Resource Letter GPP-1: Geometric Phases in Physics

Jeeva Anandan, Joy Christian, and Kazimir Wanelik

Citation: American Journal of Physics **65**, 180 (1997); doi: 10.1119/1.18570
View online: https://doi.org/10.1119/1.18570
View Table of Contents: http://aapt.scitation.org/toc/ajp/65/3
Published by the American Association of Physics Teachers

Roger H. Stuewer, *Editor*
School of Physics and Astronomy, 116 Church Street
University of Minnesota, Minneapolis, Minnesota 55455

This is one of a series of Resource Letters on different topics intended to guide college physicists, astronomers, and other scientists to some of the literature and other teaching aids that may help improve course content in specified fields. [The letter E after an item indicates elementary level or material of general interest to persons becoming informed in the field. The letter I, for intermediate level, indicates material of somewhat more specialized nature; and the letter A, indicates rather specialized or advanced material.] No Resource letter is meant to be exhaustive and complete; in time there may be more than one letter on some of the main subjects of interest. Comments on these materials as well as suggestions for future topics will be welcomed. Please send such communications to Professor Roger H. Stuewer, Editor, AAPT Resource Letters, School of Physics and Astronomy, 116 Church Street SE, University of Minnesota, Minneapolis, MN 55455.

Resource Letter GPP-1: Geometric Phases in Physics

Jeeva Anandan, Joy Christian, and Kazimir Wanelik
Department of Theoretical Physics, University of Oxford, Oxford OX1 3NP, United Kingdom

(Received 2 April 1996; accepted 7 October 1996)

This Resource Letter provides a guide to the literature on the geometric angles and phases in classical and quantum physics. Journal articles and books are cited for the following topics: anticipations of the geometric phase, foundational derivations and formulations, books and review articles on the subject, and theoretical and experimental elaborations and applications.

I. INTRODUCTION

Suppose a system undergoes an evolution so that after some time it returns to its original state. We shall call such an evolution a *cyclic* evolution. If the system is classical, then it is impossible to say from its initial and final states that it has undergone any evolution. However, the wave function of a quantal system retains a memory of its motion in the form of a geometric phase factor. This phase factor can be measured by interfering the wave function with another coherent wave function enabling one to discern whether or not the system has undergone an evolution. Therefore geometric phase factors are ''signatures'' of quantum motion. The adjective ''geometric'' emphasizes that such phase factors depend only on the loop in the quantum-mechanical state space—the set of rays of the Hilbert space, sometimes called the projective Hilbert space. In particular, geometric phases are independent of parametrization of the path in the projective Hilbert space, and therefore of the speed at which it has been traversed.

As early as 1956,[1] in a classic paper on phase shifts in nonquantal polarized light,[2] S. Pancharatnam anticipated the quantal geometric phases. He was only 22 years of age at the time. He studied the problem of determining the phase change undergone by polarized light after it has passed through a sequence of polarizers such that its final polarization is the same as its initial polarization. To describe how the phase of polarized light changes under passage through a polarizer, Pancharatnam needed to define the *phase difference* between two different polarization states. He reasoned that the most natural way to accomplish this task is to ask what would happen if two such states were brought to interfere with each other, and accordingly he proposed the following definition: The polarization states of any two monochromatic beams of light with the same momenta are *in phase* if the superposition of the two has the maximum possible intensity. Let $|A\rangle$ and $|B\rangle$ represent the polarization state vectors of photons in the two beams. Since the intensity of their superposition is proportional to

$$(\langle A|+\langle B|)(|A\rangle+|B\rangle)=2+2|\langle A|B\rangle|\cos\{\mathrm{ph}\langle A|B\rangle\}, \tag{1}$$

according to his convention $|A\rangle$ and $|B\rangle$ are *in phase* when their scalar product $\langle A|B\rangle$ is real and positive, or equivalently, when $\mathrm{ph}\langle A|B\rangle=0$. Incidentally, since orthogonal states do not interfere, this convention breaks down for such states, and the phase difference between them remains undefined. In the general case of *nonorthogonal* states, it is natural to identify the phase difference between $|A\rangle$ and $|B\rangle$ with the phase $\mathrm{ph}\langle A|B\rangle$ of their scalar product.

Pancharatnam used this definition of the phase difference to analyze an experiment involving a sequence of changes in polarization of a beam of classical light by sending it through suitable polarizers. His experiment consisted of three sequential changes in polarization, from $|A\rangle$ to $|B\rangle$ to $|C\rangle$ and back to a state $|A'\rangle$ of the initial polarization. It is easy to show that in such a scheme each successive state remains in phase with the previous one. Now, the label A used here to describe a state of a polarized wave of light represents a set of values (the eigenvalues of a complete set of commuting observables) required to specify this state uniquely. In Pancharatnam's experiment all but one of these values—including the one that specifies the polarization—were returned to their original values, with the phase of polarization being the only exception. Thus Pancharatnam's evolution was not cyclic in the sense described above. Indeed, in what follows, the classical phase difference he observed will be shown to come from the quantum mechanical phase difference between the initial and final one-photon states:

$$\langle\psi|\psi'\rangle=\exp(-(i/2)\alpha_{ABC}),\tag{2}$$

where α_{ABC} is the solid angle subtended by the geodesic triangle ABC on the Poincaré sphere (whose points, as is well known, represent all conceivable forms of polarization states). For simplicity, we ignore the dynamical phase difference due to the fixed frequency of the photon. Remarkably enough, Pancharatnam not only anticipated the quantal geometric phases, but also was able to corroborate his theory experimentally.

Another geometric phase given by a solid-angle formula analogous to (2) was put forward in 1984 by M. V. Berry (who was unaware of Pancharatnam's work) in a seminal paper on the quantum-mechanical adiabatic theorem.[7] He investigated the nonrelativistic Schrödinger evolution

$$i\frac{d}{dt}|\psi(t)\rangle=H(\mathbf{R}(t))|\psi(t)\rangle\tag{3}$$

of a quantal system in a slowly changing environment described by a set of N time-dependent parameters $\mathbf{R}(t)=(R_1(t),R_2(t),\ldots,R_N(t))$, with the initial state

$$|\psi(0)\rangle=|n;\mathbf{R}(0)\rangle\tag{4}$$

being the stationary state given by the time-independent Schrödinger equation

$$H(\mathbf{R}(0))|n;\mathbf{R}(0)\rangle=E_n(\mathbf{R}(0))|n;\mathbf{R}(0)\rangle.\tag{5}$$

If $H(\mathbf{R}(t))$ is nondegenerate and slowly varying, then it is known that the time-evolving Schrödinger state $|\psi(t)\rangle$ remains an eigenstate of the instantaneous Hamiltonian $H(\mathbf{R}(t))$. More precisely,

$$|\psi(t)\rangle=\exp\left[-i\int_0^t ds E_n(\mathbf{R}(s))\right]\times\exp\{ib[n,\mathbf{R}(t)]\}|E_n(\mathbf{R}(t))\rangle,\tag{6}$$

where

$$b[n;\mathbf{R}(t)]=\int_0^t ds\langle n;\mathbf{R}(s)|i\frac{d}{ds}|n;\mathbf{R}(s)\rangle,\tag{7}$$

or, equivalently,

$$b[n;\mathbf{R}(t)]=\int_{\mathbf{R}(0)}^{\mathbf{R}(t)} d\mathbf{R}'\cdot\langle n;\mathbf{R}'|i\nabla_{\mathbf{R}'}|n;\mathbf{R}'\rangle,\tag{7$'$}$$

where $\nabla_{\mathbf{R}}$ is the gradient operator in the parameter space $\mathbf{R}$. This is, of course, just the time-honored adiabatic theorem.

Berry's investigations, however, went beyond the usual formulation of the adiabatic theorem captured in (6) and (7). He considered the case of an adiabatic transport around a *closed* path,

$$\rho_t=\{\mathbf{R}(t)|\mathbf{R}(T)=\mathbf{R}(0);0<t<T\},\tag{8}$$

in the parameter space, and made the crucial observation that in such an adiabatic setup the phase factor $e^{ib[n,\mathbf{R}(t)]}$ is not integrable, i.e., in general it cannot be written as a function of $\mathbf{R}$, and in particular is not single-valued under continuation around the loop: $\exp\{ib[n;\mathbf{R}(T)]\}\neq\exp\{ib[n;\mathbf{R}(0)]\}$. Moreover, it is easy to see that (7$'$) can be re-expressed in the form

$$b[n;\rho]=\oint_\rho d\mathbf{R}\cdot\langle n;\mathbf{R}|i\nabla_{\mathbf{R}}|n;\mathbf{R}\rangle,\tag{9}$$

from which it is evident that $b[n;\mathbf{R}(T)]$, or the Berry phase as it is now called, is independent of parametrization: $b[n;\mathbf{R}(T)]=b[n;\rho]$, where ρ denotes the unparametrized loop corresponding to ρ_t. In particular, unlike the usual dynamical phase $\{-\int ds E_n\}$, the Berry phase $b[n;\rho]$ is independent of the rate at which the state of the system traverses around ρ.

To illustrate his findings Berry analyzed the example of a spin-s particle interacting with a magnetic field $\mathbf{B}$ through the Hamiltonian

$$H(\mathbf{B})=\kappa\mathbf{B}\cdot\mathbf{S},\tag{10}$$

where κ is a constant involving the gyromagnetic ratio, and $\mathbf{S}$ is the vector spin operator whose components have $2s+1$ eigenvalues n lying between $-s$ and $+s$ with integer spacing. The eigenvalues of $H(\mathbf{B})$ are, of course,

$$E_n(\mathbf{B})=\kappa Bn,\tag{11}$$

with $B=|\mathbf{B}|$. Now, if one identifies the components of the external magnetic field $\mathbf{B}$ with the parameter space $\mathbf{R}$, then Berry's formula is easily applicable to this case. In particular, (9) gives the geometric phase change of an eigenstate $|n;\mathbf{B}(t)\rangle$ of $H(\mathbf{B}(t))$ as $\mathbf{B}(t)$ is slowly transported—and hence the spin is slowly precessed—around a loop γ in the $\mathbf{B}$ space. Berry was able to show that, in that case,

$$\exp\{ib[n;\mathbf{B}(T)]\}=\exp(-in\alpha_\gamma),\tag{12}$$

where α_γ is the solid angle subtended by the loop γ at $\mathbf{B}=0$. In particular, when $s=\frac{1}{2}$ and the initial state is $|\frac{1}{2};\mathbf{B}(0)\rangle$ (''spin up'' along $\mathbf{B}$), then the right-hand side of (12) takes the form of the right-hand side of the observation (2) of Pancharatnam—namely, $\exp[-(i/2)\alpha]$. To establish an analogy between (2) and (12) it suffices now to identify $|\psi(0)\rangle=|\psi\rangle$ and $|\psi(T)\rangle=|\psi'\rangle$, and note that the left-hand side of (12) can be rewritten, after the dynamical phase is removed, in the form $\langle\psi(0)|\psi(T)\rangle$.

A simple explanation of the beautiful result (12) was given in 1987 by J. Anandan and L. Stodolsky.[36] They considered a sphere whose points represented the possible directions of the magnetic field $\mathbf{B}$. In the above example of Berry, it is sufficient for the direction of $\mathbf{B}$ to trace a closed curve $\hat\gamma$ on this sphere in order for each eigenstate to acquire a geometric phase. In other words, it is not necessary for $\mathbf{B}(t)$ to form a closed curve; it is sufficient if merely the directions of $\mathbf{B}(0)$ and $\mathbf{B}(T)$ coincide. Anandan and Stodolsky then considered a Cartesian triad with its origin on $\hat\gamma(t)$ and its z axis in the radial direction of the sphere (the direction of $\mathbf{B}(t)$ and the spin axis). If the triad is moved along $\hat\gamma(t)$ so that the x,y axes are parallel-transported along the surface of the sphere, then when the triad returns to the original point $\hat\gamma(0)=\hat\gamma(T)$, it will have rotated about its z axis by the solid angle α subtended by $\hat\gamma$ at the center of the sphere. Now, relative to the triad, each eigenstate individually should acquire only the usual dynamical phase factor because the triad has no angular velocity about the spin axis. Consequently, the additional phase factor acquired by the eigenstate must be interpreted as the *geometric* phase factor due to the rotation of the triad given by $\exp(i\alpha J_z|n\rangle=\exp(i\alpha n)|n\rangle$, where J_z generates rotation about the z axis of the triad.

For an arbitrary cyclic evolution in any Hilbert space, the above angle α generalizes to a set of angles $\alpha_1,\ldots,\alpha_N$. These are the geometric quantum angles introduced by Anandan,[14] which perhaps provides the deepest approach so far to the geometric phase. The geometric phases acquired by a complete set of orthogonal states $\{|n\rangle\}$ are now obtained by the '1action on each $|n\rangle$ by $\exp(i\Sigma_{k=1}^N\alpha_kJ_k)$, where the elements of the set $\{J_k\}$ commute among themselves. In the classical limit the geometric angles $\{\alpha_k\}$ reduce to the classical angles of J. H. Hannay,[10] while the observables $\{J_k\}$ become the corresponding action variables that are in involution with each other.

We now illustrate the usefulness of geometric angles by providing a quantum-mechanical explanation of the above-mentioned experiment of Pancharatnam. For this purpose, we need to generalize his classical electromagnetic polarized wave, with fixed momentum **p**, passing through an arbitrary number of polarizers, such that the final polarization is the same as the initial polarization. A classical electromagnetic wave is an approximation of a coherent state in quantum electrodynamics. In the Coulomb gauge the quantized vector-potential for the electromagnetic field may be written as

$$\mathbf{A}=\sum_{\mathbf{k}}\sum_{\lambda=1}^{2}[a_{\mathbf{k},\lambda}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)] + a^{\dagger}_{\mathbf{k},\lambda}\exp[-i(\mathbf{k}\cdot\mathbf{x}-\omega t)]]e_{\mathbf{k},\lambda}, \tag{13}$$

where $\mathbf{k}$ is the momentum vector, $\omega=|\mathbf{k}|$ is the frequency, $e_{\mathbf{k},\lambda}$ are real orthogonal polarization vectors perpendicular to $\mathbf{k}$, and $a_{\mathbf{k},\lambda}$, $a^{\dagger}_{\mathbf{k},\lambda}$ are the annihilation and creation operators for the mode $(\mathbf{k},\lambda)$. The electric and magnetic fields corresponding to $\mathbf{A}$ are $\mathbf{E}=-\partial A/\partial t$ and $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, respectively. The coherent state corresponding to the electromagnetic wave considered by Pancharatnam is then

$$|z_1,z_2,\mathbf{p}\rangle=\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)] \times\exp(z_1a^{\dagger}_{\mathbf{p},1}+z_2a^{\dagger}_{\mathbf{p},2})|0\rangle, \tag{14}$$

which is an eigenstate of $a_{\mathbf{p},\lambda}$ with eigenvalues z_λ. Therefore,

$$\langle z_1,z_2,\mathbf{p}|\mathbf{A}|z_1,z_2,\mathbf{p}\rangle=2\{|z_1|\cos(\mathbf{p}\cdot\mathbf{x}-\omega t+\theta_1)\mathbf{e}_{\mathbf{p},1} +|z_2|\cos(\mathbf{p}\cdot\mathbf{x}-\omega t+\theta_2)\mathbf{e}_{\mathbf{p},2}\}, \tag{15}$$

where θ_1 and θ_2 are the phases of z_1 and z_2, respectively. It follows that $|z_1|\omega$ and $|z_2|\omega$ are the amplitudes of the electric field $\mathbf{E}$ in the directions of $\mathbf{e}_{\mathbf{p},1}$ and $\mathbf{e}_{\mathbf{p},2}$, respectively.

We may represent the polarization state of a one-photon state $(z_1a^{\dagger}_{\mathbf{p},1}+z_2a^{\dagger}_{\mathbf{p},2})|0\rangle$ as a two-component spinor

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

in a two-dimensional vector space with the usual inner product, which makes it a Hilbert space. The corresponding projective Hilbert space is the Poincaré sphere. As each photon corresponding to the mode $(\mathbf{p},\mu)$ passes through the polarizer it undergoes a transition to a state $(\mathbf{p},\mu')$. The new state is obtained by simply projecting the old state onto the state that passes through the polarizer. It can be shown that this corresponds to parallel-transporting the old state-vector along the shorter geodesic joining the two points on the Poincaré sphere representing the two polarization states.[43] Therefore, using arguments similar to those used by Anandan and Stodolsky,[36] the final state obtained after a sequence of such polarization changes that return the photons to their initial polarization state is given by the action of the operator $\exp(i\alpha J)$ on the initial photon state. Here α is the solid angle subtended by the geodesic polygon defined by the sequence of the polarization states on the Poincaré sphere, and $J=N/2$, N being the number operator for the initial and final mode $(\mathbf{p},\mu)$. As a result, the final state of the electromagnetic field is

$$\exp(i\alpha J)|z_1,z_2,\mathbf{p}\rangle=\left|z_1\exp i\frac{\alpha}{2},\quad z_2\exp i\frac{\alpha}{2},\mathbf{p}\right\rangle \tag{16}$$

Finally, in this resultant state we have

$$\langle z_1,z_2,\mathbf{p}|\mathbf{A}|z_1,z_2,\mathbf{p}\rangle =2\left\{|z_1|\cos\left(\mathbf{p}\cdot\mathbf{x}-\omega t+\theta_1+\frac{\alpha}{2}\right)\mathbf{e}_{\mathbf{p},1} +|z_2|\cos\left(\mathbf{p}\cdot\mathbf{x}-\omega t+\theta_2+\frac{\alpha}{2}\right)\mathbf{e}_{\mathbf{p},2}\right\}. \tag{17}$$

Comparison of this expression with Eq. (15) shows that $\alpha/2$ is the phase Pancharatnam observed in his classical experiment. A similar explanation can be given to the experiment of Tomita and Chiao,[117] except in their case we would have $J=N$ for the photon since it is a spin-1 particle.

Having obtained these results using the operator $\exp(i\alpha J)$, which depends on the geometric angle α, we may generalize them to an arbitrary superposition of number eigenstates with the same polarization. In general, such a state would not be a coherent state and cannot therefore be represented by a classical electromagnetic wave. Nevertheless, the geometric part of the evolution may be obtained by taking the expectation value of $\exp(i\alpha J)$ with respect to the initial state.

The above-mentioned geometric treatment of Berry's phase by Anandan and Stodolsky suggests that the geometric phase is associated with the motion of a quantum system and not with the particular Hamiltonian used to achieve this motion. This is the basic idea used by Aharonov and Anandan[11] in obtaining a geometric phase, which, since it is associated with the motion of the quantum-mechanical state itself, does not require an adiabatically varying Hamiltonian (environment). However, if an adiabatically varying Hamiltonian is used to implement this motion, then this geometric phase is the same as Berry's phase. They defined the evolution of a normalized state $|\psi(t)\rangle$ to be cyclic in the interval $[0,T]$ if and only if

$$|\psi(T)\rangle=\exp[i\phi(0,T)]|\psi(0)\rangle, \tag{18}$$

where $\phi(0,T)$ is a real number. Equivalently, this can be re-expressed with the help of the unitary time evolution operator $U(0,t)$ in the form

$$U(0,T)|\psi(0)\rangle=\exp[i\phi(0,T)]|\psi(0)\rangle. \tag{19}$$

It follows from this equation that for an initial state to evolve cyclicly in the interval $[0,T]$ under the time-evolution operator $U(0,t)$, it is necessary and sufficient for it to be an eigenstate of the operator $U(0,T)$. Incidentally, this assures the existence of cyclic evolutions as defined above at least in the finite-dimensional case. According to Aharonov and Anandan, the geometric contribution to $\phi(0,T)$, denoted by β, is

$$\exp(i\beta)=\exp\left[i\phi(0,T)+i\int_0^T ds\langle\psi(s)|i\frac{d}{ds}|\psi(s)\rangle\right] \tag{20}$$

or, equivalently,

$$\exp(i\beta)=\langle\psi(0)|\psi(T)\rangle \times\exp\left[i\int_0^T ds\langle\psi(s)|i\frac{d}{ds}|\psi(s)\rangle\right], \tag{20$'$}$$

which reduces to the Berry's phase factor in the adiabatic limit.[11,67] What is more, just as Berry's phase, it is independent of the choice of parametrization or the speed at which the path $|\psi(t)\rangle$ is traversed. More significantly, Aharonov and Anandan demonstrated that β is projective-geometric in nature, i.e., it is the same for all paths $|\phi(t)\rangle$ that project to the same path in the projective Hilbert space. In other words, it is the same for any two motions $|\phi(t)\rangle$ and $|\psi(t)\rangle$ such that $p_{\phi(t)}=p_{\psi(t)}$, where p_α denotes a ray corresponding to a vector $|\alpha\rangle$, namely,

$$p_\alpha=\{|\beta\rangle \,|\, |\beta\rangle=z|\alpha\rangle;\ \ z\in\mathbb{C}\}. \tag{21}$$

The above two properties imply that $\beta=\beta[p_\psi]$, and suggest that β may have a geometric interpretation in terms of paths in the projective Hilbert space. Indeed, it may be geometrically understood as the *anholonomy* with respect to the natural connection on the projective Hilbert space. This interpretation generalizes an earlier differential-geometric interpretation of the Berry phase given by B. Simon in 1983.[8]

Berry's 1984 paper was concerned with nondegenerate states undergoing adiabatic evolution. In the same year F. Wilczek and A. Zee reported on how the theory can be generalized to include the adiabatic evolution of *degenerate* quantum states.[9] They showed that in the case of a d-fold degeneracy, Berry's phase factor of the nondegenerate case, $\exp(ib[n;\rho])$, is generalized to a $d\times d$ unitary matrix, which is now called the non-Abelian Berry phase or the Wilczek–Zee phase. More precisely, if the initial state is one of the eigenstates belonging to an orthonormal set of eigenstates of $H(\mathbf{R}(0))$ with a d-fold degenerate eigenvalue $E_n(\mathbf{R}(0))$, i.e.,

$$H(\mathbf{R}(0))|l;\mathbf{R}(0)\rangle=E_n(\mathbf{R}(0))|l;\mathbf{R}(0)\rangle, \tag{22}$$

with $l=1,2,\ldots,d$, then

$$|\psi(t)\rangle=\exp\left[-i\int_0^t ds E_n(\mathbf{R}(s))\right] \times\sum_{l'=1}^{d} D_{l'l}(\mathbf{R}(t))|l;\mathbf{R}(t)\rangle. \tag{23}$$

Here, the matrix D is a path-ordered exponential integral

$$D[\mathbf{R}(t)]=\mathscr{P}\exp\left\{i\int_{\mathbf{R}(0)}^{\mathbf{R}(t)} d\mathbf{R}' A(\mathbf{R}')\right\}, \tag{24}$$

with

$$A_{l'l}(\mathbf{R}(t))=\langle l';\mathbf{R}|i\nabla_{\mathbf{R}}|l;\mathbf{R}\rangle, \tag{25}$$

and $\mathscr{P}$ represents the path-ordering. The non-Abelian phase factor $D[l;\mathbf{R}(T)]$ is a unitary matrix, and may be denoted by $D[l;\rho]$ because it too is independent of parameterization or the speed with which a particular path is traversed and is therefore geometric.

Berry's 1984 paper, and the other reports discussed above, were followed by a great number of papers on the subject. They can be divided into two broad groups. The first contains contributions that reformulate or generalize Berry's findings, while the second contains papers in which geometric phases are identified or measured in a great number of apparently disparate physical phenomena, or in which attempts are made to use these phases to explain unresolved physical questions. The bibliography that follows is selective and by no means exhaustive since there are hundreds of research papers on the subject.

We conclude by remarking that, there are at least four different reasons for the phenomenal success of the concepts related to geometric phases. First, these concepts are exceptionally clear and have a very elegant geometric interpretation in terms of anholonomies and connections (gauge fields). Second, geometric phases have a certain unifying character that enables one to relate many apparently disparate phenomena. Third, these phases can be observed and, indeed, various predictions of the geometric phases have been amply corroborated. Finally, and perhaps most importantly, these concepts reassert the importance and fruitfulness of geometric ideas in physical theories.

ACKNOWLEDGMENTS

The work of one of the authors (JA) was partially supported by NSF Grant No. PHY-9307708 and ONR Grant No. R & T 3124141.

II. ANTICIPATIONS OF THE GEOMETRIC PHASE

1. Actually, there are some Russian theoretical papers, collected in Ref. 18 below, which anticipate the geometric phase as early as the 1940s.

2. ''Generalized theory of interference, and its applications,'' S. Pancharatnam, Proc. Indian Acad. Sci. A **44**, 247–262 (1956). See also Ref. 43 below for the formulation of Pancharatnam's phase in quantum theoretical language. (E)

3. ''Significance of electromagnetic potentials in the quantum theory,'' Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485–491 (1959). (I)

4. ''Intersection of potential energy surfaces in polyatomic molecules,'' G. Herzberg and H. C. Longuet-Higgins, Disc. Faraday Soc. **35**, 77–82 (1963). (I)

5. ''Spin–orbit coupling and the intersection of potential energy surfaces in polyatomic molecules,'' A. J. Stone, Proc. R. Soc. London, Ser. A **351**, 141–150 (1976). (I).

6. ''On the determination of Born–Oppenheimer nuclear motion wave functions including complications due to conical intersections and identical nuclei,'' C. A. Mead, J. Chem. Phys. **70**, 2284–2296 (1979). (I)

III. FOUNDATIONAL DERIVATIONS AND FORMULATIONS

7. ''Quantum phase factors accompanying adiabatic changes,'' M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45–57 (1984). (I)

8. ''Holonomy, the quantum adiabatic theorem, and Berry's phase,'' B. Simon, Phys. Rev. Lett. **51**, 2167–2170 (1983). (A)

9. ''Appearance of gauge structure in simple dynamical systems,'' F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111–2114 (1984). (I)

10. ''Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian,'' J. H. Hannay, J. Phys. A **18**, 221–230 (1985). (I)

11. ''Phase change during a cyclic quantum evolution,'' Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593–1596 (1987). (I)

12. ''General setting for Berry's phase,'' J. Samuel and R. Bhandari, Phys. Rev. Lett. **60**, 2339–2342 (1988). (A)

13. ''Non-adiabatic non-Abelian geometric phase,'' J. Anandan, Phys. Lett. A **133**, 171–175 (1988). (I)

14. ''Geometric angles in quantum and classical physics,'' J. Anandan, Phys. Lett. A **129**, 201–207 (1988). (I)

15. ''Geometrical phases from global gauge invariance of non linear classical field theories,'' J. C. Garrison and R. Y. Chiao, Phys. Rev. Lett. **60**, 165–168 (1988). (I)

16. ''Comment on geometric phases for classical field theories,'' J. Anandan, Phys. Rev. Lett. **60**, 2555 (1988). (I)

IV. BOOKS AND REVIEW ARTICLES

17. **Geometric Phases in Physics**, edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989). This book contains original reprints of many pioneering articles on the subject with some introductory comments on each sub-topic, from elementary to advanced. (E, I, A)

18. **Topological Phases in Quantum Theory**, edited by B. Markovski and S. I. Vinitsky (World Scientific, Singapore, 1989). This book traces anticipations of the geometric phase emphasizing Russian contributions to the subject and contains many less well-known reprints. (E, I, A)

19. ''Berry's topological phase in quantum-mechanics and quantum-field theory,'' I. J. R. Aitchison, Phys. Scr. T **23**, 12–20 (1988). (I)

20. ''Adiabatic quantum transport in multiply connected systems,'' J. E. Avron, B. Zur, and A. Raveh, Rev. Mod. Phys. **60**, 873–915 (1988). (A)

21. ''Berry phase,'' J. W. Zwanziger, M. Koenig, and A. Pines, Annu. Rev. Phys. Chem. **41**, 601–646 (1990). This review contains an extensive list of articles up to 1990. (I)

22. ''Anticipations of the geometric phase,'' M. V. Berry, Phys. Today **43**, 34–40 (1990). (E)

23. ''Topological phases in quantum-mechanics and polarization optics,'' S. I. Vinitskii, V. L. Derbov, V. M. Dubovik, B. L. Markovski, and Y. P. Stepanovskii, Uspekhi (Sov. Phys.) **33**, 403–428 (1990). (I)

24. ''The geometric phase,'' J. Anandan, Nature **360**, 307–313 (1992). (E)

25. ''The geometric phase in molecular-systems,'' C. A. Mead, Rev. Mod. Phys. **64**, 51–85 (1992). (I)

26. **Quantum Mechanics**, A. Böhm (Springer-Verlag, New York, 1993). The last two chapters of this book contain a detailed textbook introduction to the subject. (E)

27. ''Pancharatnam memorial issue,'' Guest Editors: S. Ramaseshan and R. Nityananda, Curr. Sci. India **67**, 217–294 (1994). This special issue contains many original contributions in addition to some review articles. (E)

V. ELABORATIONS AND APPLICATIONS

A. Theoretical articles

28. ''Fractional statistics and the Quantum Hall-effect,'' D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722–723 (1984). (A)

29. ''Hamiltonian Interpretation of Anomalies,'' P. Nelson and L. Alvarez-Gaumé, Commun. Math. Phys. **99**, 103–114 (1985). (A)

30. ''Classical adiabatic angles and quantal adiabatic phase,'' M. V. Berry, J. Phys. A **18**, 15–27 (1985). (I)

31. ''Semiclassical quantization with a quantum adiabatic phase,'' H. Kuratsuji and S. Iida, Phys. Lett. A **111**, 220–222 (1985). (I)

32. ''Classical and quantum adiabatic invariants,'' E. Gozzi, Phys. Lett. B **165**, 351–354 (1985). (I)

33. ''Quantum holonomy and the chiral gauge anomaly,'' A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **55**, 927–930 (1985). (A)

34. ''Effective action for adiabatic process—dynamical meaning of Berry and Simon phase,'' H. Kuratsuji and S. Iida, Prog. Theor. Phys. **74**, 439–445 (1985). (I)

35. ''The interference of polarized light as an early example of Berry's phase,'' S. Ramaseshan and R. Nityananda, Curr. Sci. India **55**, 1225–1226 (1986). (E)

36. ''Some geometrical considerations of Berry phase,'' J. Anandan and L. Stodolsky, Phys. Rev. D **35**, 2597–2600 (1987) (I)

37. ''Classical adiabatic holonomy and its canonical structure,'' E. Gozzi and W. D. Thacker, Phys. Rev. D **35**, 2398–2406 (1987). (I)

38. ''Berry phases, magnetic monopoles, and Wess–Zumino terms or how the skyrmion got its spin,'' I. J. R. Aitchison, Acta Phys. Pol. B **18**, 207–235 (1987). (A)

39. ''Non-Abelian Berry phase effects and optical-pumping of atoms,'' J. Segert, Ann. Phys. **179**, 294–312 (1987). (I)

40. ''Non-Abelian Berry phase, accidental degeneracy, and angular-momentum,'' J. Segert, J. Math. Phys. **28**, 2102–2114 (1987). (I)

41. ''A topological investigation of the quantum adiabatic phase,'' E. Kiritsis, Commun. Math. Phys. **111**, 417–437 (1987). (A)

42. ''Direct calculations of the Berry phase for spins and helicities,'' T. F. Jordan, J. Math. Phys. **28**, 1759–1760 (1987). (E)

43. ''The adiabatic phase and the Pancharatnam phase for polarized-light,'' M. V. Berry, J. Mod. Opt. **34**, 1401–1407 (1987). (E)

44. ''Cyclic geometrical quantum phases—group-theory derivation and manifestations in atomic physics,'' C. Bouchiat, J. Phys. **48**, 1401–1406 (1987). (I)

45. ''Interpreting the anholonomy of coiled light,'' M. V. Berry, Nature **326**, 277–278 (1987). (I)

46. ''Quantum phase corrections from adiabatic iteration,'' M. V. Berry, Proc. R. Soc. London, Ser. A **414**, 31–46 (1987). (I)

47. ''Appearance of gauge potentials in atomic collision physics,'' B. Zygelman, Phys. Lett. A **125**, 476–481 (1987). (I)

48. ''Molecular Kramers degeneracy and non-Abelian adiabatic phase-factors,'' C. A. Mead, Phys. Rev. Lett. **59**, 161–164 (1987). (I)

49. ''Berry geometrical phase and the sequence of states in the Jahn–Teller effect,'' F. S. Ham, Phys. Rev. Lett. **58**, 725–728 (1987). (I)

50. ''Induced gauge-fields in a nongauged quantum system,'' H. Z. Li, Phys. Rev. Lett. **58**, 539–542 (1987). (I)

51. ''Geometrical description of Berry's phase,'' D. N. Page, Phys. Rev. A **36**, 3479–3481 (1987). (A)

52. ''Geometric quantum phase and angles,'' J. Anandan and Y. Aharonov, Phys. Rev. D **38**, 1863–1870 (1988). (I)

53. ''Berry phase, locally inertial frames, and classical analogs,'' M. Kugler and S. Shtrikman, Phys. Rev. D **37**, 934–937 (1988). (E)

54. ''Berry phase and unitary transformations,'' T. F. Jordan, J. Math. Phys. **29**, 2042–2052 (1988). (I)

55. ''Non-integrable quantum phase in the evolution of a spin-1 system,'' C. Bouchiat and G. W. Gibbons, J. de Phys. **49**, 187–199 (1988). (I)

56. ''Complex geometrical phases for dissipative systems,'' J. C. Garrison and E. M. Wright, Phys. Lett. A **128**, 177–181 (1988). (I)

57. ''Non-Abelian gauge structure in nuclear-quadrupole resonance,'' A. Zee, Phys. Rev. A **38**, 1–6 (1988). (I)

58. ''Effective action for a nonadiabatic quantum process,'' A. Bulgac, Phys. Rev. A **37**, 4084–4089 (1988). (I)

59. ''Cyclic evolution in quantum-mechanics and the phases of Bohr-Sommerfeld and Maslov,'' R. G. Littlejohn, Phys. Rev. Lett. **61**, 2159–2162 (1988). (I)

60. ''Geometric canonical phase-factors and path-integrals,'' H. Kuratsuji, Phys. Rev. Lett. **61**, 1687–1690 (1988). (I)

61. ''Classical non-adiabatic angles,'' M. V. Berry and J. H. Hannay, J. Phys. A **21**, L325–L331 (1988). (I)

62. ''The Berry phase and the Hannay angle,'' G. Ghosh and B. Dutta-Roy, Phys. Rev. D **37**, 1709–1711 (1988). (I)

63. ''Non-Abelian geometric phase from incomplete quantum measurements,'' J. Anandan and A. Pines, Phys. Lett. A **141**, 335–339 (1989). (I)

64. ''Non local aspects of quantum phases,'' J. Anandan, Ann. Inst. Henri Poincaré **49**, 271–286 (1988). (I)

65. ''On removing Berry phase,'' G. Giavarini, E. Gozzi, D. Rohrlich, and W. D. Thacker, Phys. Lett. A **138**, 235–241 (1989). (I)

66. ''Berry phase and Fermi–Walker parallel transport,'' R. Dandoloff, Phys. Lett. A **139**, 19–20 (1989). (I)

67. ''The Berry phase as an appropriate correspondence limit of the Aharonov–Anandan phase in a simple model,'' J. Christian and A. Shimony, in **Quantum Coherence**, edited by J. Anandan (World Scientific, Singapore, 1990), pp. 121–135. (E)

68. ''Berry phase, interference of light-beams, and the Hannay angle,'' G. S. Agarwal and R. Simon, Phys. Rev. A **42**, 6924–6927 (1990). (E)

69. ''Connections of Berry and Hannay type for moving Lagrangian submanifolds,'' A. Weinstein, Adv. Math. **82**, 133–159 (1990). (A)

70. ''On quantum holonomy for mixed states,'' L. Dabrowski and H. Grosse, Lett. Math. Phys. **19**, 205–210 (1990). (I)

71. ''Geometric phase in neutron interferometry,'' A. G. Wagh and V. C. Rakhecha, Phys. Lett. A **148**, 17–19 (1990). (I)

72. ''Geometric phase for cyclic motions and the quantum state-space metric,'' J. Anandan, Phys. Lett. A **147**, 3–8 (1990). (I)

73. ''Non-Abelian geometric phase and long-range atomic forces,'' B. Zygelman, Phys. Rev. Lett. **64**, 256–259 (1990). (I)

74. ''A geometric approach to quantum mechanics,'' J. Anandan, Found. Phys. **21**, 1265–1284 (1991). (A)

75. ''How much does the rigid body rotate—a Berry phase from the 18th-century,'' R. Montgomery, Am. J. Phys. **59**, 394–398 (1991). (E)

76. ''A gauge field governing parallel transport along mixed states,'' A. Uhlmann, Lett. Math. Phys. **21**, 229–236 (1991). (A)

77. ''Born–Oppenheimer revisited,'' Y. Aharonov, E. Benreuven, S. Popescu, and D. Rohrlich, Nucl. Phys. B **350**, 818–830 (1991). (I)

78. ''Budden and Smith additional memory and the geometric phase,'' M. V. Berry, Proc. R. Soc. London, Ser. A **431**, 531–537 (1990). (I)

79. ''On the real and complex geometric phases,'' I. J. R. Aitchison and K. Wanelik, Proc. R. Soc. London, Ser. A **439**, 25–34 (1992). (I)

80. ''A group theoretical treatment of the geometric phase,'' E. C. G. Sudarshan, J. Anandan, and T. R. Govindarajan, Phys. Lett. A **164**, 133–137 (1992). (A)

81. ''Origin of the geometric forces accompanying Berry's geometric potentials,'' Y. Aharonov and A. Stern, Phys. Rev. Lett. **69**, 3593–3597 (1992). (A)

82. ''Geometric phase and symmetries in dissipative systems,'' A. S. Landsberg, Phys. Rev. Lett. **69**, 865–868 (1992). (A)
83. ''Berry phase, motive forces, and mesoscopic conductivity,'' A. Stern, Phys. Rev. Lett. **68**, 1022–1025 (1992). (A)
84. ''Geometric phase, geometric distance, and length of the curve in quantum evolution,'' A. K. Pati, J. Phys. A **25**, L1001–L1008 (1992). (I)
85. ''The geometric phase for chaotic systems,'' J. M. Robbins and M. V. Berry, Proc. R. Soc. London, Ser. A **436**, 631–661 (1992). (A)
86. ''Berry phase and Euclidean path integral,'' T. Kashiwa, S. Nima, and S. Sakoda, Ann. Phys. **220**, 248–273 (1992). (A)
87. ''Cyclic states, Berry phases and the Schrödinger operator,'' A. N. Seleznyova, J. Phys. A **26**, 981–1000 (1993). (A)
88. ''Connection between solitons and geometric phases,'' R. Balakrishnan, Phys. Lett. A **180**, 239–243 (1993). (A)
89. ''Geometric phase for the relativistic Klein–Gordon equation,'' J. Anandan and P. O. Mazur, Phys. Lett. A **173**, 116–120 (1993). (A)
90. ''Non-Abelian Berry phases in Baryons,'' H. K. Lee, M. A. Nowak, M. Rho, and I. Zahed, Ann. Phys. **227**, 175–205 (1993). (A)
91. ''Interplay of Aharonov–Bohm and Berry phases,'' B. Reznik and Y. Aharonov, Phys. Lett. B **315**, 386–391 (1993). (A)
92. ''Nonlinearity of Pancharatnam's topological phase,'' H. Schmitzer, S. Klein, and W. Dultz, Phys. Rev. Lett. **71**, 1530–1533 (1993). (A)
93. ''Berry phase and the magnus force for a vortex line in a superconductor,'' P. Ao and D. J. Thouless, Phys. Rev. Lett. **70**, 2158–2161 (1993). (A)
94. ''Quantum kinematical approach to the geometric phase,'' N. Mukunda and R. Simon, Ann. Phys. **228**, 205–268 (1993). (I)
95. ''Geometric phase in vacuum instability—applications in quantum cosmology,'' D. P. Datta, Phys. Rev. D **48**, 5746–5750 (1993). (A)
96. ''On geometric phases and dynamical invariants,'' D. B. Monteoliva, H. J. Korsch, and J. A. Nunez, J. Phys. A **27**, 6897–6906 (1994). (I)
97. ''Symplectic structure for the non-Abelian geometric phase,'' D. Chruscinski, Phys. Lett. A **186**, 1–4 (1994). (A)
98. ''Spin–orbit interaction and Aharonov–Anandan phase in mesoscopic rings,'' T. Z. Qian and Z. B. Su, Phys. Rev. Lett. **72**, 2311–2315 (1994). (A)
99. ''Adiabatic approximation and Berry's phase in the Heisenberg picture,'' Y. Brihaye and P. Kosinski, Phys. Lett. A **195**, 296–300 (1994). (I)
100. ''Topological interpretations of quantum Hall conductance,'' D. J. Thouless, J. Math. Phys. **35**, 5362–5372 (1994). (A)
101. ''Geometric-phase effects in laser dynamics,'' V. Y. Toronov and V. Derbov, Phys. Rev. A **50**, 878–881 (1994). (A)
102. ''Aharonov–Bohm and Berry phases for a quantum cloud of charge,'' Y. Aharonov, S. Coleman, A. S. Goldhaber, S. Nussinov, S. Popescu, B. Reznik, D. Rohrlich, and L. Vaidman, Phys. Rev. Lett. **73**, 918–921 (1994). (I)
103. ''*S*-matrix as geometric phase factor,'' R. G. Newton, Phys. Rev. Lett. **72**, 954–956 (1994). (A)
104. ''Macroscopic polarization in crystalline dielectrics—the geometric phase approach,'' R. Resta, Rev. Mod. Phys. **66**, 899–915 (1994). (A)
105. ''Geometric phases and Mielnik's evolution loops,'' D. J. Fernandez, Int. J. Theor. Phys. **33**, 2037–2047 (1994). (A)
106. ''Geometric phase of polarized hydrogenlike atoms in an external magnetic-field,'' Z. Tang and D. Finkelstein, Phys. Rev. Lett. **74**, 3134–3137 (1995). (A)
107. ''The geometric vector potential in molecular-systems with arbitrarily many identical nuclei,'' B. Kendrick and C. A. Mead, J. Chem. Phys. **102**, 4160–4168 (1995). (A)
108. ''Geometric phase effects for wave-packet revivals,'' C. Jarzynski, Phys. Rev. Lett. **74**, 1264–1267 (1995). (A)
109. ''New derivation of the geometric phase,'' A. K. Pati, Phys. Lett. A **202**, 40–45 (1995). (I)
110. ''Gravity and geometric phases,'' A. Corichi and M. Pierri, Phys. Rev. D **51**, 5870–5875 (1995). (A)
111. ''Nature of geometric (Berry) potentials,'' A. Krakovsky and J. L. Birman, Phys. Rev. A **51**, 50–53 (1995). (I)
112. ''Chern–Simons term induced from a topological phase on the Wilson fermion vacuum functional,'' Z. S. Ma, S. S. Wu, and H. Z. Li, Phys. Rev. D **52**, 337–339 (1995). (A)
113. ''Photon-echoes and Berry phase,'' R. Friedberg and S. R. Hartmann, Phys. Rev. A **52**, 1601–1608 (1995). (I)
114. ''Gauge-invariant reference section and geometric phase,'' A. K. Pati, J. Phys. A **28**, 2087–2094 (1995). (I)
115. ''An adiabatic phase in scattering,'' G. Ghosh, Phys. Lett. A **210**, 40–44 (1996). (I)

B. Experimentals articles

116. ''Manifestations of Berry's topological phase for the photon,'' R. Y. Chiao and Y. S. Wu, Phys. Rev. Lett. **57**, 933–936 (1986). (I)
117. ''Observation of Berry's topological phase by use of an optical fiber,'' A. Tomita and R. Y. Chiao, Phys. Rev. Lett. **57**, 937–940 (1986). (I)
118. ''Berry phase in magnetic-resonance,'' D. Suter, G. C. Chingas, R. A. Harris, and A. Pines, Mol. Phys. **61**, 1327–1340 (1987). (I)
119. ''Adiabatic rotational splitting and Berry's phase in nuclear-quadrupole resonance,'' R. Tycko, Phys. Rev. Lett. **58**, 2281–2284 (1987). (I)
120. ''Manifestation of Berry topological phase in neutron spin rotation,'' T. Bitter and D. Dubbers, Phys. Rev. Lett. **59**, 251–254 (1987). (I)
121. ''Path-dependence of the geometric rotation of polarization in optical fibers,'' F. D. M. Haldane, Opt. Lett. **11**, 730–732 (1986). (E)
122. ''Measurement of the Pancharatnam phase for a light-beam,'' T. H. Chyba, L. Mandel, L. J. Wang, and R. Simon, Opt. Lett. **13**, 562–564 (1988). (E)
123. ''Observation of non-integrable geometric phase on the Poincaré sphere,'' R. Bhandari, Phys. Lett. A **133**, 1–3 (1988). (I)
124. ''Observation of Berry geometrical phase in electron-diffraction from a screw dislocation,'' D. M. Bird and A. R. Preston, Phys. Rev. Lett. **61**, 2863–2866 (1988). (I)
125. ''Observation of topological phase by use of a laser interferometer,'' R. Bhandari and J. Samuel, Phys. Rev. Lett. **60**, 1211–1213 (1988). (E)
126. ''Observation of a topological phase by means of a nonplanar Mach-Zehnder Interferometer,'' R. Y. Chiao, A. Antaramian, H. Nathel, S. R. Wilkinson, K. M. Ganga, and H. Jiao, Phys. Rev. Lett. **60**, 1214–1217 (1988). (E)
127. ''Study of the Aharonov–Anandan quantum phase by NMR interferometry,'' D. Suter, A. Pines, and K. T. Mueller, Phys. Rev. Lett. **60**, 1218–1220 (1988). (I)
128. ''Measurement of the Berry phase with polarized neutrons,'' D. Dubbers, Physica B/C **151**, 93–95 (1988). (I)
129. ''Measurement of Berry phase for noncyclic evolution,'' H. Weinfurter and G. Badurek, Phys. Rev. Lett. **64**, 1318–1321 (1990). (I)
130. ''Geometric phase experiments in optics—a unified description,'' R. Bhandari, Curr. Sci. **59**, 1159–1167 (1991). (I)
131. ''Observation of a nonclassical Berry phase for the photon,'' P. G. Kwiat and R. Y. Chiao, Phys. Rev. Lett. **66**, 588–591 (1991). (I)
132. ''Manifestations of Berry's phase in image-bearing optical beams,'' M. Segev, R. Solomon, and A. Yariv, Phys. Rev. Lett. **69**, 590–592 (1992). (A)
133. ''A geometric-phase interferometer,'' P. Hariharan and M. Roy, J. Mod. Opt. **39**, 1811–1815 (1992). (I)
134. ''The geometric phase—a simple optical demonstration,'' P. Hariharan, Am. J. Phys. **61**, 591–594 (1993). (E)
135. ''The geometric phase observations at the single-photon level,'' P. Hariharan, M. Roy, P. A. Robinson, and J. W. Obyrne, J. Mod. Opt. **40**, 871–877 (1993). (A)
136. ''Scheme for measuring a Berry phase in an atom interferometer,'' M. Reich, U. Sterr, and W. Ertmer, Phys. Rev. A **47**, 2518–2522 (1993). (A)
137. ''Non-cyclic geometric phases in a proposed two-photon interferometric experiment,'' J. Christian and A. Shimony, J. Phys. A **26**, 5551–5567 (1993). (I)
138. ''The geometric phase—interferometric observations with white-light,'' P. Hariharan, K. G. Larkin, and M. Roy, J. Mod. Phys. **41**, 663–667 (1994). (I)
139. ''Observation of a nonlocal Pancharatnam phase-shift in the process of induced coherence without induced emission,'' T. P. Grayson, J. R. Torgerson, and G. A. Barbosa, Phys. Rev. A **49**, 626–628 (1994). (A)
140. ''On measuring the Pancharatnam phase. I. Interferometry,'' A. G. Wagh and V. C. Rakhecha, Phys. Lett. A **197**, 107–111 (1995). (I)
141. ''On measuring the Pancharatnam phase. II. SU(2) polarimetry,'' A. G. Wagh and V. C. Rakhecha, Phys. Lett. A **197**, 112–115 (1995). (I)
142. ''Observation of nonadiabatic geometrical effects in a time-of-flight experiment with polarized neutrons,'' D. A. Korneev, V. I. Bodnarchuk, and L. S. Davtyan, Physica B **213**, 993–995 (1995). (I)