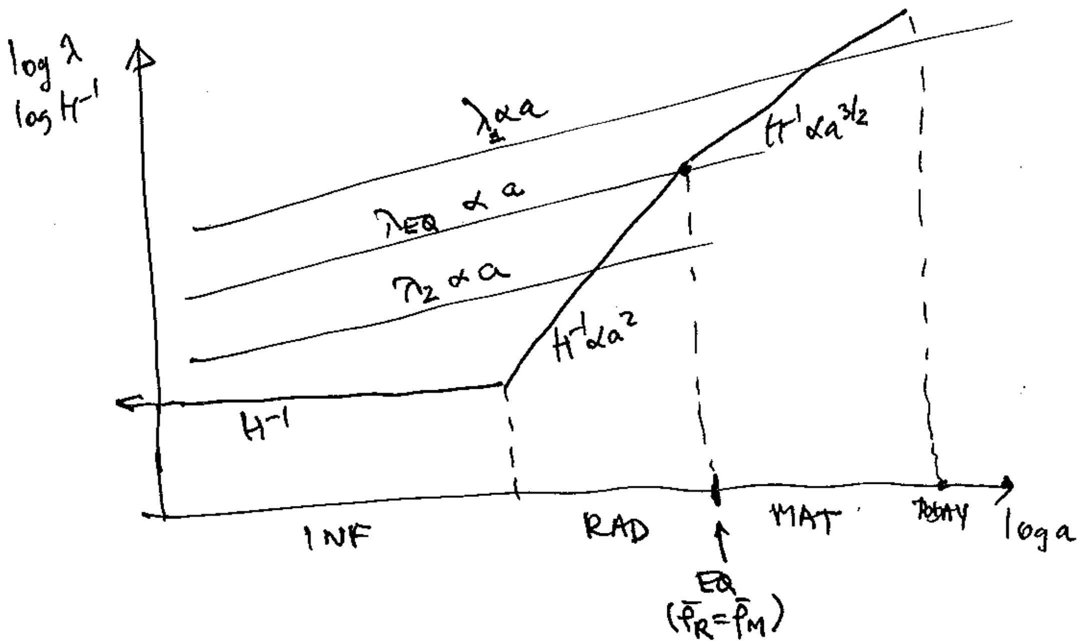
The density fluctuation spectrum @ late times

We now follow the matter power spectrum from inflationary induced curvature perturbations at super-Hubble scales down to sub-Hubble scales at present times.

Remember the basic picture for fluctuation scales compared to H^{-1}:



We show different characteristic perturbations:

i) λ_1 enters H^{-1} while universe is MAT

ii) $\lambda_2 \ll \lambda_1$ enters H^{-1} while RAD

iii) the "boundary" case is wavelength λ_{EQ} which crosses H^{-1} at equality $\bar{\rho}_R = \bar{\rho}_M$.

We derived already that at super-Hubble scales ($\lambda \gg H^{-1}$) the curvature perturbation ξ is conserved and proportional to the newtonian potential Φ:

$$\lambda \gg H^{-1}:\ \xi' = 0 \qquad \text{and} \qquad -\xi = \frac{5+3w}{3+2w}\,\Phi \qquad \left(\begin{array}{c}\text{no aniso stress}\\ w = \text{const.}\end{array}\right)$$

We now must derive what is the behavior of e.g. Φ for $\lambda \ll H^{-1}$ and for that we need to consider separately the case where we are inside H^{-1} when RAD or MAT.

Now:
$$\begin{cases} 1+z_{eq} = 2.396 \times 10^4\, \Omega_m^0 h^2 = 3258\, \dfrac{\Omega_m^0 h^2}{0.136} \\ \left(\dfrac{H_{eq}}{H_0}\right)^2 = \dfrac{2\Omega_m^{(0)}}{a_{EQ}^3} \end{cases} \Rightarrow \quad k_{EQ} = a_{EQ} H(a_{EQ}) = H_0 \sqrt{\frac{2\Omega_m^{(0)}}{a_{EQ}}} = 0.014\, \frac{\Omega_m^0 h^2}{0.136}\, \frac{0.7}{h}\, \frac{h}{\mathrm{Mpc}}$$

As we discuss below Φ stays constant for modes $k \ll k_{EQ}$ that become sub-Hubble during MAT era (this follows from the fact that $\delta_M \propto a$ when $\Omega_m = 1$ and the Poisson eqn: $\nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \delta \sim \frac{1}{a} a = \text{const.}$), while Φ decays as $\frac{\ln a}{a^2}$ for $k \gg k_{EQ}$ modes that become sub-Hubble during RAD era.

Therefore $k \gg k_{EQ}$ modes get suppressed with respect to $k \ll k_{EQ}$ modes by the factor

$$\ln\left[\frac{a_{EQ}}{a_e(k)}\right] \left(\frac{a_e(k)}{a_{EQ}}\right)^2 \qquad < 1$$

where $a_e(k)$ is the scale factor at the time the mode k crosses Hubble, during RAD era:

$$\lambda_{comov}\, a_e(k) = \frac{2\pi}{k} a_e(k) = H^{-1}[a_e(k)] \propto t_e \underset{\substack{\uparrow \\ a \propto t^{1/2} \text{ during RAD}}}{\propto} a_e^2(k)$$

$$\Rightarrow \quad a_e(k) \propto \frac{1}{k}$$

$$\Rightarrow \quad \frac{a_e(k)}{a_{EQ}} = \frac{k_{EQ}}{k}$$

Then the suppression factor is $\ln\left(\frac{k}{k_{EQ}}\right) \left(\frac{k_{EQ}}{k}\right)^2 \qquad (k \gg k_{EQ})$

So for $k \gg k_{EQ}$ we have at late times (when all modes are inside Hubble during MAT)

$$P_\Phi^{late}(k) \propto P_\Phi(k) \left[\ln\left(\frac{k}{k_{EQ}}\right)\left(\frac{k_{EQ}}{k}\right)^2\right]^2 \qquad (k \gg k_{EQ})$$

or, for matter fluctuations, (for $k \gg \mathcal{H}$ $k^2 \Phi \sim \delta$)

$$P_\delta^{late}(k) = \begin{cases} A\, k^{n_s} & k \ll k_{EQ} \\ B\, k^{n_s - 4} \ln^2\left(\frac{k}{k_{EQ}}\right) & k \gg k_{EQ} \end{cases}$$

It is customary to define a <u>transfer function</u> $T(k)$

$$\Phi^{late}(k) = -\frac{3}{5}\,\zeta(k)\,T(k) \quad \Rightarrow \quad \delta(k) = -\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2 \frac{1}{\Omega_m}\,\Phi^{late}(k)$$

$$\Phi(k) = -\left(\frac{5+3w}{3+3w}\right)^{-1}\zeta(k) \quad \text{at } k \ll \mathcal{H}$$

$$= \frac{2}{5}\left(\frac{k}{\mathcal{H}}\right)^2 \frac{1}{\Omega_m}\,T(k)\,\zeta(k)$$

$$\Rightarrow \quad P_\delta(k) = \frac{4}{25}\left(\frac{k}{\mathcal{H}}\right)^4 \frac{1}{\Omega_m^2}\,T^2(k)\,\overset{A_s k^{n_s-4}}{\overset{\shortparallel}{P_\zeta(k)}}$$

or $$\boxed{P_\delta(k) \equiv A\,T^2(k)\,k^{n_s}}$$

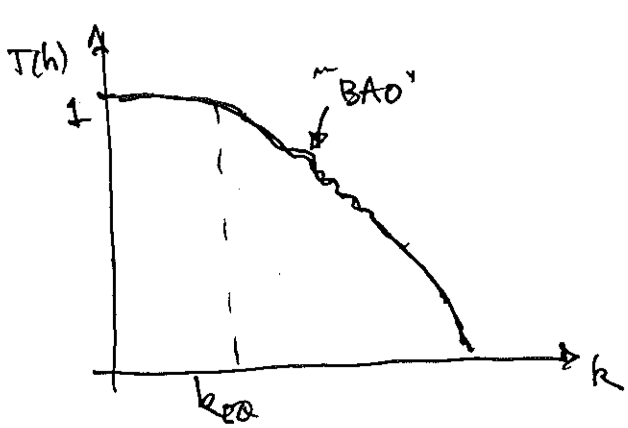
$$T(k) = \begin{cases} 1 & k \ll k_{eq} \\ \ln\left(\frac{k}{k_{eq}}\right)\left(\frac{k_{eq}}{k}\right)^2 & k \gg k_{eq} \end{cases}$$

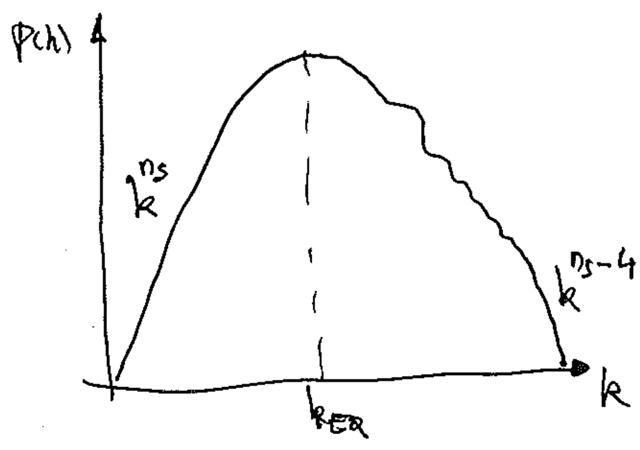
For example, a classic fit for $T(k)$ is given by the (BBKS) form:

$$T(q) = \frac{\ln(1+2.34q)}{2.34q}\left[1+3.89q+(16.1q)^2+(5.46q)^3+(6.71q)^4\right]^{-1/4}$$

where $q = \frac{k}{\Gamma h}$ $\quad \Gamma \equiv \Omega_m h \; e^{-\Omega_b - \sqrt{2h}\,\Omega_b/\Omega_m} \approx \Omega_m h$

In practice $T(k)$ can only be calculated precisely by using Boltzmann solvers for radiation (photons + neutrinos), dark matter and baryons (using numerical codes). The fitting above ignores acoustic oscillations, which are very important. So in practice $T(k)$ has oscillations that the dark matter inherits from the baryon-photon fluid:





Matter and radiation ($\lambda \ll H^{-1}$)

Consider the two-fluid system with non-relativistic cold dark matter and photons
($w = c_s^2 = 0$) ($w = c_s^2 = 1/3$)

At sub-Hubble scales we have the Poisson eqn.

$$-k^2 \Phi = 4\pi G a^2 (\rho_m \delta_m + \rho_r \delta_r) = \frac{3}{2} \mathcal{H}^2 (\Omega_m \delta_m + \Omega_r \delta_r)$$

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 (\rho_m + \rho_r)$$

Therefore for matter we have ($\lambda \ll H^{-1}$)

during RAD $\Omega_m \simeq 0$, $\Omega_r \simeq 1$

$$\frac{\partial^2 \delta_m}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_m}{\partial \tau} = \frac{3}{2} \mathcal{H}^2 (\Omega_m \delta_m + \Omega_r \delta_r) \simeq \frac{3}{2} \mathcal{H}^2 \Omega_r \delta_r$$

but radiation undergoes acoustic oscillations,

$$\frac{\partial^2 \delta_r}{\partial \tau^2} + \frac{k^2}{3} \delta_r = 0 \quad \Rightarrow \quad \langle \delta_r \rangle \simeq 0$$

averaged over oscillations

$$\Rightarrow \quad \delta_m'' + \mathcal{H} \delta_m' \simeq 0$$

$$\Rightarrow \quad \delta_m = C_1 + C_2 \int \frac{d\tau}{a}$$

during RAD $a \propto t^{1/2}$ and $a\, d\tau = dt$ $\Rightarrow$ $a \propto \tau$

$$\Rightarrow \quad \delta_m = c_1 + c_2 \ln a$$

so matter fluctuations only grow logarithmically.

CD Matter and baryons ($\lambda \ll H^{-1}$)

$\Omega_c \delta_c \gg \Omega_b \delta_b$

$$\delta_c'' + \mathcal{H} \delta_c' = \frac{3}{2} \mathcal{H}^2 (\Omega_c \delta_c + \Omega_b \delta_b) \simeq \frac{3}{2} \mathcal{H}^2 \Omega_c \delta_c$$

(same eqn as for single fluid)

$$\delta_b'' + \mathcal{H} \delta_b' + k^2 c_s^2 \delta_b = \frac{3}{2} \mathcal{H}^2 (\Omega_c \delta_c + \Omega_b \delta_b) \simeq \frac{3}{2} \mathcal{H}^2 \Omega_c \delta_c$$

$\Rightarrow$ baryons fall into potential wells of CDM

after DEC

let's work through this; we assume that $\delta_c \propto a\, \delta_0$, the growing mode in the $\Omega_m = 1$ case (which is a reasonable approx. though not great for $z \approx z_{DEC}$, recall $z_{EQ} \sim 3000$, $z_{DEC} \sim 10^3$). For the baryons pressure we have

$$\nabla^2 \frac{\delta p}{\bar{\rho}} \rightarrow c_s^2 (-k^2)\, \delta_b$$

After decoupling the sound speed for baryons drops significantly from $c_s^2 = \frac{1}{3(1+R)}$ to that appropriate for a monoatomic ideal gas (hydrogen) with adiabatic index $\gamma = 5/3$, e.g. $\rho_b = n m_b + \frac{3}{2} n kT$ and $p = nkT$ $\Rightarrow$ $c_s^2 = \frac{5}{3} \frac{k_B T}{m_b} \propto \frac{1}{a}$ for $z \lesssim z_{DEC}$ ($T_b \propto \frac{1}{a^2}$ eventually at $z \sim 100$)

Now assume $\Omega_m = 1$, so $a \propto \tau^2$ and $\mathcal{H} = \frac{2}{\tau}$

$$\Rightarrow \quad \tau^2 \delta_b'' + \underbrace{\tau^2 \mathcal{H}}_{2\tau} \delta_b' + \underbrace{\tau^2 k^2 c_s^2}_{\sim \tau^2 \frac{1}{\tau^2}} \delta_b = \frac{3}{2} \underbrace{\mathcal{H}^2 \tau^2}_{\frac{4}{\tau^2}\tau^2} \delta_c$$

look at homogeneous solution first. Its of the form $\tau^m \frac{d^m}{d\tau^m} \delta_b$ for each term so it admits power-law solutions τ^n:

$$n(n-1) + 2n + \frac{5}{3} \frac{k_B T_*}{m_b} (k \tau_*)^2 = 0$$

$T_* = T_b(\tau^*)$
k_B = Boltzman
$* \simeq$ dec, rec

we can rewrite last term introducing the Jeans wave number k_J:

$$k_J^2 c_s^2 \equiv \frac{3}{2} \mathcal{H}^2 \quad \Rightarrow \quad k_J^2 = \frac{3 \mathcal{H}^2}{2 c_s^2} = \frac{6}{\tau^2} \frac{1}{c_s^2}$$

$$\Rightarrow \quad \frac{5}{3} \frac{k_B T_*}{m_b} (k \tau_*)^2 = 6 \frac{k^2}{k_J^2}$$

[if $\Omega_b = 1$, no DM $\Rightarrow$ δ_b terms in eqs. of motion are $c_s^2 (k_J^2 - k^2)\, \delta_b$, which gives competition between pressure and gravity.

$$\Rightarrow \quad n^2 + n + 6 (k/k_J)^2 = 0$$

$$\Rightarrow \quad n = \frac{-1 \pm \sqrt{1 - 24 (k/k_J)^2}}{2}$$

So at most the homogenous solution is (roughly) constant without forcing term due to CDM, as expected since baryon self-gravity is pretty small (when $\rho_b \ll \rho_c$). The interesting part then is the particular solution,

$$\underbrace{\delta_b(\vec{x})}_{\text{ansatz}} = A \underbrace{\delta_c^*(\vec{x})}_{\text{DM}_0} \underbrace{(\tau/\tau_*)^2}_{\substack{\text{DM growth} \\ (a \propto \tau^2)}} \quad \Rightarrow \quad \left(2 + 4 + 6\frac{k^2}{k_J^2}\right) A = 6$$

$$\Rightarrow \quad A = \frac{1}{1 + k^2/k_J^2}$$

$$\Rightarrow \quad \delta_b(k) = \frac{\delta_c^* (\tau/\tau_*)^2}{1 + k^2/k_J^2} + \delta_b^{hom}(k) \simeq \frac{\delta_c(k,\tau)}{1 + k^2/k_J^2}$$

So at large scales ($k \ll k_J$) the baryons follow the dark matter.