

# SYMMETRY AND SYMMETRY-BREAKING BIFURCATIONS IN FLUID DYNAMICS

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## 1. INTRODUCTION

The recognition that fluid-dynamical models can yield solutions with less symmetry than the governing equations is not new. Jacobi's discovery that a rotating fluid mass could have equilibrium configurations lacking rotational symmetry is a famous nineteenth-century example. In modern terminology, Jacobi's asymmetric equilibria appear through a symmetry-breaking bifurcation from a family of symmetric equilibria as the angular momentum (the "bifurcation parameter") increases above a critical value (the "bifurcation point"). Chandrasekhar (1969) gives a brief historical account of this discovery.

In this example, as in many others, the presence of symmetry breaking was discovered by solving specific model equations. In contrast to this specificity, it is widely recognized that symmetry-breaking bifurcations are of frequent occurrence in a variety of nonlinear, nonequilibrium physical settings—fluids, chemical reactions, plasmas, and biological systems, to

mention some diverse examples. It is worthwhile, therefore, to view symmetry breaking in fluids as representative of a ubiquitous phenomenon and to review a recent theoretical approach capable of distinguishing general properties of such bifurcations from specific quantitative features that inevitably vary among systems. In this approach, group theory is a key tool, permitting the effect of symmetry on a bifurcation to be disentangled from the detailed physics of the problem. In a parallel development, progress in multiparameter bifurcation theory has permitted detailed investigations of complicated transitions found in fluids. In this article, we survey these developments in a selective fashion that complements the coverage of the related reviews by Michel (1980), Stewart (1988), Ahlers (1989), and Newell (1989).

The basic techniques for analyzing transitions between different dynamical states are those of bifurcation theory. However, in its classical form for nonsymmetric systems, this theory often excludes "exceptional" cases that turn out to be "typical" in the presence of an appropriate symmetry. The evident need for a more general theory has led to the development of *equivariant bifurcation theory* as a distinct body of mathematics providing a more flexible and more powerful approach to analyzing symmetry-breaking bifurcations. (The term "equivariant" is used synonymously with "covariant" or, less precisely, "symmetric.") This development reflects the efforts of many mathematicians and may be followed in the books by Vanderbauwhede (1982), Sattinger (1983), and Golubitsky et al (1988) and the recent survey by Stewart (1988). For the applications-minded consumers of this new theory, there have been several benefits. First, there are now tools permitting a more systematic study of the role played by symmetry in restricting the range of dynamical behavior available to the system in a given transition. The dynamical equations are required only for certain specific model-dependent calculations that must be performed to distinguish among the allowed scenarios. A second important benefit of equivariant bifurcation theory is the identification of a growing number of novel dynamical phenomena whose existence is fundamentally related to the presence of symmetry, including rotating waves (Ruelle 1973), modulated waves (Renardy 1982, Rand 1982), slow "phase" drifts along directions of broken symmetry (Krupa 1990), stable heteroclinic cycles (Field 1980, Guckenheimer & Holmes 1988, Melbourne et al 1989), and symmetry-increasing bifurcations (Chossat & Golubitsky 1988). Applications now range from the Taylor-Couette system (Chossat & Iooss 1985, Golubitsky & Stewart 1986a, Chossat et al 1987, Golubitsky & Langford 1988), Tollmien-Schlichting waves (Barkley 1990), binary fluid convection (Knobloch et al 1986a, Knobloch 1986a), surface-wave interactions (Crawford et al 1989, 1990b, Silber & Knobloch 1989), and rotating fluid drops

(Lewis et al 1987) to viscous sedimentation (Golubitsky et al 1990) and a dynamical model of a turbulent boundary layer (Aubry et al 1988).

The aims of this article are twofold: (a) to describe equivariant bifurcation theory sufficiently to allow the reader to appreciate what is meant by a “model-independent” analysis of a symmetry-breaking bifurcation, and (b) to introduce some of the applications just mentioned to illustrate how such an analysis works in practice. In particular, our choice of applications is aimed at illustrating the utility of the “lattice of isotropy subgroups” as a conceptual organizing point for the analysis of mode interactions in symmetric systems. In the next section we review briefly the basic concepts in bifurcation theory. In Section 3 we describe the modifications that are necessary in the presence of symmetry and introduce some of the tools used to study bifurcation problems with symmetry. Both Hopf and steady-state bifurcations with symmetry are discussed in Section 4, followed in Section 5 by several examples of mode interactions. In Section 6 we describe the effects of imperfect symmetry on several important bifurcations. Examples from hydrodynamics are used throughout to illustrate the basic issues and to indicate how the techniques we describe can help to understand them.

## 2. FUNDAMENTALS

Bifurcation theory provides a way of introducing a small parameter into the analysis of nonlinear differential equations. This parameter is an internal one, and it specifies the proximity to a bifurcation value. A system, or a differential equation, depending on a set of parameters  $\lambda = \{\lambda_1, \lambda_2, \dots\}$  is said to undergo a bifurcation at  $\lambda = 0$  when a qualitative change in the dynamics takes place as  $\lambda$  passes through  $\lambda = 0$ . There are two types of bifurcation: local ones, recognized by a change in stability of a solution; and global ones, whose existence is not revealed by a local analysis (Guckenheimer & Holmes 1986, Wiggins 1989). Near a local bifurcation ( $0 \leq |\lambda| \ll 1$ ) a partial differential equation can, under certain circumstances discussed below, be reduced to a finite-dimensional system of ordinary differential equations describing the slow evolution of the modes that are close to marginal stability. The dimension of the reduced system is equal to the number of eigenvalues on the imaginary axis (counting multiplicity) at  $\lambda = 0$ . Consequently, the bifurcation analysis offers a dramatic simplification of the equation of motion. In systems depending on several parameters it is possible to arrange for several eigenvalues to lie on the imaginary axis simultaneously. Such a situation is called a *mode interaction*. For nearby parameter values, the resulting low-dimensional system then describes finite-dimensional but potentially complex dynamics

of the partial differential equations. A basic strategy of bifurcation theory is therefore to seek out multiple bifurcations in an attempt to get a finite-dimensional description of more complicated nonlinear behavior. To this end, it is often useful to vary even those parameters that cannot readily be changed in an experiment. In the same spirit, one seeks *degenerate bifurcations*, which arise when the coefficients of certain critical nonlinear terms vanish at  $\lambda = 0$ . In both cases this approach enables one to bring various secondary or subsidiary bifurcations down to small amplitudes, where they are accessible to bifurcation analysis. Empirically, one finds that the bifurcation behavior detected by such methods often persists for parameter values  $\lambda$  substantially different from  $\lambda = 0$  (cf Knobloch et al 1986b, Knobloch & Moore 1990).

Consider, first, bifurcation from an equilibrium. The construction of the amplitude equations requires that the number of eigenvalues on the imaginary axis is finite, with all remaining eigenvalues bounded away from it. In this case the center manifold theorem applies (Guckenheimer & Holmes 1986), and the analysis takes place in five steps:

1. The linear-stability problem is solved, and the (finite number of) marginally stable modes at  $\lambda = 0$  are identified; these marginal degrees of freedom are isolated by putting the linear terms into Jordan block form.
2. The nonlinear terms governing the time-asymptotic evolution of the marginally stable modes are computed by center manifold reduction.
3. Near-identity nonlinear coordinate changes are then performed to put these equations into a simple form called a *normal form*.
4. The normal form is *unfolded* by adding small linear and nonlinear terms describing the effects of varying  $\lambda$  away from  $\lambda = 0$ .
5. The unfolded system is then truncated at some order, and the resulting dynamics is analyzed; once the truncated system is understood, the effects of the neglected higher-order terms are considered.

At  $\lambda = 0$ , step (2) above results in a system of the form

$$\dot{a} = M(0)a + N(0, a), \quad (2.1a)$$

where  $M(0)$  is an  $n \times n$  matrix, all of whose eigenvalues lie on the imaginary axis, and  $N(0, a)$  denotes nonlinear terms. Here  $a$  is the (real or complex) vector of mode amplitudes for the critical modes, and  $a = 0$  corresponds to the original equilibrium. The normal-form transformations  $a = \Phi(a')$  alluded to in step (3) are used to simplify the form of  $N(0, a)$  and are constructed iteratively:  $\Phi(a') = a' + \Phi_2(a') + \Phi_3(a') + \dots$ , where  $\Phi_2(a')$  is chosen to eliminate as many quadratic terms in  $N(0, a)$  as possible,  $\Phi_3(a')$  as many of the cubic terms as possible, and so forth. The resulting form of Equation (2.1a) in  $a'$  variables is called a (Poincaré-Birkhoff)

normal form for the bifurcation; it need not be unique. There is an important subtlety in this procedure. The transformation  $\Phi$  representing the change of variables that puts the reduced system (2.1a) in normal form to all orders is a formal one, and the series for  $\Phi$  typically diverges. Thus, while we can specify which terms will not appear in the normal form for (2.1a), the required change of variables to remove them to all orders does not generally exist. Under these circumstances, one must settle for a transformation  $\Phi$  that takes (2.1a) to normal form up to some finite order. This finite-order approximation to the reduced system is then unfolded and analyzed, but the perturbing effect of the neglected higher order remainder must be considered in reaching final conclusions. A similar procedure applies to systems describing discrete time evolution, as in a periodically forced system analyzed using the “stroboscopic” map (cf Crawford et al 1989). Then (2.1a) is replaced by

$$a_{m+1} = M(0)a_m + N(0, a_m), \quad (2.1b)$$

where  $a_m$  represents the state of the system after  $m$  periods of the forcing, and all the eigenvalues of  $M(0)$  lie on the unit circle. Similar maps arise in the study of bifurcations from periodic oscillations (Guckenheimer & Holmes 1986).

The structure of the normal form depends on the nature of the eigenvalues of  $M(0)$  (Guckenheimer & Holmes 1986, Elphick et al 1987) and on the presence of any symmetries. The form of the unfolding additionally requires knowledge of whether or not  $a = 0$  is a solution for all  $\lambda$  near 0. Consequently, once the nature of the bifurcation problem is specified, the equations of motion and associated boundary conditions enter only into the determination of the normal-form coefficients. It is therefore possible and advantageous to analyze the normal-form dynamics first; such analysis reveals the range of dynamics associated with the bifurcation and also indicates which coefficients distinguish between various types of behavior. The calculation of these coefficients is inherently model dependent; explicit expressions for the coefficients in a normal form can only be derived after a specific model for the system has been selected. In many applications, the ability to construct a normal form without needing to explicitly calculate the coefficients is a considerable advantage (cf Barkley 1990).

An important feature of the normal-form transformations invoked above is that they preserve the dynamics. On the other hand, the normal form that one obtains typically involves an arbitrary truncation; in addition the unfolding procedure is not systematic. For problems where the bifurcating states may be viewed as steady states, Liapunov-Schmidt reduction (e.g. Golubitsky & Schaeffer 1985) can be employed instead, and the techniques of singularity theory can then be used to find normal

forms of finite order with a systematic construction of a “universal unfolding.” This is achieved by allowing a wider class of coordinate changes. In singularity theory the transformations preserve equilibria but do not in general preserve dynamics and hence the stability properties of equilibria. In many cases, however, stability can be deduced by nonsystematic means. For a recent application of Liapunov-Schmidt reduction to bifurcation from Couette flow in the Taylor-Couette system, see Golubitsky & Langford (1988).

In contrast to traditional amplitude expansions, both Poincaré-Birkhoff and singularity-theory normal-form techniques avoid an explicit scaling of the equations of motions and the necessity of specifying a priori the nature of the solution being sought, and they are also more systematic. In addition, stability relative to the competing critical modes is automatically taken into account, and in some cases the resulting description is valid in a neighborhood of the mode-interaction point rather than being merely asymptotic.

There are two important circumstances under which the center-manifold reduction does not apply, both of which arise in translation- and rotation-symmetric systems. Rotational symmetry introduces a continuum of marginally stable wavenumbers; consequently, the number of eigenvalues on the imaginary axis is uncountable. Although this difficulty does not arise in systems that are translation-symmetric only, the second difficulty—that there is a continuous band of unstable wavenumbers for  $\lambda > 0$ —remains. Thus at  $\lambda = 0$ , stable eigenvalues accumulate on the imaginary axis, and the time-scale separation required for the center manifold theorem is absent. Under these conditions, envelope equation methods become invaluable (Newell 1989). For the spatially periodic patterns considered below, these difficulties do not arise, however.

### 3. SYMMETRY AND BIFURCATION

Symmetries are ubiquitous in fluid dynamics and may arise either from the geometry of an experimental apparatus or through the assumptions made in a theoretical or numerical study of the system. A typical example of the latter, and one not generally appreciated, is provided by the use of periodic boundary conditions in translation-symmetric systems. The assumption of periodic boundary conditions introduces the group  $SO(2)$  of proper rotations of a circle into the analysis, or the group  $O(2)$  of rotations and reflections of a circle if the system is also symmetric under reflection. Other symmetries can be introduced by the adoption of Neumann or Dirichlet boundary conditions (Dangelmayr & Armbruster 1986, Crawford et al 1990a). Mathematically what matters are the symmetries

of (2.1) or, more precisely, the symmetries of its normal form. Suppose the unfolded normal form

$$\dot{a} = V(\lambda, a) \equiv M(\lambda)a + N(\lambda, a) \tag{3.1}$$

has symmetry  $\Gamma$ . The group  $\Gamma$  may be represented by matrices acting on the amplitude variables  $a$ . For the groups we consider, there is no loss of generality in assuming that these matrices are orthogonal, i.e.  $|\det \gamma| = 1$  for all  $\gamma \in \Gamma$ . For example, if  $\Gamma$  is  $SO(2)$  and  $a \in \mathbf{R}^2$ , we have the standard representation by rotation matrices,

$$R(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \tag{3.2}$$

acting on the plane. We say that the vector field  $V(\lambda, a)$  has symmetry  $\Gamma$  if

$$\gamma \cdot V(\lambda, a) = V(\lambda, \gamma \cdot a) \tag{3.3}$$

for each element  $\gamma \in \Gamma$  of the group. Here  $\gamma \cdot a$  denotes multiplication of the vector  $a$  by the appropriate matrix  $\gamma$ . Thus if  $a(t)$  solves (3.1), so does  $\gamma \cdot a(t)$ .

Symmetric systems typically exhibit more complicated bifurcations than nonsymmetric systems. For example, symmetry may force two eigenvalues to cross the imaginary axis simultaneously as a single parameter is varied; without symmetry, the same phenomenon would require the judicious variation of two parameters. In addition, symmetry may force the absence of certain terms in the amplitude equations, introducing degeneracies into the bifurcation analysis. Thus, symmetries typically force both multiple and degenerate bifurcations. At the same time, however, group theory helps in studying the resulting dynamics and in so doing renders the complex behavior more accessible to analysis than in a corresponding nonsymmetric problem.

### 3.1 Examples

The following three simple bifurcations illustrate some of the ways that symmetry may enter a bifurcation problem:

**STEADY-STATE BIFURCATION WITH REFLECTION SYMMETRY** Consider two-dimensional Bénard convection with identical boundary conditions at rigid sidewalls  $x = \pm L/2$  in the horizontal direction, and let  $\lambda = 0$  represent the Rayleigh number at the onset of convection. The fluid equations are symmetric with respect to spatial reflection  $\kappa: x \rightarrow -x$ , and the boundary conditions and the basic conduction state are both invariant under reflection; hence, the onset of convection leads to a bifurcation problem with

reflection symmetry. We denote this two-element group by  $\Gamma = Z_2(\kappa)$ . Let  $\theta(x, t)$  be the scalar field describing the deviation of the temperature from pure conduction, and suppose that the convective instability corresponds to a single mode  $f(x)$  with a zero eigenvalue at criticality. We suppress the dependence on the vertical coordinate on the assumption that there are no additional symmetries associated with the vertical direction. Then near  $\lambda = 0$ ,  $\theta(x, t)$  takes the form

$$\theta(x, t) = a(t)f(x) + (\text{remaining stable modes}), \tag{3.4}$$

where

$$\dot{a} = V(\lambda, a), \tag{3.5}$$

and  $V(\lambda, a)$  is a real-valued function such that

$$V(\lambda, 0) = 0 \quad \text{and} \quad \frac{\partial V}{\partial a}(0, 0) = 0. \tag{3.6}$$

Since  $V(\lambda, a)$  may be derived by a procedure that preserves the symmetries of the original problem, it will satisfy (3.3) with respect to the appropriate representation of the reflection symmetry:

$$(\kappa\theta)(x, t) = \theta(-x, t). \tag{3.7}$$

The effect of  $\kappa$  on the marginal convective mode determines this representation. The normal modes can be written as eigenfunctions of  $\kappa$ , and since  $\kappa^2 = 1$ , there are two possibilities for  $f(x)$ :

$$(\kappa f)(x) = f(x) \quad (\text{even}) \tag{3.8}$$

or

$$(\kappa f)(x) = -f(x), \quad (\text{odd}). \tag{3.9}$$

When  $f$  is even, the bifurcation does not break the reflection symmetry. In this case the term  $a(t)f(x)$  in (3.4) is invariant with respect to  $\kappa$ , and consequently the induced group-transformation law for  $a(t)$  in (3.5) is trivial:

$$\kappa \cdot a = a. \tag{3.10}$$

When  $f$  is odd, the symmetry is broken and we have  $\kappa(a(t)f(x)) = -a(t)f(x)$ ; the transformation law for  $a(t)$  in (3.5) is now nontrivial:

$$\kappa \cdot a = -a. \tag{3.11}$$

If (3.10) is the correct group action, the symmetry (3.3) imposes no



constraints and one expects  $V(\lambda, a)$  to satisfy the nondegeneracy conditions

$$\frac{\partial^2 V}{\partial \lambda \partial a}(0, 0) \neq 0, \quad \frac{\partial^2 V}{\partial a^2}(0, 0) \neq 0; \tag{3.12}$$

this convective instability is a transcritical bifurcation. If, however, the symmetry is broken, then equivariance with respect to (3.11) requires that

$$V(\lambda, -a) = -V(\lambda, a); \tag{3.13}$$

hence  $V(\lambda, a) = h(\lambda, a^2)a$ , and the second condition in (3.12) fails. Now one expects a nondegenerate pitchfork bifurcation provided that

$$\frac{\partial^3 V}{\partial a^3}(0, 0) \neq 0 \tag{3.14}$$

holds. In this latter case there are two new branches of symmetry-breaking states that bifurcate from  $a = 0$  at  $\lambda = 0$ . These are interchanged by reflection and describe equilibria with upflow at one wall and downflow at the other; when  $f$  is even, either upflow or downflow at both walls is preferred.

**STEADY-STATE BIFURCATION WITH CIRCULAR SYMMETRY** Consider next the same example but now with periodic boundary conditions in the horizontal direction. In particular, let us assume that  $\theta(x+L) = \theta(x)$ . This changes the symmetry of the problem from  $Z_2(\kappa)$  to  $O(2)$  and modifies the primary bifurcation—with periodic boundary conditions the convective instability cannot break the reflection symmetry.

In addition to the reflection symmetry  $\kappa$ , the problem is now symmetric with respect to translations  $T_l: x \rightarrow x+l \pmod L$ . As a consequence, at  $\lambda = 0$  there are two linearly independent convective modes. If we suppress inessential variables as before, these take the form  $f_1(x) \sim e^{ikx}$  and  $f_2(x) \sim e^{-ikx}$ . The corresponding eigenvalues must be equal because the eigenfunctions are related by the symmetry

$$(\kappa f_1)(x) = e^{ik(-x)} = f_2(x). \tag{3.15}$$

Provided  $k \neq 0$ , the  $O(2)$  symmetry forces the critical eigenvalue to have multiplicity two. The modal expansion (3.4) now becomes

$$\theta(x, t) = \frac{1}{2}[a(t)e^{ikx} + \bar{a}(t)e^{-ikx}] + (\text{remaining stable modes}). \tag{3.16}$$

As before, the group action for the reduced equation  $\dot{a} = V(\lambda, a)$  is determined from the transformation properties of the linear eigenfunctions  $e^{\pm ikx}$  in (3.16):

$$\begin{aligned} \kappa \cdot a &= \bar{a}, \\ T_l \cdot a &= e^{ikl} a. \end{aligned} \tag{3.17}$$

Note that introducing real coordinates ( $a = a_r + ia_i$ ) converts (3.17) into a matrix representation of  $O(2)$  on the  $(a_r, a_i)$  plane in which  $T_l$  has the form (3.2) with  $\theta = kl$  and

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.18}$$

The symmetry of  $V(\lambda, a)$  now requires

$$\begin{aligned} \overline{V(\lambda, a)} &= V(\lambda, \bar{a}), \\ e^{ikl} V(\lambda, a) &= V(\lambda, e^{ikl} a), \end{aligned} \tag{3.19}$$

and so  $V(\lambda, a) = h(\lambda, |a|^2)a$ , where  $h(\lambda, r)$  is a real-valued function satisfying  $h(0, 0) = 0$ . In the reduced system, new equilibria are found by solving  $h(\lambda, |a|^2) = 0$  near  $(\lambda, |a|^2) = (0, 0)$ . The solution,

$$|a|^2 \simeq \left( \frac{-\lambda}{\frac{\partial h}{\partial r}(0, 0)} \right), \quad \frac{\partial h}{\partial r}(0, 0) \neq 0, \tag{3.20}$$

describes a family of new equilibria parametrized by the phase of the amplitude  $a$ . Although the bifurcation breaks the translational symmetry, each member of this family still possesses a reflectional symmetry.

Note that if only translation symmetry were present, then periodic boundary conditions would lead to an  $SO(2)$ -symmetric bifurcation problem. The convective modes  $e^{\pm ikx}$  persist but are no longer related by (3.15); hence, their eigenvalues are no longer forced to be equal by symmetry. Indeed, when  $k \neq 0$ , pure  $SO(2)$  symmetry typically results in complex-conjugate pairs of eigenvalues [cf (3.26)], and Ruelle (1973) has pointed out that  $SO(2)$ -breaking instabilities typically lead to rotating waves.

**HOPF BIFURCATION WITHOUT SYMMETRY** Even when (3.1) lacks symmetry, it is often the case that the normal form appropriate to the problem is symmetric. Indeed it is this property that makes the study of a normal form simpler (Elphick et al 1987, Stewart 1988). A familiar example arises when the loss of stability occurs as a result of “overstability” or Hopf bifurcation. At  $\lambda = 0$  there is now a pair of purely imaginary eigenvalues  $\pm i\omega_0$ , with all other eigenvalues bounded away from the imaginary axis. The linear problem has the solution  $\text{Re}\{a(t)f(x)\}$ , where  $\dot{a} = i\omega_0 a$ . The normal-form transformation  $a = \Phi(a')$  followed by unfolding shows that

the nonlinear equation for  $\dot{a}$  near  $\lambda = 0$  can always be put into the form (Guckenheimer & Holmes 1986)

$$\begin{aligned} \dot{r} &= g(\lambda, r^2)r, \\ \dot{\phi} &= \Omega(\lambda, r^2), \end{aligned} \tag{3.21}$$

where  $a' = re^{i\phi}$ ,  $g(0, 0) = 0$ , and  $\Omega(0, 0) = \omega_0$ . Equations (3.21) exhibit the *phase-shift symmetry*  $\phi \rightarrow \phi + \phi_0$ , denoted by  $S^1$  to distinguish it from the spatial symmetry  $SO(2)$  discussed above. Recall that the transformation  $\Phi$  is a formal one, so that the phase-shift symmetry of (3.21) will not in general reflect an exact symmetry of  $V(\lambda, a)$  in the original variables.

### 3.2 Symmetry and Linear Theory

We return to the question of how a symmetry  $\Gamma$  affects the transitions in the dynamics of (3.1). In considering the consequences of (3.3) it is helpful to first describe the effects on the linear terms in the problem. By differentiating (3.3) with respect to  $a$  and then setting  $a = 0$ , we obtain the condition

$$\gamma \cdot M(\lambda) = M(\lambda) \cdot \gamma, \quad \gamma \in \Gamma, \tag{3.22}$$

on the matrix  $M(\lambda)$  in (3.1); the fact that  $M(\lambda)$  must commute with the matrices of the group constrains the form of  $M(\lambda)$  and the properties of the marginal eigenvalues. These consequences of (3.3) for the linear problem are well understood by virtue of standard results in group-representation theory. We briefly discuss here the relevant conclusions, referring the reader to Golubitsky et al (1988) for further details.

An eigenvalue  $\sigma(\lambda)$  of  $M(\lambda)$  determines an associated linear eigenspace  $E_\sigma \subset \mathbf{R}^n$  whose definition depends on whether  $\sigma(\lambda)$  is real or complex:

$$\begin{aligned} \sigma(\lambda) \in \mathbf{R}, \quad E_\sigma &\equiv \{w \in \mathbf{R}^n \mid [M(\lambda) - \sigma(\lambda)I] \cdot w = 0\}, \\ \sigma(\lambda) \notin \mathbf{R}, \quad E_\sigma &\equiv \{w \in \mathbf{R}^n \mid [M(\lambda) - \overline{\sigma(\lambda)}I][M(\lambda) - \sigma(\lambda)I] \cdot w = 0\}, \end{aligned} \tag{3.23}$$

where  $I$  is the identity matrix. When  $\sigma(\lambda)$  is real, the corresponding eigenvectors span  $E_\sigma$ ; when  $\sigma(\lambda)$  is complex, the eigenvectors are complex and their real and imaginary parts span  $E_\sigma$ . In both cases, if the eigenvalue happens to have a larger algebraic multiplicity than geometric multiplicity, the definition of  $E_\sigma$  must be appropriately enlarged to include the generalized eigenvectors. When  $M(\lambda)$  satisfies (3.22),  $E_\sigma$  is left invariant by  $\Gamma$ —i.e. for a vector  $w \in E_\sigma$  and a symmetry operation  $\gamma \in \Gamma$ , the transformed vector  $\gamma \cdot w$  still belongs to  $E_\sigma$ . By virtue of this invariance we can discuss the matrix representation of  $\Gamma$  obtained by restricting to vectors in  $E_\sigma$ . This restriction allows us to associate the eigenvalue  $\sigma$  with an irreducible

representation of the group  $\Gamma$ ; the nature and implications of this association comprise the constraints that (3.22) imposes on  $M(\lambda)$ .

A matrix representation of  $\Gamma$  on  $\mathbf{R}^n$  is *irreducible* if the only proper subspace of  $\mathbf{R}^n$  left invariant by  $\Gamma$  is the origin  $\{0\}$ . In the example (3.2) of  $SO(2)$  on  $\mathbf{R}^2$ , the representation is clearly irreducible: excepting  $\{0\}$ , proper subspaces of  $\mathbf{R}^2$  are lines through the origin, and no such line is left invariant by all rotations. By contrast, the representation of the reflection group  $Z_2(\kappa)$  on  $\mathbf{R}^2$  given by (3.18) is not irreducible, since the  $a_r$ -axis and the  $a_i$ -axis are each left invariant. Thus we can decompose  $\mathbf{R}^2$  into these two  $Z_2(\kappa)$ -invariant subspaces:

$$\mathbf{R}^2 = (a_r, 0) \oplus (0, a_i). \quad (3.24)$$

In this case, by restricting the  $Z_2$  action to each of these subspaces we obtain one-dimensional irreducible representations of  $Z_2(\kappa)$ . These are distinct (“nonisomorphic”), since on  $(a_r, 0)$  the action of  $\kappa$  is multiplication by  $+1$  and on  $(0, a_i)$  we multiply by  $-1$ .

The decomposition in (3.24) is a particularly simple example of a general result—namely, that any reducible representation can be decomposed into a “sum” of irreducible representations by simply breaking invariant subspaces into smaller and smaller invariant subspaces until only irreducible components remain. For the (compact Lie) groups that we consider, the irreducible representations that can arise in this way are well understood (Golubitsky et al 1988, Bröcker & tom Dieck 1985). This decomposition is very useful for analyzing the stabilities of the new asymmetric states that arise when the bifurcation breaks the original symmetry (see below). Finally, it is useful to characterize an irreducible representation by the type of matrices that commute with the representation [i.e. satisfy (3.22)]. A representation is *absolutely irreducible* if (a) it is irreducible and (b) Equation (3.22) forces  $M(\lambda)$  to have the diagonal form

$$M(\lambda) = m(\lambda)I, \quad m(\lambda) \in \mathbf{R}. \quad (3.25)$$

For example, the representation of  $O(2)$  in (3.17) is absolutely irreducible, but the irreducible representation of  $SO(2)$  in (3.2) is not.

When we restrict the symmetry matrices  $\gamma$  to act on vectors in an eigenspace  $E_\sigma$ , we obtain a representation of  $\Gamma$  by  $n \times n$  matrices, where  $n$  is the dimension of  $E_\sigma$ . The character of this  $n$ -dimensional representation depends on whether the eigenvalue  $\lambda$  is real or complex. When  $\sigma$  is real, the representation carried by  $E_\sigma$  is typically an absolutely irreducible representation of  $\Gamma$  (Golubitsky et al 1988, Prop. 3.2). If the absolutely irreducible representation of  $\Gamma$  entering the bifurcation problem in this way has dimension greater than one, then the critical eigenvalue  $\sigma$  is forced to have a multiplicity equal to the dimension of the representation [see (3.25)].

This mechanism for producing degenerate eigenvalues is well known in atomic physics, and it is the most important consequence that a symmetry has for the linear structure of a bifurcation problem. This phenomenon has already appeared in our discussion of steady-state bifurcation with circular symmetry. In that problem, the critical zero eigenvalue has multiplicity two, since the eigenvectors  $e^{ikx}$  and  $e^{-ikx}$  are linearly independent. Within the context of circular symmetry, this degeneracy is unavoidable, since the two-dimensional representation in (3.17) on the eigenspace is absolutely irreducible for  $k \neq 0$ .

When the eigenvalue  $\sigma$  is complex, there are typically two possibilities for the  $n$ -dimensional representation on  $E_\sigma$ . If  $E_\sigma$  transforms irreducibly under  $\Gamma$ , then the representation is necessarily irreducible but not absolutely irreducible, since the restriction of  $M(\lambda)$  to  $E_\sigma$  is not of the form (3.25). The second possibility is that  $E_\sigma$  is not irreducible; in this case, one expects a decomposition of the form  $E_\sigma = V \oplus W$ , where each of the subspaces  $V$  and  $W$  transforms according to the same absolutely irreducible representation of  $\Gamma$  (Golubitsky & Stewart 1985). The first possibility is illustrated by the rotation symmetry  $SO(2)$ . For this group, all (nontrivial) irreducible representations are of the form (3.2), and the eigenspaces  $E_\sigma$  encountered when  $V(\lambda, a)$  has rotational symmetry are typically two dimensional. Upon restriction to  $E_\sigma$ , the matrix  $M(\lambda)$  has the form

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \tag{3.26}$$

with eigenvalues  $\mu \pm i\omega$ . The second possibility is illustrated by  $O(2)$ -symmetric Hopf bifurcation, which we discuss in Section 4.1.

### 3.3 Symmetry Breaking and Stability

When an equilibrium undergoes a symmetry-breaking bifurcation, new fluid states appear that have less symmetry and frequently more complicated dynamics. The loss of symmetry is manifested by the appearance of a new pattern. A theory of this process should address various fundamental questions; for example,

1. What forms of symmetry breaking are possible?
2. Which patterns actually occur, and which will be stable?
3. How does the symmetry of a pattern constrain the dynamics?

Complete answers to such questions require a nonlinear analysis; nevertheless, the extent to which linear theory shapes these answers is significant because linear theory (e.g. the linear eigenfunctions of the critical modes) specifies how the group acts on the amplitudes of the critical modes, and

it is this action in (3.3) that determines how the symmetry influences the center manifold dynamics. Recent developments in equivariant bifurcation theory have improved our ability to use group theory in the analysis of these questions. In this section we review several of the group-theoretic concepts that have proved particularly effective in applications. For a somewhat different approach, employing crystallographic terminology, see McKenzie (1988).

**GENERATORS FOR INVARIANTS AND EQUIVARIANTS** The implications of (3.3) for the linear terms in  $V(\lambda, a)$  were discussed in the previous section. The constraints imposed on the nonlinear terms are also important and can qualitatively change the behavior expected from the system. For example, when the action of  $\kappa$  in our discussion of steady-state bifurcation with reflection symmetry was nontrivial, one of the nondegeneracy conditions (3.12) failed, and a pitchfork bifurcation replaced a transcritical bifurcation.

There is a systematic way to formulate the amplitude equations (3.1) that accounts for symmetries automatically. This formulation requires one to determine a set of “generators” for invariant functions and equivariant vector fields (Golubitsky et al 1988). A symmetric or equivariant vector field satisfies (3.3), whereas an invariant function  $g(\lambda, a)$  satisfies

$$g(\lambda, a) = g(\lambda, \gamma \cdot a) \quad (3.27)$$

for all  $\gamma \in \Gamma$ . For a given representation of a compact Lie group, there always exists a finite set of invariant polynomials  $\{u_1(a), u_2(a), \dots, u_k(a)\}$  such that any smooth invariant function  $g(\lambda, a)$  may be expressed in terms of this generating set:

$$g(\lambda, a) = h(\lambda, u_1(a), u_2(a), \dots, u_k(a)), \quad (3.28)$$

where  $h(\lambda, x_1, x_2, \dots, x_k)$  is an appropriately chosen function of  $(k+1)$  arguments. Similarly, a finite set of equivariant polynomial vector fields  $\{V_1(a), V_2(a), \dots, V_m(a)\}$  can be determined such that any equivariant vector field (3.3) may be described in terms of  $m$  appropriately chosen invariant functions  $\{g_1(\lambda, a), g_2(\lambda, a), \dots, g_m(\lambda, a)\}$ :

$$V(\lambda, a) = \sum_{j=1}^m g_j(\lambda, a) V_j(a). \quad (3.29)$$

Each  $g_j(\lambda, a)$  is in turn expressible in terms of the invariant generators (3.28). In practice, writing the amplitude equations in this fashion is an efficient way to explicitly account for the constraints on  $V(\lambda, a)$  due to (3.3). For example, when  $\Gamma = Z_2(\kappa)$  acts on  $a \in \mathbf{R}$  by (3.11), the invariant functions are generated by

$$u_1(a) = a^2, \tag{3.30}$$

and the equivariants are generated by

$$V_1(a) = a. \tag{3.31}$$

From (3.28) and (3.29), an arbitrary reflection-symmetric vector field (3.13) may be represented as

$$V(\lambda, a) = h(\lambda, a^2)a \tag{3.32}$$

for an appropriately chosen  $h$ . The action (3.17) of  $O(2)$  on  $\mathbf{R}^2$  provides a second example. In this case,

$$u_1(a) = |a|^2 \tag{3.33}$$

generates the invariants and

$$V_1(a) = a \tag{3.34}$$

generates the equivariants. The resulting representation,

$$V(\lambda, a) = h(\lambda, |a|^2)a, \tag{3.35}$$

has already been introduced following (3.19).

**LATTICE OF ISOTROPY SUBGROUPS** The forms of possible symmetry breaking are specified by determining which subgroups of  $\Gamma$  can describe the remaining symmetry of the new fluid states. The points  $a \in \mathbf{R}^n$  in (3.1) represent time-asymptotic (and possibly time-dependent) configurations of the fluid. The symmetry of a given configuration  $a$  is measured by its *isotropy subgroup*  $\Sigma_a \subseteq \Gamma$ , given by

$$\Sigma_a \equiv \{\gamma \in \Gamma \mid \gamma \cdot a = a\}. \tag{3.36}$$

Obviously if  $a = 0$ , corresponding to the original fluid equilibrium, then  $\Sigma_0 = \Gamma$ , but for  $a \neq 0$ ,  $\Sigma_a$  is usually a proper subgroup of  $\Gamma$ . By calculating  $\Sigma_a$  for all  $a \in \mathbf{R}^n$ , we determine the possible ways that the symmetry  $\Gamma$  can break. In practice this is accomplished by considering not the individual points but rather their *group orbits*. The group orbit of  $a$  consists of all points  $a'$  related to  $a$  by a symmetry transformation—that

$$a' = \gamma \cdot a \tag{3.37}$$

for some  $\gamma \in \Gamma$ . If  $a$  and  $a'$  are related in this way, then their isotropy subgroups are conjugate to one another, i.e.

$$\Sigma_a = \gamma^{-1} \Sigma_{a'} \gamma. \tag{3.38}$$

Thus, the possibilities of symmetry breaking can be classified, up to conjugacy, by determining how the group orbits of  $\Gamma$  partition  $\mathbf{R}^n$  and then

calculating the isotropy subgroup for only one representative point of each distinct orbit. The resulting hierarchy of subgroups forms the *lattice of isotropy subgroups*. The action of  $O(2)$  on  $\mathbf{R}^2$  in (3.19) provides a simple example of this construction. The origin  $a = 0$  determines a group orbit of a single point, and the isotropy of this point is  $O(2)$ . Any other point  $a_0 \neq 0$  determines a group orbit that is a circle of radius  $|a_0|$ :

$$O(2) \text{ orbit of } a_0 = \{a \in \mathbf{R}^2 \mid |a| = |a_0|\}. \tag{3.39}$$

If we take  $(a_r, a_i) = (|a_0|, 0)$  to be the representative point for these orbits, then each representative has an isotropy subgroup given by the reflection  $\kappa$ :

$$\Sigma_{(|a_0|, 0)} = Z_2(\kappa). \tag{3.40}$$

Hence every nonzero point has a remaining symmetry that is simply  $Z_2(\kappa)$  up to conjugacy. The resulting lattice in Figure 1 contains only one proper subgroup. This is a group-theoretic derivation of our previously stated conclusion that the new equilibria (3.20) break the translation symmetry but not the reflection symmetry.

**FIXED-POINT SUBSPACES** The isotropy subgroups specify the possible patterns consistent with the symmetry group  $\Gamma$ . To know if these patterns can in fact occur requires that we determine if there are solutions to the equations of motion (3.1) with a given isotropy  $\Sigma$ . Knowing in advance that solutions with symmetry  $\Sigma$  are sought simplifies the task of solving the equations, because it is often possible to work in a lower dimensional setting than the full center manifold (3.1). This setting is the *fixed-point subspace*  $\text{Fix}(\Sigma)$  for the isotropy subgroup of interest, defined by

$$\text{Fix}(\Sigma) \equiv \{a \in \mathbf{R}^n \mid \sigma \cdot a = a \text{ for all } \sigma \in \Sigma\}. \tag{3.41}$$

If the symmetry group of a point  $a \in \mathbf{R}^n$  is large enough to include  $\Sigma$ , then this point will belong to  $\text{Fix}(\Sigma)$ . Most importantly,  $\text{Fix}(\Sigma)$  is a linear space and it is invariant under the dynamics of (3.1). Linearity follows from the assumption that the group  $\Gamma$  acts linearly on  $\mathbf{R}^n$ . If  $a, a' \in \text{Fix}(\Sigma)$  are two



Figure 1 Isotropy lattice for  $O(2)$  on  $\mathbf{R}^2$ . The arrow indicates that  $Z_2(\kappa)$  is contained in  $O(2)$ .



points in  $\text{Fix}(\Sigma)$ , then  $a+a' \in \text{Fix}(\Sigma)$  also; to check this, we simply note that for  $\sigma \in \Sigma$ ,

$$\sigma \cdot (a+a') = \sigma \cdot a + \sigma \cdot a' = a+a'. \tag{3.42}$$

Hence  $a+a' \in \text{Fix}(\Sigma)$  by (3.41). The dynamical invariance of  $\text{Fix}(\Sigma)$  follows from the symmetry of  $V(\lambda, a)$  in (3.3). If  $a \in \text{Fix}(\Sigma)$ , then

$$\sigma V(\lambda, a) = V(\lambda, \sigma \cdot a) = V(\lambda, a) \tag{3.43}$$

for all  $\sigma \in \Sigma$ ; thus,  $V(\lambda, a)$  belongs to  $\text{Fix}(\Sigma)$  if  $a$  does, and the trajectory given by the time evolution of  $a$  remains in  $\text{Fix}(\Sigma)$ . Hence, to find solutions with symmetry  $\Sigma$ , we may restrict (3.1) to  $\text{Fix}(\Sigma)$ ,

$$\dot{a} = V(\lambda, a), \quad a \in \text{Fix}(\Sigma) \subseteq \mathbf{R}^n, \tag{3.44}$$

and analyze the solutions to this restricted dynamical system. For example, to study solutions with  $\Sigma = Z_2(\kappa)$  in the problem of steady-state bifurcation with  $O(2)$  symmetry (3.19), we note from (3.41) that

$$\text{Fix}(Z_2(\kappa)) = \{a \in \mathbf{R}^2 \mid a = \kappa \cdot a = \bar{a}\}; \tag{3.45}$$

hence, it is sufficient to restrict  $V(\lambda, a)$  to the real axis:

$$\dot{a}_r = V(\lambda, a_r), \quad a_r \in \text{Fix}(Z_2(\kappa)) = \mathbf{R}. \tag{3.46}$$

This reduces the problem to one dimension, and in the notation of (3.20) nontrivial equilibria with  $Z_2(\kappa)$  symmetry are given by

$$a_r = \pm \left( \frac{-\lambda}{\frac{\partial h}{\partial r}(0,0)} \right)^{1/2}. \tag{3.47}$$

Note that these two solutions are related by a rotation and so lie on the same group orbit.

**ISOTYPIC DECOMPOSITIONS** If new solutions are found in  $\text{Fix}(\Sigma)$ , then it is important to determine their stabilities. For equilibria of differential equations or periodic cycles (fixed points, two-cycles, three-cycles, etc) of discrete maps, the symmetry  $\Sigma$  of the solution can be exploited to simplify the calculation of the eigenvalues governing stability. The idea is the same for continuous or discrete systems, so we consider only (3.1). If  $a_0 \in \text{Fix}(\Sigma)$  is an equilibrium for (3.1),

$$V(\lambda, a_0) = 0, \tag{3.48}$$

then its linear stability is determined by the eigenvalues of the Jacobian

matrix  $DV(\lambda, a_0)$  evaluated at  $a = a_0$ . This matrix must commute with  $\Sigma$ , i.e.

$$\sigma \cdot DV(\lambda, a_0) = DV(\lambda, a_0) \cdot \sigma \tag{3.49}$$

for all  $\sigma \in \Sigma$ , by the same reasoning that led to (3.22). The previous discussion in Section 3.2 applies to (3.49), with  $\Sigma$  replacing  $\Gamma$  as the relevant group. However, for  $\Gamma$  in (3.22), the representation is likely to be irreducible, since  $M(\lambda)$  often represents a single eigenvalue; this is not usually the case in (3.49), since we are linearizing about a bifurcating solution of lower symmetry. Indeed, if  $\text{Fix}(\Sigma)$  is a proper subspace of  $\mathbf{R}^n$ , then the representation of  $\Sigma$  must be reducible. The decomposition of  $\mathbf{R}^n$  into irreducible subspaces for  $\Sigma$ , as in (3.24), provides useful information about the eigenvalues of  $DV(\lambda, a_0)$  and thus about the stability of  $a_0$ . Let

$$\mathbf{R}^n = X_1 \oplus X_2 \oplus \dots \oplus X_m \tag{3.50}$$

denote the decomposition of  $\mathbf{R}^n$  into irreducible subspaces for the representation of  $\Sigma$  in (3.49)—that is, each subspace  $X_i$  ( $i = 1, \dots, m$ ) transforms under an irreducible representation of  $\Sigma$ . It may happen that two or more of these subspaces transform according to the same irreducible representation; if this occurs, we combine all the subspaces belonging to the same representation. With this adjustment we have the *isotypic decomposition* of  $\mathbf{R}^n$ :

$$\mathbf{R}^n = W_1 \oplus W_2 \oplus \dots \oplus W_d \quad (d \leq m), \tag{3.51}$$

where  $W_j$  contains all the subspaces in (3.50) that transform by the  $j$ th irreducible representation of  $\Sigma$ . The property that makes this latter decomposition useful is

$$DV(\lambda, a_0): W_j \rightarrow W_j, \quad j = 1, \dots, d, \tag{3.52}$$

i.e.  $DV(\lambda, a_0)$  carries each subspace  $W_j$  into itself. Consequently, if we calculate the stability matrix using coordinates given by the  $W_j$ , then  $DV(\lambda, a_0)$  will appear in block-diagonal form. As an illustration, let  $a_0$  represent a  $Z_2(\kappa)$ -symmetric equilibrium in (3.20). The irreducible decomposition of  $\mathbf{R}^2$  by  $Z_2(\kappa)$  is given in (3.24); since each of the irreducible representations is distinct, this is also the isotypic decomposition of  $\mathbf{R}^2$ . Thus, if we use coordinates  $(a_r, a_i)$  for  $\mathbf{R}^2$ , the matrix  $DV(\lambda, a_0)$  will be in diagonal form (since here each  $W_j$  is one dimensional):

$$DV(\lambda, a_0) \cdot a = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} a_r \\ a_i \end{pmatrix}. \tag{3.53}$$

From (3.53) it follows that the stability eigenvalues of  $a_0$  are both real,

which precludes a secondary Hopf bifurcation. Furthermore, we may calculate them if desired without needing to explicitly evaluate the off-diagonal terms. For a  $2 \times 2$  matrix, such elaborate considerations are wholly unnecessary, but in more complicated bifurcations the simplification obtained by choosing coordinates according to the isotypic decomposition can be decisive in the calculation of stability.

**GROUP ORBITS OF SOLUTIONS** If  $a(t)$  solves (3.1), then for each symmetry transformation  $\gamma \in \Gamma$ ,  $\gamma \cdot a(t)$  is also a solution. Whenever the state  $a(t)$  does not exhibit the full symmetry, then the resulting group orbit of  $a(t)$ ,

$$\Gamma \cdot a(t) = \{ \gamma \cdot a(t) | \gamma \in \Gamma \}, \tag{3.54}$$

is a family of solutions that have the same stability properties and similar dynamics.

If  $a(t)$  breaks a continuous symmetry, then the group orbit  $\Gamma \cdot a(t)$  will have continuous components. This forces  $a(t)$  to have directions of neutral linear stability. For example, the  $Z_2(\kappa)$ -symmetric equilibrium  $a_r$  produced in an  $O(2)$  steady-state bifurcation [see (3.47)] breaks the continuous symmetry  $T_1$  [Equation (3.17)]. This equilibrium forms a circle of equilibria,  $T_i \cdot a_r = e^{ikl} a_r$ , under translations  $T_i$ ; this feature forces the direction along the circle, i.e.

$$\frac{d}{dt}(T_1 \cdot a_r)|_{t=0} = ika_r, \tag{3.55}$$

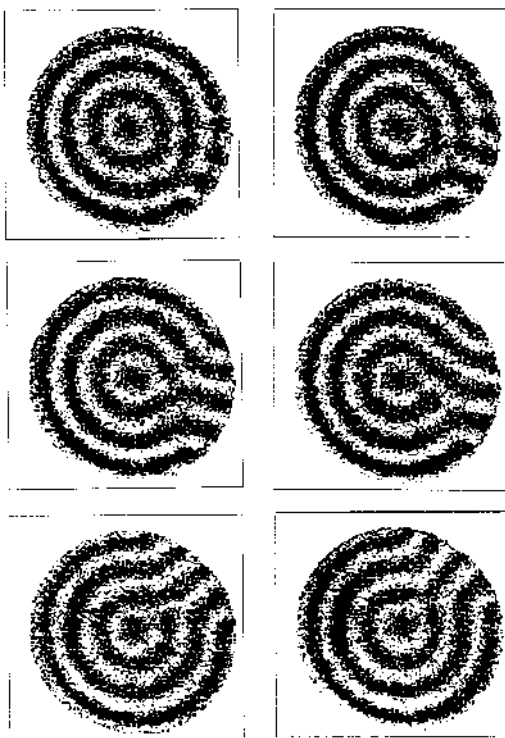
to be an eigenvector with eigenvalue zero. Note that since the vector  $ia_r$  points in the  $a_i$  direction, it corresponds to the eigenvalue  $\mu_2$  in (3.53), i.e.  $\mu_2 = 0$ .

Since the direction along the group orbit corresponds to marginal stability, subsequent instabilities may lead to nontrivial dynamics in this direction. In the present example a subsequent steady-state bifurcation breaking the remaining  $Z_2(\kappa)$  symmetry of the equilibrium will generically lead to new states called traveling (or rotating) waves that drift either clockwise or counterclockwise in the direction of the group orbit. Their drift speed varies as the square root of the distance from the  $Z_2(\kappa)$ -symmetry-breaking bifurcation. Note that in these circumstances a steady-state bifurcation results in time dependence. Examples of this behavior are provided by the bifurcation from steady convection rolls to traveling-wave convection (see below), and by the bifurcation from standing waves to modulated waves. An additional illustration is provided by a study of Steinberg et al (1985) of convection in a circular container. As the Rayleigh number is raised, a pattern of steady concentric rolls shifts off center, breaking  $O(2)$  symmetry. It remains  $Z_2$  symmetric until a defect spontaneously develops and the

pattern starts to rotate. When the defect heals, restoring  $Z_2$  symmetry, the pattern stops rotating (see Figure 2). Similar azimuthal drifts were also observed by Gollub & Meyer (1983). More general discussions of these phenomena are given by Golubitsky et al (1988) and Krupa (1990), and references therein.

#### 4. SINGLE-MODE THEORY

In systems without symmetry, the notion of a “mode” is intuitive and generally refers to an eigenvalue and a single solution to the appropriate linear problem. In the presence of symmetry this idea is more subtle, since



*Figure 2* Time evolution of a convection pattern in a circular container at fixed Rayleigh number, with time increasing from *left to right (top to bottom)*. The shadowgraph images were recorded at  $t = 5, 17, 29, 160, 1013,$  and  $1630$  (in units of the vertical diffusion time). The reflection symmetry of the pattern is broken between  $t = 29$  and  $t = 1630$  by the spontaneous nucleation of a defect, and a slow counterclockwise rotation is observed. At  $t = 1630$  the  $Z_2(\kappa)$  symmetry is restored and the rotation ceases (courtesy of G. Ahlers).

an eigenvalue often corresponds to a set of linear solutions associated with an irreducible representation of the group. For example, in the case of  $O(2)$ ,  $e^{\pm ikx}$  are solutions corresponding to one eigenvalue and a single irreducible representation. In the presence of symmetry, it is natural to associate the term “mode” with the set of solutions corresponding to a given eigenvalue. With this terminology, “single-mode theory” refers to bifurcation problems involving a single irreducible representation of  $\Gamma$ . Mode interactions are then problems involving two or more irreducible representations. Thus, the  $O(2)$ -symmetric Takens-Bogdanov bifurcation may be considered a mode interaction between two (isomorphic) representations of  $O(2)$  (see Section 5.3).

In nonsymmetric bifurcation problems there are standard theorems specifying sufficient conditions for an instability to yield new branches of solutions (Guckenheimer & Holmes 1986). Often these conditions are not satisfied when the corresponding instability occurs in a symmetric system. For example, these theorems assume that the marginal eigenvalues are simple, which necessarily excludes many symmetry-breaking bifurcations. For symmetric systems, generalizations of these theorems are available both for steady-state bifurcation (i.e. real eigenvalues) and for Hopf bifurcation (i.e. complex eigenvalues). In each case, attention is focused on isotropy subgroups  $\Sigma$  that are large enough to define fixed-point subspaces with “minimal” dimensions:  $\dim \text{Fix}(\Sigma) = 1$  for steady-state bifurcation, and  $\dim \text{Fix}(\Sigma) = 2$  for Hopf bifurcation. The bifurcation problem obtained by restricting to these low-dimensional subspaces may be solved without difficulty. We describe below two results that are obtained in this way: the Hopf-bifurcation theorem of Golubitsky & Stewart (1985), and the equivariant branching lemma for steady-state bifurcation (Cicogna 1981, Vanderbauwhede 1982).

#### 4.1 Hopf Bifurcation With Symmetry

In this bifurcation a complex-conjugate pair of eigenvalues  $(\sigma, \bar{\sigma})$  [ $\sigma(\lambda) = \mu(\lambda) + i\omega(\lambda)$ ] crosses the imaginary axis transversely at  $\lambda = 0$ , i.e.  $\mu(0) = 0$  and

$$\frac{d\mu}{d\lambda}(0) \neq 0. \quad (4.1)$$

In the presence of a symmetry group  $\Gamma$  these eigenvalues may readily have a multiplicity greater than one, and we assume that the representation of  $\Gamma$  on  $E_\sigma$  corresponds to one of the two situations described in Section 3.2: Either  $E_\sigma$  is irreducible but not absolutely irreducible, or else  $E_\sigma = V \oplus W$  and the subspaces  $V$ ,  $W$  carry the same absolutely irreducible representation of  $\Gamma$ .

Since the normal form for the Hopf bifurcation has an additional  $S^1$  phase-shift symmetry, the full symmetry of the problem is described by the group  $\Gamma \times S^1$ . The Hopf-bifurcation theorem for symmetric systems (Golubitsky & Stewart 1985) predicts under appropriate conditions the bifurcation of periodic orbits with symmetry  $\Sigma \subset \Gamma \times S^1$ . Two distinct actions of the phase-shift symmetry play a role in the theorem. First, there is the  $S^1$  action on the amplitude variables of the normal form; a specific example was given in Section 3.1 for (3.21). More generally, if  $M(0) = DV(0, 0)$  is the matrix with imaginary eigenvalues  $\pm i\omega(0)$ , and  $0 \leq \phi_0 < 2\pi/\omega(0)$  denotes an element of  $S^1$ , then for  $a \in E_\sigma$  the normal form will be equivariant with respect to the action

$$\phi_0 \cdot a \equiv e^{\phi_0 M(0)} \cdot a, \tag{4.2}$$

where  $e^{\phi_0 M(0)}$  is the matrix obtained by exponentiating  $\phi_0 M(0)$ . When  $E_\sigma$  is two dimensional and

$$M(0) = \begin{pmatrix} i\omega(0) & 0 \\ 0 & -i\omega(0) \end{pmatrix}, \tag{4.3}$$

this result yields the normal-form symmetry for the nonsymmetric Hopf bifurcation. The second phase-shift action enters because Hopf bifurcation leads to periodic solutions. Let  $a_\tau(t)$  denote a periodic solution with period  $\tau$ . Then  $(\gamma, \phi_0) \in \Gamma \times S^1$  acts on  $a_\tau$  by

$$(\gamma, \phi_0) \cdot a_\tau(t) = \gamma \cdot a_\tau(t + \phi_0), \tag{4.4}$$

where  $0 \leq \phi_0 < \tau$ . This action makes precise the notion that  $\phi_0$  shifts the phase of the oscillation. The isotropy subgroup  $\Sigma_{a_\tau} \subset \Gamma \times S^1$  of a periodic solution is defined using this latter  $S^1$  action:

$$\Sigma_{a_\tau} = \{(\gamma, \phi_0) | (\gamma, \phi_0) \cdot a_\tau = a_\tau\}. \tag{4.5}$$

Golubitsky & Stewart (1985) proved that if the action of  $\Gamma \times S^1$  on  $E_\sigma$  has an isotropy subgroup  $\Sigma$  [in the sense of (3.36)] satisfying  $\dim \text{Fix}(\Sigma) = 2$ , then there exists a unique branch of small-amplitude periodic solutions to (3.1) with period near  $2\pi/\omega(0)$  bifurcating at  $\lambda = 0$ . In addition, the isotropy subgroup of these periodic solutions [in the sense of (4.5)] is  $\Sigma$ .

*O(2)*-SYMMETRIC HOPF BIFURCATION A particularly common illustration of this theorem is the case of circular symmetry where the relevant action of  $O(2) \times S^1$  has two (nonconjugate) isotropy subgroups with two-dimensional fixed-point subspaces. Each subgroup gives rise to a new branch of periodic solutions; physically, these correspond to standing and traveling

waves. We shall summarize this theory and then describe three fluid instabilities that yield standing and traveling waves by this mechanism.

Although the discussion that follows is couched in terms relevant to translation-symmetric systems with periodic boundary conditions, most of it applies equally as well to systems in which the circular symmetry is of geometric origin. In both cases the critical linear eigenfunctions determine whether the bifurcation breaks the symmetry. If the eigenfunctions are  $O(2)$  symmetric, then the standard (nonsymmetric) theory applies [cf (3.21)]; in contrast, when the critical modes lack  $O(2)$  symmetry [e.g.  $e^{ikx}f(z)$  for two-dimensional convection with periodic boundary conditions in  $x$ , or  $J_l(kr)e^{il\theta}$  for a circular system], then the action of  $O(2)$  on the critical amplitudes is nontrivial.

When the  $O(2)$  symmetry is broken, the general solution at  $\lambda = 0$  to the linear problem can be written as an arbitrary superposition of counter-propagating traveling waves:

$$\theta(x, z, t) = \text{Re}\{a_1 e^{ikx} + a_2 e^{-ikx}\} f(z), \tag{4.6}$$

where

$$\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \begin{pmatrix} -i\omega(0) & 0 \\ 0 & -i\omega(0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{4.7}$$

Thus  $(a_1, a_2) \in \mathbb{C}^2$  are the (complex) amplitudes of right- and left-propagating traveling waves, and the center manifold is four dimensional. The action of  $O(2)$  on the mode amplitudes  $(a_1, a_2)$  follows directly from (4.6):

$$\begin{aligned} \text{translations } T_l: x \rightarrow x+l, \quad (a_1, a_2) &\rightarrow (e^{ikl}a_1, e^{-ikl}a_2), \\ \text{reflection } \kappa: x \rightarrow -x, \quad (a_1, a_2) &\rightarrow (a_2, a_1). \end{aligned} \tag{4.8}$$

This representation of  $O(2)$  on  $\mathbb{C}^2$  is reducible and decomposes into two copies of the absolutely irreducible representation (3.17). This decomposition becomes explicit if we define new variables  $v \equiv a_1 + \bar{a}_2$  and  $w \equiv -i(a_1 - \bar{a}_2)$ . Then  $\mathbb{C}^2$  may be written as the direct sum of these two subspaces:

$$\mathbb{C}^2 = (v, 0) \oplus (0, w). \tag{4.9}$$

In addition, each of these two-dimensional subspaces transforms according to (3.17).

The normal form will be equivariant with respect to the action (4.2), i.e. with respect to the  $S^1$  action

$$\psi \cdot (a_1, a_2) = e^{-i\psi}(a_1, a_2) \tag{4.10}$$

for  $0 \leq \psi < 2\pi$ . This  $O(2) \times S^1$  action has two maximal isotropy subgroups,  $Z_2(\kappa) \times S^1$  and  $\tilde{S}\tilde{O}(2)$  (up to conjugacy), where  $\tilde{S}\tilde{O}(2) \equiv \{(T, \psi) \in O(2) \times S^1 \mid \psi = k\ell\}$ . Both subgroups have two-dimensional fixed-point subspaces,

$$\begin{aligned} \text{Fix}(Z_2(\kappa) \times S^1) &= \{(a_1, a_2) \mid a_1 = a_2\}, \\ \text{Fix}(\tilde{S}\tilde{O}(2)) &= \{(a_1, a_2) \mid a_2 = 0\}; \end{aligned} \tag{4.11}$$

consequently, the theorem of Golubitsky & Stewart (1985) guarantees the bifurcation of periodic orbits with each symmetry. The  $Z_2(\kappa) \times S^1$  solutions are standing waves (SW), and the  $\tilde{S}\tilde{O}(2)$  solutions are traveling waves (TW). Since the SW break the translation symmetry, there is a whole circle of them distinguished by the spatial phase. In contrast, the TW are isolated solutions that break the reflection symmetry but are invariant under translation followed by appropriate time evolution.

In terms of the two basic  $O(2) \times S^1$  invariants,  $u_1 = |a_1|^2 + |a_2|^2$  and  $u_2 = (|a_1|^2 - |a_2|^2)^2$ , and the two equivariants,

$$V_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{and} \quad V_2 = (|a_1|^2 \quad |a_2|^2) \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix}, \tag{4.12}$$

the unfolded  $O(2) \times S^1$ -equivariant normal form takes the form

$$\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = h_1(\lambda, u_1, u_2)V_1 + h_2(\lambda, u_1, u_2)V_2, \tag{4.13}$$

where  $h_1, h_2$  are complex-valued invariant functions such that  $h_1(\lambda, 0, 0) = \gamma(\lambda) - i\omega(\lambda)$ . In terms of the polar variables defined by  $a_1 = x_1 e^{i\theta_1}$ ,  $a_2 = x_2 e^{i\theta_2}$ , Equations (4.13) become, assuming that  $\text{Re}(h_{1,\lambda}(0)) = 1$ ,

$$\begin{aligned} \dot{x}_1 &= \{\lambda + bx_1^2 + (a+b)x_2^2\}x_1 + \mathcal{O}(5) + \lambda\mathcal{O}(3), \\ \dot{x}_2 &= \{\lambda + (a+b)x_1^2 + bx_2^2\}x_2 + \mathcal{O}(5) + \lambda\mathcal{O}(3), \end{aligned} \tag{4.14}$$

together with two decoupled equations for  $\theta_1, \theta_2$ . This decoupling is a consequence of the translation and phase-shift symmetries. Here  $\mathcal{O}(n)$  denotes terms of order  $n$  in  $(x_1, x_2)$ , and  $a, b$  are real coefficients. Under the nondegeneracy hypotheses  $a \neq 0$ ,  $b \neq 0$ , and  $a + 2b \neq 0$ , the higher order terms can be omitted. The results of analyzing the resulting equations are summarized in Figure 3. Observe that as a consequence of the symmetry multiple solution branches bifurcate simultaneously, that a stable solution exists only if all branches bifurcate supercritically, and that the stable one has larger amplitude as measured by  $u_1$  (Knobloch 1986a; see also Ruelle 1973, Golubitsky & Stewart 1985). Since  $u_1$  is proportional, in



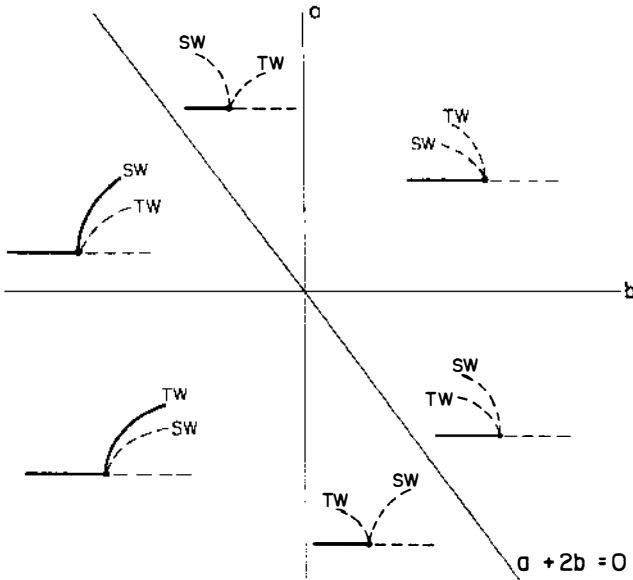


Figure 3 Bifurcation diagrams for  $O(2)$ -symmetric Hopf bifurcation from the normal form (4.14); stable (unstable) solutions are indicated by *solid* (*dashed*) branches. The two traveling-wave solutions are related by reflection symmetry and hence are indicated by a single branch.

convection, to the time-averaged heat flux, this problem provides an example where the stable state is the one transporting the most heat. Counterexamples to a more general conjecture along this line (Malkus 1954) do exist, however (see, for example, Frick et al 1983). It should be noted that if the imposed spatial period is an integer multiple of  $2\pi/k$ , both the TW and SW branches may subsequently lose stability to subharmonic perturbations.

More complex behavior can be located if the nondegeneracy conditions are relaxed. Near  $b = 0$  one discovers a secondary saddle-node bifurcation on the TW branch; near  $a + 2b = 0$  there is a secondary saddle-node bifurcation on the SW branch; and near  $a = 0$  a secondary branch of modulated traveling waves  $(a_1, a_2) = (x_1 e^{i\theta_1(t)}, x_2 e^{i\theta_2(t)})$ ,  $x_1 \neq x_2$ ,  $\dot{\theta}_1 \neq \dot{\theta}_2$ , connects the TW and SW branches. These waves, hereafter MW, bifurcate from the TW branch in a secondary Hopf bifurcation; they bifurcate from the SW branch in a pitchfork bifurcation. Because there is a whole circle of SW depending on the spatial phase, and because the instability breaks the remaining reflection symmetry, this bifurcation leads to drifting (either to the left or the right) standing waves. The stability of the MW depends

on seventh-order terms in (4.14) (Knobloch 1986b). Finally, analysis of the degeneracy  $a = b = 0$  reveals that the MW can undergo a tertiary Hopf bifurcation giving rise to three-frequency waves. A complete classification with a distinguished parameter through codimension two, and without one through codimension three, is available (Golubitsky & Roberts 1987, Crawford & Knobloch 1988a).

*Spiral vortices and ribbons in Taylor-Couette flow* In the Taylor-Couette system, one studies fluid flow between two concentric cylinders as the angular frequencies  $\Omega_i$  and  $\Omega_o$  of the inner and outer cylinders are varied (DiPrima & Swinney 1981). (Other adjustable parameters such as  $\eta$ , the radius ratio for the two cylinders, also play a significant role.) Taylor's investigation with  $\Omega_o = 0$  studied the transition between basic Couette flow and Taylor vortex flow as  $\Omega_i$  was increased. Later work (cf Langford et al 1988) revealed a counterrotating regime (i.e.  $\Omega_o < 0$  and  $\Omega_i \geq 0$ ) in which the initial transition from Couette flow was a Hopf bifurcation leading to a time-periodic flow.

In models that adopt periodic boundary conditions in the axial direction, the bifurcation from Couette flow has an  $O(2)$  axial symmetry and an  $SO(2)$  azimuthal symmetry. For the Hopf bifurcations that occur in the counterrotating regime, the azimuthal  $SO(2)$  symmetry coincides with the  $S^1$  phase-shift symmetry and thus does not alter the  $O(2)$  normal-form theory. The Hopf bifurcation breaks the  $O(2)$  symmetry, and the resulting branches of TW and SW are identified as spiral vortices and a flow called ribbons that does not propagate in the axial direction (Chossat & Iooss 1985). Calculations of the nonlinear coefficients to determine stability have found both super- and subcritical branching and have predicted that either spirals or ribbons can be stable depending on  $\eta$  (Demay & Iooss 1984, Golubitsky & Langford 1988). Both types of waves occur experimentally, although ribbons have not been found at the theoretically predicted parameter values (R. Tagg, private communication).

*Convection in binary fluids systems* Binary fluid mixtures with a sufficiently negative Soret coefficient undergo a Hopf bifurcation from the conduction state when the Rayleigh number  $\mathcal{R}$  reaches a critical value  $\mathcal{R}_0$ . When attention is restricted to two-dimensional motions with periodic boundary conditions, the system acquires  $O(2)$  symmetry. Since the bifurcation at  $\mathcal{R}_0$  breaks this symmetry, two branches of nontrivial solutions, TW and SW, bifurcate simultaneously. The system with idealized (stress-free, fixed temperature and concentration) boundary conditions at top and bottom is nongeneric in that the coefficient  $b \equiv 0$ . When  $a < 0$ , this implies that SW are unstable with respect to TW perturbations (cf Figure 3), a finding that agrees with numerical simulation (Knobloch et al 1986a). With experi-

mental boundary conditions,  $b$  is positive (unless the system is very close to the codimension-two Takens-Bogdanov point—see Schöpf & Zimmermann 1989) and the TW branch bifurcates subcritically, typically (though not always) acquiring stability at a secondary saddle-node bifurcation. The coefficient  $a$  is typically negative, but the SW are unstable. A realization of exact  $O(2)$  symmetry is provided by experiments in narrow annular cells (Kolodner et al 1988), although the fact that the imposed spatial period is many times the pattern wavelength allows for additional phenomena. Very similar conclusions apply to doubly diffusive convection in which the solute concentration gradient is imposed at the boundaries. With idealized boundary conditions a study of the degeneracy  $a = b = 0$  revealed a hysteretic transition between stable SW and TW, suggesting the presence of an unstable MW branch (Deane et al 1987). The coefficients  $a, b$  have also been calculated for convection in a vertical (Dangelmayr & Knobloch 1986) and horizontal magnetic field (Knobloch 1986c), and for rotating convection (Knobloch & Silber 1990).

*Oscillatory instability of convection rolls* The oscillatory instability is a well-known instability of convection rolls in low-Prandtl-number fluids (Busse 1978). It is a Hopf bifurcation and has a nonzero wavenumber along the axes of the rolls. Since the rolls are translation- and reflection-invariant along their axes, their symmetry properties in the axial coordinate are identical to those of the equilibrium states in the previous two examples. Therefore, with periodic boundary conditions in the axial direction the system has  $O(2)$  symmetry. At the Hopf bifurcation this symmetry is broken, and both TW and SW branches bifurcate simultaneously—i.e. the oscillatory instability can take the form of traveling or standing oscillations. For the stress-free case, Fauve et al (1987) showed that the instability is a long-wavelength one, and that it evolves into TW (see also Tveitereid et al 1986). With no-slip boundary conditions, stable TW have also been found (Clever & Busse 1987; corrected in Clever & Busse 1989), but the coefficients  $a, b$  of the theory and, in particular, their dependence on both the background roll wavenumber and the instability wavenumber have not been computed, even though they are critical for the selection between TW and SW (cf Tveitereid et al 1986). Recent experiments (Croquette & Williams 1989) show that this system behaves in all respects like binary fluid systems near the Hopf bifurcation from the conduction state. Similar considerations apply to the oscillatory instability in an imposed uniform vertical or horizontal magnetic field (Clever & Busse 1989, Busse & Clever 1989).

**OTHER SYMMETRIES** The generic Hopf bifurcation with  $D_n$  symmetry (rotation through  $2\pi/n$  and reflection of a regular  $n$ -gon) was analyzed

by Golubitsky & Stewart (1986b). The particular case  $n = 4$  applies to overstable systems in a square container; here it is possible to get a bifurcation from the trivial state directly to chaos (Swift 1988). Partial analysis with  $D_n \times T^2$  ( $n = 4, 6$ ), corresponding to a doubly periodic lattice with square and hexagonal unit cells, was initiated by Swift (1984) and completed by Silber (1989,  $n = 4$ ) and Roberts et al (1986,  $n = 6$ ). In these cases there are generically 5 and 11 primary branches, respectively, corresponding to a variety of spatially periodic traveling and standing patterns. For an application, see Renardy & Renardy (1988).

## 4.2 Steady-State Bifurcation With Symmetry

In a generic steady-state bifurcation, a real eigenvalue  $\mu(\lambda)$  crosses zero at  $\lambda = 0$  with

$$\frac{d\mu}{d\lambda}(0) \neq 0. \quad (4.15)$$

Let (3.1) denote the amplitude equation on the center manifold with  $a \in \mathbf{R}^n \equiv E_\sigma$ , and consider the generic situation in which  $\Gamma$  acts absolutely irreducibly on  $E_\sigma$ . With these assumptions, the *equivariant branching lemma* states that for each isotropy subgroup  $\Sigma \subset \Gamma$  with a one-dimensional fixed-point subspace, i.e.

$$\dim \text{Fix}(\Sigma) = 1, \quad (4.16)$$

the bifurcation yields a unique branch of equilibria with symmetry  $\Sigma$  (Cicogna 1981, Vanderbauwhede 1982). For  $O(2)$ -symmetric steady-state bifurcation, the representation (3.17) is absolutely irreducible and  $\Sigma = Z_2(\kappa)$  satisfies (4.16) [cf (3.45)]. Hence, this lemma guarantees the existence of the new equilibrium (3.47). When there are several subgroups  $\Sigma$  satisfying (4.16), this lemma can detect the emergence of multiple new patterns from a single instability.

**STEADY-STATE BIFURCATION WITH SQUARE SYMMETRY** Symmetries are particularly important in studies of spatially periodic patterns in translation- and rotation-symmetric systems. We restrict our attention to solutions that lie on a square lattice defined by four wavevectors selected from the circle of marginally stable wavevectors. In this case, the eigenspace for the marginal modes is four-dimensional,

$$\theta(x, y, z, t) = \text{Re} \{ a_1 e^{ik_x x} + a_2 e^{ik_y y} \} f(z), \quad (4.17)$$

and we can construct, via center-manifold reduction, equations for the complex amplitudes  $a_1, a_2$  subject to the requirement that they commute

with the following action of the (generators of the) group  $D_4 \times T^2$  of symmetries of the square lattice:

$$\text{rotation through } \frac{\pi}{2}: x \rightarrow y, \quad (a_1, a_2) \rightarrow (a_2, \bar{a}_1), \quad (4.18)$$

$$\text{reflection: } y \rightarrow -y, \quad (a_1, a_2) \rightarrow (a_1, \bar{a}_2), \quad (4.19)$$

$$\text{translation: } \begin{cases} x \rightarrow x + s, \\ y \rightarrow y + t, \end{cases} \quad (a_1, a_2) \rightarrow (e^{iks} a_1, e^{ikt} a_2). \quad (4.20)$$

Then

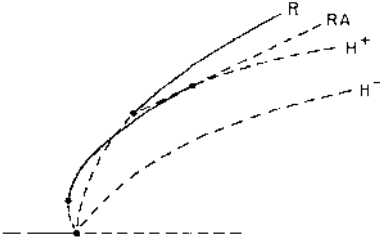
$$\begin{aligned} \dot{a}_1 &= g(\lambda, |a_1|^2, |a_2|^2) a_1, \\ \dot{a}_2 &= g(\lambda, |a_2|^2, |a_1|^2) a_2, \end{aligned} \quad (4.21)$$

where  $g$  is a real-valued function. In terms of the real variables  $a_j = r_j e^{i\theta_j}$ , ( $j = 1, 2$ ), these equations reduce to Equations (4.14), together with  $\dot{\theta}_1 = \dot{\theta}_2 = 0$ . The analysis is therefore identical to that for the Hopf bifurcation, provided that the identification of TW with rolls (R), SW with squares (S), and MW with cross rolls (CR) is made (Swift 1984). Note that the rolls and squares both have one-dimensional fixed-point subspaces, and hence their simultaneous bifurcation is guaranteed by the equivariant branching lemma. As before, a stable pattern is produced at the bifurcation provided that both the R and S branches bifurcate supercritically, and again the stable pattern transports more heat. The symmetries of the patterns are different, however. The symmetry of rolls is  $Z_2 \times SO(2)$ , whereas that of squares is  $D_4$ ; there is a circle's worth of rolls, and a torus' worth of squares, given by the spatial phases  $\theta_1, \theta_2$ . Note that when the translation symmetry is absent, Equations (4.21) still apply, but that  $a_1$  and  $a_2$  are now real. In this case the center eigenspace is two dimensional rather than four dimensional. Note also that in systems with an additional reflection symmetry  $\kappa: z \rightarrow -z$ ,  $f(-z) = -f(z)$ ,  $\kappa$  acts on  $(a_1, a_2)$  by  $\kappa: (a_1, a_2) \rightarrow (-a_1, -a_2)$ . This symmetry is equivalent, however, to the translation  $(s, t) = (\pi/k, \pi/k)$  and so has no effect on the amplitude equations. The theory applies, therefore, equally well to systems with distinct or identical boundary conditions at top and bottom (Silber & Knobloch 1988). The same theory also applies to rotating convection, although the symmetry is now  $Z_4 \times T^2$ , where  $Z_4$  is the group of proper rotations of a square (Goldstein et al 1990).

The coefficients  $a$  and  $b$  have been calculated for doubly diffusive convection (Swift 1984, Nagata & Thomas 1986), for binary fluid convection (Silber & Knobloch 1988), and for rotating convection (Goldstein

et al 1990), all with idealized boundary conditions. Additional examples are provided by penetrative convection (Matthews 1988), by Rayleigh-Bénard convection with imperfectly conducting boundaries (Jenkins & Proctor 1984), and with temperature-dependent viscosity (Jenkins 1987), although in the latter the stability of the crossrolls is determined incorrectly (cf Knobloch 1986b). Similar calculations have been carried out for convection in a porous medium (Rudraiah & Srimani 1980) and in a vertical magnetic field (Rudraiah et al 1985), but the above theory was not used to deduce the relative stability between rolls and squares.

**STEADY-STATE BIFURCATION WITH HEXAGONAL SYMMETRY** In this case we must distinguish between the generic problem (exemplified by non-Boussinesq convection), the degenerate problem (which arises when non-Boussinesq effects are absent but the boundary conditions at top and bottom are different), and the symmetric problem (when they are identical). When restricted to doubly periodic functions in the former two cases, the symmetry group is  $D_6 \times T^2$ ; in the remaining case it is  $D_6 \times T^2 \times Z_2$ . We restrict attention to the hexagonal lattice defined by selecting six wave vectors from the circle of marginally stable wave vectors. The group  $D_6 \times T^2$  has two maximal isotropy subgroups with one-dimensional fixed-point subspaces, corresponding to rolls and hexagons. Here one must distinguish between hexagons with upflow ( $H^+$ ) or downflow ( $H^-$ ) in their centers, these being unrelated by symmetry. There are no primary branches with submaximal isotropy. Abstract theory shows that since the generic normal form contains quadratic equivariants, all three branches ( $R, H^\pm$ ) are unstable (Ihrig & Golubitsky 1984). When the quadratic terms are absent (i.e. the linear problem at the level of the partial differential equations is self-adjoint), there is in addition a primary branch of rectangles (RA). Unlike the other three branches, the RA branch has a submaximal isotropy subgroup. A necessary and sufficient condition for a stable pattern is that all four branches bifurcate supercritically. With small non-Boussinesq terms the RA branch is no longer a primary branch but appears instead in secondary bifurcations from the three remaining primary branches (Buzano & Golubitsky 1983, Golubitsky et al 1984). Additional secondary bifurcations involving a branch of triangles (T) may also occur. This analysis completes the bifurcation diagram obtained earlier by Busse (1978) (Figure 4). In the symmetric case there are four primary branches  $R, H^\pm, RT,$  and  $PQ$ , with  $RT$  and  $PQ$  denoting, respectively, regular triangles and an equal-amplitude rectangle solution called patchwork quilt (Golubitsky et al 1984), all of which have one-dimensional fixed-point subspaces and hence are guaranteed by the equivariant branching lemma. In contrast to the earlier cases, here the  $H^\pm$  are taken into one another by



*Figure 4* Bifurcation diagram for the nearly degenerate  $D_6 \times T^2$  symmetric steady-state bifurcation appropriate to slightly non-Boussinesq Rayleigh-Bénard convection. Stability is transferred with increasing Rayleigh number from hexagons (here  $H^+$ ) to rolls (R) via an unstable branch of rectangles (RA) (after Golubitsky et al 1984).

the  $Z_2$  symmetry and so lie on a single group orbit. The PQ is always unstable, while the relative stability between  $H^\pm$  and RT depends on fifth-order terms. So far, only the cubic terms have been calculated in applications. These show that for Rayleigh-Bénard convection, as well as for binary fluid mixtures with idealized boundary conditions, rolls are stable (Schlüter et al 1965, Silber & Knobloch 1988). Application of the theory to long-wavelength Bénard convection and to binary fluid convection with experimental boundary conditions has also been made (Knobloch 1989, 1990) but requires analyzing an additional degeneracy.

At present there is no rigorous procedure for determining the stability of periodic patterns with respect to perturbations that do not lie on the lattice, or for determining the relative stability of patterns that lie on different lattices (e.g. squares and hexagons). In particular, counter-examples to the “relative-stability” criterion of Malkus & Veronis (1958) are known.

## 5. MODE INTERACTIONS

Bifurcations involving the simultaneous criticality of two eigenvalues are termed mode interactions. Typically these are codimension-two bifurcations, because one usually requires two independently adjustable parameters to arrange such a coincidence. When there is a symmetry group, each eigenvalue corresponds to an irreducible representation. This dependence of the basic instabilities on the group-theoretic characterization of the critical eigenspaces makes the theory of mode interactions with symmetry very rich. If the representations are changed, the center manifold dimension, normal-form equations, and dynamics can vary considerably. Although a comprehensive theory of mode interactions with symmetry does not exist, a number of specific interactions have been studied, with particular emphasis on circular symmetry (Golubitsky et al 1988). In this section we consider several examples of such mode interactions in fluids.

Since these interactions bring together multiple irreducible repre-

sentations, the resulting (reducible) representation often generates a lattice of isotropy subgroups with multiple maximal subgroups containing further submaximal subgroups. In the examples we describe below one finds that generically the primary bifurcations lead to solutions with maximal isotropy subgroups. Secondary bifurcations then give solutions with submaximal isotropy subgroups. Such regularities in the appearance of symmetry-breaking solutions have motivated conjectures that maximal subgroups would always be favored (Golubitsky 1983). However, counterexamples in which primary branches of solutions have submaximal isotropy subgroups do exist. Field (1989) gives an up-to-date mathematical discussion of these results (see also Field & Richardson 1990). Nevertheless, when it can be constructed, the isotropy lattice provides a valuable way to organize the bifurcation analysis, and we emphasize this theme in the discussion below.

### 5.1 *Parametrically Excited Surface Waves*

Experimental work on parametrically excited surface waves goes back at least to Faraday, and the subject has been recently reviewed by Miles & Henderson (1990). We focus here on the experiment performed by Ciliberto & Gollub (1985) in which a circular container filled with fluid oscillates vertically with frequency  $f_0$ . When the amplitude of the forcing exceeds a critical value, the surface of the fluid breaks up into a pattern of standing waves with azimuthal wavenumber  $k$  and frequency  $f_0/2$ . By adjusting  $f_0$  it is possible to have two such instabilities, with wavenumbers  $k$  and  $l$ , occur simultaneously. Ciliberto & Gollub studied the mode interaction obtained in this fashion for wavenumbers  $k = 4$ ,  $l = 7$  and found several different dynamical states, including a chaotic regime. They summarized the situation in the parameter-space diagram of Figure 5. The occurrence of chaotic dynamics in the immediate neighborhood of the initial instability is a significant feature of this experiment.

Theoretical analyses have all assumed that fluid behavior near the mode interaction point is described by a finite-dimensional model for the relevant mode amplitudes. In linear approximation, neglecting viscosity, these amplitudes satisfy a Mathieu equation. One approach to the mode interaction has been to derive (or else postulate) nonlinear terms coupling the Mathieu equations for the amplitudes of the surface modes with wavenumbers  $k$  and  $l$  (Ciliberto & Gollub 1985, Meron & Procaccia 1986, Umeki & Kambe 1989, Kambe & Umeki 1990). Alternatively, one can view this experiment as a mode interaction between two period-doubling bifurcations in the stroboscopic map (Crawford et al 1989); the bifurcation theory for this mode interaction has been developed in detail, taking into account the  $O(2)$  symmetry of the experiment (Crawford et al 1990b). Our



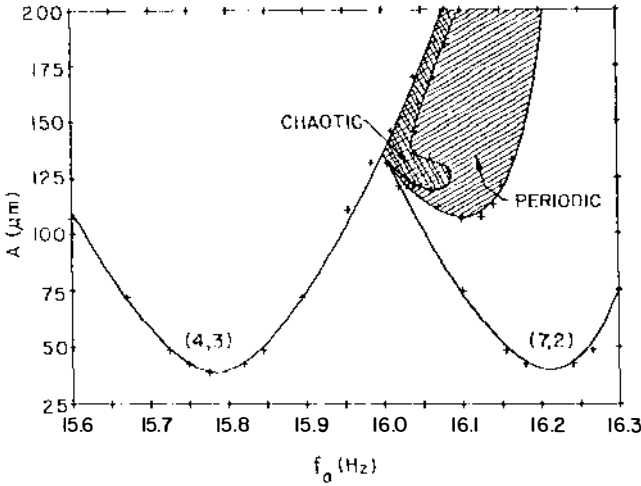


Figure 5 Parameter space for the  $[k = 4, l = 7]$  mode interaction in the Faraday experiment with a circular container of inner radius 6.35 cm and fluid depth of 1 cm. The vertical oscillation has amplitude  $A$  and frequency  $f_0$ . For each standing wave (pure mode) the notation  $(l, n)$  indicates the azimuthal wavenumber  $l$  and the radial node number  $n$ . Parameter regions are labeled by the observed type of surface-wave dynamics; below the onset of the standing waves, the surface remains essentially flat. The crosses are experimentally determined points on the stability boundaries (courtesy of J. Gollub).

discussion below follows this latter formulation; the relation between the various theoretical approaches has been analyzed in the appendix of Crawford et al (1990b).

The stroboscopic map is obtained by integrating the dynamical equations for the fluid through one period of the external drive. Let the state of the fluid at time  $t_n = (1/f_0)n$  be represented by  $\Psi_n \equiv [\zeta(r, \theta, t_n), \mathbf{u}(r, \theta, z, t_n)]$ , where  $\zeta(r, \theta, t)$  describes the surface deformation and  $\mathbf{u}(r, \theta, z, t)$  denotes the fluid velocity. The evolution of the fluid from  $t_n$  to  $t_{n+1}$  defines the stroboscopic map:  $\Psi_{n+1} = \mathcal{F}(\Psi_n)$ . The circularly symmetric basic state  $\Psi^{(0)}$  is a fixed point of the map:  $\mathcal{F}(\Psi^{(0)}) = \Psi^{(0)}$ ; similarly, a standing wave with frequency  $f_0/2$  corresponds to a two-cycle for  $\mathcal{F}$ . Thus, the parametric instabilities leading to these standing waves correspond to period-doubling bifurcations for  $\mathcal{F}$ . For the linearized map  $D\mathcal{F}(\Psi^{(0)})$  a period-doubling bifurcation implies that there is a real eigenvalue of  $D\mathcal{F}(\Psi^{(0)})$  passing through  $-1$ . Thus, near the mode interaction point we expect  $D\mathcal{F}(\Psi^{(0)})$  to have two eigenvalues simultaneously near  $-1$ , corresponding to the two standing waves with wavenumbers  $k$  and  $l$ . For each eigenvalue there will be two eigenfunctions of the form  $\psi_k(r, z) \cos(k\theta)$  and  $\psi_k(r, z) \sin(k\theta)$ ,

where  $\psi_k(r, z)$  is a real-valued (multicomponent) field. It is convenient to introduce complex amplitudes  $(z, w)$  for the critical modes and write

$$\Psi_n = \frac{1}{2}(z_n e^{ik\theta} \psi_k + w_n e^{i\theta} \psi_l + \text{c.c.}) + (\text{all other modes})_n. \tag{5.1}$$

The circular and reflection symmetries,  $\phi: \theta \rightarrow \theta + \phi$ ,  $\kappa: \theta \rightarrow -\theta$ , determine the action of  $O(2)$  on the amplitudes of the critical modes:

$$\begin{aligned} \phi \cdot (z, w) &= (e^{ik\phi} z, e^{i\phi} w), \\ \kappa \cdot (z, w) &= (\bar{z}, \bar{w}). \end{aligned} \tag{5.2}$$

As in continuous time systems, near the mode-interaction point the map  $\mathcal{F}$  reduces to the finite-dimensional map

$$\begin{pmatrix} z_{n+1} \\ w_{n+1} \end{pmatrix} = f(z_n, w_n). \tag{5.3}$$

Since the original system is rotation- and reflection-symmetric,  $f$  commutes with  $\phi$  and  $\kappa$ :

$$\begin{aligned} \phi \cdot f(z, w) &= f(\phi \cdot (z, w)), \\ \kappa \cdot f(z, w) &= f(\kappa \cdot (z, w)). \end{aligned} \tag{5.4}$$

In addition, to describe the interaction of the period-doubling bifurcations requires that we take  $f(z, w) = (-z, -w) + \dots$  for the linear terms of the reduced map. For the wavenumbers in the experiment, these properties imply that in normal form  $f$  may be written in terms of four (undetermined) real-valued invariant functions  $P, Q, R, S$ :

$$f(z, w) = \begin{pmatrix} P(u_1, u_2, u_3^2)z + Q(u_1, u_2, u_3^2)u_3 \bar{z}^6 w^4 \\ R(u_1, u_2, u_3^2)w + S(u_1, u_2, u_3^2)u_3 z^7 \bar{w}^3 \end{pmatrix}, \tag{5.5}$$

where  $u_1 = |z|^2$ ,  $u_2 = |w|^2$ , and  $u_3 = z^7 \bar{w}^4 + \bar{z}^7 w^4$  are the fundamental invariants for the action (5.2). The symmetry of the normal form is  $O(2) \times Z_2(-I)$ , since terms that are even under  $(z, w) \rightarrow (-z, -w)$  have been removed. Consequently, the amplitudes in (5.5) are related to  $(z, w)$  in (5.1) by a near-identity transformation.

Crawford et al (1990b) analyzed the two-parameter unfolding defined by

$$\begin{aligned} P(0, 0, 0) &= -(1 + \lambda - \mu), \\ R(0, 0, 0) &= -(1 + \lambda). \end{aligned} \tag{5.6}$$

In this unfolding the  $k = 4$  standing-wave instability occurs for  $\lambda = \mu$ , and the  $l = 7$  instability occurs when  $\lambda = 0$ . Physically,  $\lambda$  plays the role of

forcing amplitude and  $\mu$  corresponds to the adjustable forcing frequency; the mode-interaction point is  $(\lambda, \mu) = (0, 0)$ .

The isotropy lattice for the action (5.2) is shown in Figure 6. The maximal subgroups  $\tilde{D}_{2n}$  ( $n = k, l$ ) are generated by the spatial reflection  $\kappa$  and  $-\phi_{2n}$ , where  $\phi_{2n}$  is rotation through  $\pi/n$ . These symmetries describe the two pure standing waves ( $k = 4$  and  $l = 7$ ) producing the mode interaction. The submaximal subgroups are generated by  $\kappa$  and  $\tilde{\kappa} \equiv -\kappa$ . Physically,  $\tilde{\kappa}$  describes the spatio-temporal symmetry of reflection followed by time evolution for one period  $1/f_0$ .

For suitable  $P, Q, R, S$ , each of the original standing waves has a secondary instability that yields two new  $f_0/2$  standing waves with  $Z_2(\kappa)$  and  $Z_2(\tilde{\kappa})$  symmetry. These new waves are mixed modes formed by an admixture of the basic wavenumbers  $k$  and  $l$ . From the mixed modes a Hopf bifurcation can occur, leading to the appearance of an attracting invariant circle. Physically, this tertiary bifurcation marks the onset of periodic amplitude oscillations, such as those observed in the "periodic" regime of the experimental phase diagram. Numerical calculations indicate that as  $\lambda$  increases further, this invariant circle becomes increasingly rough until a continuous transition to a chaotic attractor occurs (J. D. Crawford, unpublished). The location of this transition suggests that it could describe the onset of chaos in the experiments, but more work on this point is needed.

In this problem, the levels of the isotropy lattice neatly organize the progressively less symmetric states produced by successive bifurcations. The initial bifurcations lead to waves with maximal isotropy subgroups; the secondary instabilities of these waves lead to surface motions with submaximal subgroups. Whether or not the final  $Z_2$  symmetry is broken

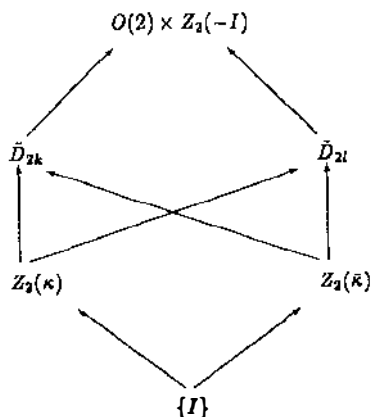


Figure 6 Isotropy lattice for the  $O(2) \times Z_2(-I)$  period-doubling mode interaction with wavenumbers  $k$  and  $l$ .

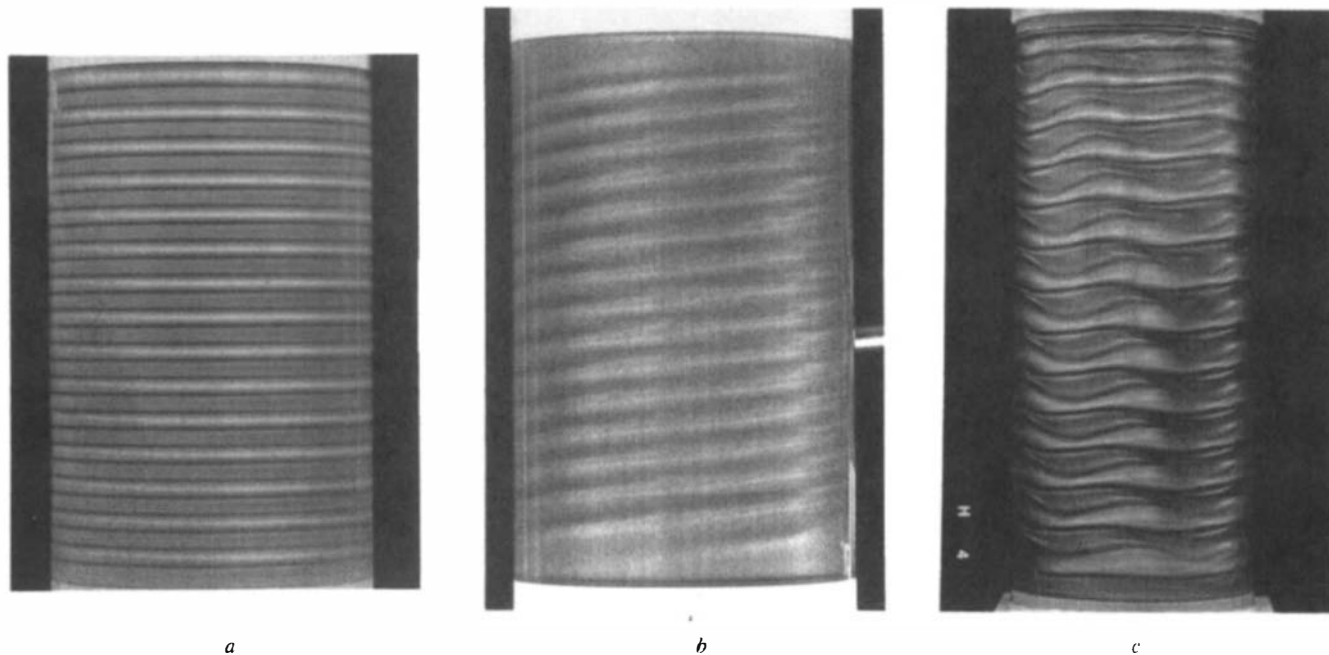
depends on the particular tertiary instability. The Hopf bifurcation to amplitude oscillations preserves the  $Z_2$  symmetry; however, there is a second possible tertiary instability that breaks the  $Z_2$  symmetry and leads to azimuthally drifting patterns. These “drifting two-cycles” were not seen near the mode-interaction point but have been observed by Gollub & Meyer (1983). In addition, Crawford et al (1990b) provide predictions of expected mixed-mode states, hysteresis effects, and mode locking. Further experimental studies of this mode interaction would be valuable.

## 5.2 *Mode Interactions in Taylor-Couette Experiments*

Numerous mode interactions occur in the counterrotating regime of the Taylor-Couette system for the initial transition from Couette flow. As the rotation frequency of the outer cylinder  $\Omega_0$  is increased, the steady-state bifurcation to Taylor vortices (with azimuthal wavenumber  $m = 0$ ) is superseded by a Hopf bifurcation, producing either spirals or ribbons (cf Section 4.1) with wavenumber  $m = 1$ . As  $\Omega_0$  is increased further, there is a sequence of bicritical points at which the Hopf bifurcation with wavenumber  $m$  coincides with another Hopf bifurcation with wavenumber  $m + 1$ . Langford et al (1988) have computed the locations of these mode-interaction points as a function of  $\Omega_0$ . At each such point two eigenvalues associated with distinct irreducible representations are simultaneously unstable.

When periodic axial boundary conditions are adopted, the bifurcations from Couette flow have  $O(2) \times SO(2)$  symmetry (cf Section 4.1). For this symmetry, the Hopf/Hopf mode interactions lead to eight-dimensional systems, which have been considered by Chossat et al (1986, 1987). The six-dimensional steady-state/Hopf interaction between Taylor vortices (Figure 7a) and spiral vortices (Figure 7b) has been analyzed by Golubitsky & Stewart (1985), Crawford et al (1988), Golubitsky & Langford (1988), and Golubitsky et al (1988). In addition, Hill & Stewart (1990) have considered the triple-mode interaction (Hopf/Hopf/steady state) relevant to the simultaneous instability of azimuthal wavenumbers  $m = 0, 1$ , and 2. Both the six- and eight-dimensional interactions have been studied experimentally (Stern 1988; R. Tagg, private communication); below, we sketch the bifurcation analysis of the six-dimensional problem.

Both the steady-state bifurcation to vortices and the Hopf bifurcation to spirals break the  $O(2)$  axial symmetry, leading to two- and four-dimensional bifurcation problems, respectively (cf Sections 3.1 and 4.1). We use  $a_0$  to denote the complex amplitude for the linear modes of the steady-state bifurcation [cf (3.16)] and  $(a_1, a_2)$  to represent the complex amplitudes of the Hopf mode [cf (4.6)]. Then the axial translations  $T_l$  and reflection  $\kappa$  act by



*Figure 7* Flows in the Taylor-Couette experiment observed in connection with the mode interaction between Taylor vortices and spiral vortices: (a) Taylor vortices; (b) spiral vortices; (c) wavy vortices (courtesy of R. Tagg).

$$\begin{aligned}
 T_l &: (a_0, a_1, a_2) \rightarrow (e^{ikl}a_0, e^{ikl}a_1, e^{-ikl}a_2), \\
 \kappa &: (a_0, a_1, a_2) \rightarrow (\bar{a}_0, a_2, a_1).
 \end{aligned}
 \tag{5.7}$$

The effect of an azimuthal rotation  $\phi : \theta \rightarrow \theta + \phi$  is

$$\phi : (a_0, a_1, a_2) \rightarrow (a_0, e^{i\phi}a_1, e^{i\phi}a_2)
 \tag{5.8}$$

for an  $m = 1$  Hopf mode. One may rescale the axial coordinate so that  $k = 1$  in (5.7); the normal form is then given by (Langford 1986):

$$\begin{pmatrix} \dot{a}_0 \\ \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \begin{pmatrix} [c^{(1)} + i\delta c^{(2)}]a_0 + [c^{(3)} + i\delta c^{(4)}]\bar{a}_0 a_1 \bar{a}_2 \\ [p^{(1)} + \delta p^{(2)} + i(q^{(1)} + \delta q^{(2)})]a_1 + [p^{(3)} + \delta p^{(4)} + i(q^{(3)} + \delta q^{(4)})]a_0^2 a_2 \\ [p^{(1)} - \delta p^{(2)} + i(q^{(1)} - \delta q^{(2)})]a_2 + [p^{(3)} - \delta p^{(4)} + i(q^{(3)} - \delta q^{(4)})]\bar{a}_0^2 a_1 \end{pmatrix},
 \tag{5.9}$$

where  $\delta = |a_2|^2 - |a_1|^2$  and the 12 real-valued invariant functions  $(c^{(i)}, p^{(i)}, q^{(i)})$  ( $i = 1, \dots, 4$ ) are functions of the five fundamental invariants of the  $O(2) \times SO(2)$  action,

$$\begin{aligned}
 u_1 &= |a_0|^2, \\
 u_2 &= |a_1|^2 + |a_2|^2, \\
 u_3 &= \delta^2, \\
 u_4 &= \text{Re}(a_0^2 \bar{a}_1 a_2), \\
 u_5 &= \delta \text{Im}(a_0^2 \bar{a}_1 a_2),
 \end{aligned}
 \tag{5.10}$$

and of the system parameters. For example,  $c^{(1)} = c^{(1)}(\Omega_i, \Omega_o, \eta, \mathbf{u})$ . Gollubsky & Langford (1988) have analyzed a three-parameter unfolding, given by

$$\begin{aligned}
 \lambda_1 &= c^{(1)}(\Omega_i, \Omega_o, \eta, 0), \\
 \lambda_2 &= p^{(1)}(\Omega_i, \Omega_o, \eta, 0), \\
 \lambda_3 &= q^{(1)}(\Omega_i, \Omega_o, \eta, 0),
 \end{aligned}
 \tag{5.11}$$

for which  $\lambda = 0$  corresponds to the mode-interaction point; the steady bifurcation occurs along  $\lambda_1 = 0$ , and the Hopf bifurcation occurs along  $\lambda_2 = 0$ .

The isotropy lattice for this bifurcation appears in Figure 8, showing three maximal subgroups and three submaximal subgroups; each of these subgroups corresponds to an identifiable fluid state, although not all are stable. As before, the primary branches correspond to states with maximal isotropy subgroups. The symmetry of the Taylor vortices is given by  $\Sigma = Z_2(\kappa) \times SO(2)$ , while spiral vortices and ribbons have  $\Sigma = \tilde{SO}(2)$  and  $\Sigma = Z_2(\kappa) \times Z_2^c$ , respectively, where  $Z_2^c$  is generated by  $(l, \phi) = (\pi, \pi)$ , the translation and rotation by  $\pi$ . In each case, these characterizations are familiar from single-mode bifurcation theory with  $O(2)$  symmetry (cf Sections 3.1 and 4.1).

The stability of these primary branches can be determined analytically using the isotropic decomposition to simplify the eigenvalue calculations. For Taylor vortices, there are two possibilities for secondary Hopf bifurcation, leading to either wavy vortices [ $\Sigma = Z_2(\kappa \cdot (\pi, \pi))$ ] or twisted vortices [ $\Sigma = Z_2(\kappa)$ ]. In each case the new secondary branch is a time-periodic flow with submaximal isotropy. From ribbons there are two possible secondary (steady-state) bifurcations to wavy vortices and twisted vortices. Finally, from spirals there is a possible secondary Hopf bifurcation to a quasi-periodic branch of modulated waves (“modulated spirals”) with  $\Sigma = Z_2^c$ . For these secondary bifurcations to occur, the low-order terms in the normal form must satisfy certain conditions, as discussed by Golubitsky & Langford (1988) and Golubitsky et al (1988).

The primary and secondary bifurcations can be analyzed by truncating the normal form at cubic order. The nonlinear coefficients required to

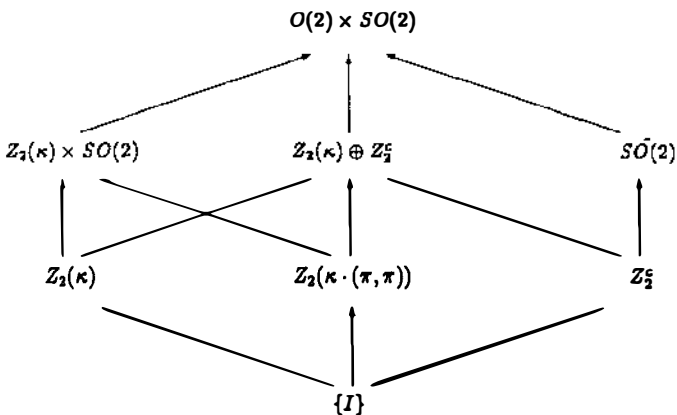


Figure 8 Isotropy lattice for the  $O(2) \times SO(2)$  steady-state Hopf mode interaction, with  $k = 1$  for the axial wavenumber.

specify this truncated model were calculated by Golubitsky & Langford (1988), starting from the Navier-Stokes equations and covering radius ratios  $0.43 \leq \eta \leq 0.98$ . This theory predicts the stable occurrence of Couette flow, spirals, Taylor vortices, and wavy vortices (Figure 7c) for appropriate values of  $\Omega_i$  and  $\Omega_o$ . In addition, regions of bistability and hysteresis involving spirals and either Taylor vortices or wavy vortices were predicted. Subsequent experiments have confirmed all of these qualitative features, although quantitative discrepancies with theoretically predicted parameter values remain (R. Tagg, private communication).

### 5.3 Takens-Bogdanov Bifurcation With $O(2)$ Symmetry

When the oscillation frequency at a nonsymmetric Hopf bifurcation vanishes, the linear problem has two zero-eigenvalues and a two-dimensional center manifold. This bifurcation was first studied by Takens (1974) and Bogdanov (1975); the case with reflection symmetry arises frequently in fluid dynamics and was analyzed by Knobloch & Proctor (1981) and Guckenheimer & Knobloch (1983).

If  $O(2)$  symmetry is present and the eigenfunctions break the symmetry, there are four zero-eigenvalues and the center manifold is four dimensional. In this case the variables  $(v, w)$  associated with the irreducible representation (4.9) are particularly convenient. At  $\lambda = 0$ , the linear problem has the solution

$$\theta(x, z, t) = \text{Re}\{v(t)e^{ikx}\}f(z), \tag{5.12}$$

where the complex amplitude  $v$  satisfies the two-dimensional system

$$\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \tag{5.13}$$

Dangelmayr & Knobloch (1987a) have shown that the unfolded normal form may be written as

$$v'' = (\mu + A|v|^2)v + \varepsilon[vw + C(v\bar{v}' + \bar{v}v')v + D|v|^2v'] + O(\varepsilon^2), \tag{5.14}$$

where the primes denote differentiation with respect to a slow time  $\varepsilon t$ ;  $(\mu, v)$  are the unfolding parameters;  $A, C,$  and  $D$  are real coefficients; and  $\varepsilon^2 \ll 1$  measures the distance in the two-dimensional parameter space from the Takens-Bogdanov (TB) point. Equation (5.14) has five types of solutions, best described in terms of the real variables defined by  $v = re^{i\theta}$ . These are (a) the trivial (conduction) state  $r = 0$ , (b) the nontrivial steady states SS ( $r' = 0, r \neq 0, \theta' = 0$ ), (c) TW ( $r' = 0, \theta' \neq 0$ ), (d) SW ( $r' \neq 0, \theta' = 0$ ), and (e) MW ( $r' \neq 0, \theta' \neq 0$ ). A complete discussion of the interconnections and relative stability of these branches has been given by Dangelmayr &



Knobloch [1987a; their Figure 8(IX-)] has been corrected by Peplowski & Haken 1988]. The analysis describes the interaction of the TW and SW with SS. The TW terminate on the SS branch in a pitchfork bifurcation, another example of a bifurcation from a group orbit. If the TW are supercritical and the SS subcritical, the TW branch must first acquire two unstable eigenvalues before it can terminate; it does so by shedding a MW branch in a secondary Hopf bifurcation. Evidence for such a bifurcation was found in the binary fluid experiment of Heinrichs et al (1987; see also Knobloch & Weiss 1987), albeit in spatially nonuniform TW. If the TW terminate on a supercritical SS branch at  $R^{TW}$ , say, they do so without change of stability; near  $R^{TW}$  their phase velocity approaches zero as  $(R - R^{TW})^{1/2}$ . In contrast, the SW terminate in a global bifurcation (if  $A > 0$ ) or a secondary Hopf bifurcation (if  $A < 0$ ) on the SS branch, as described by Knobloch & Proctor (1981).

Applications of the analysis have been made to convection in binary fluid mixtures (Knobloch 1986a), in an imposed magnetic field (Dangelmayr & Knobloch 1986, Knobloch 1986b), and in a rotating layer (Knobloch & Silber 1990). In many systems, including binary fluid convection with experimental boundary conditions, the TB point is shielded by steady-state bifurcations that set in at smaller Rayleigh number and different wavenumber (Knobloch & Moore 1988). The relevance of the TB analysis to such systems has not been completely clarified (cf Cross & Kim 1988).

## 6. IMPERFECT SYMMETRIES

In all equations the presence of symmetry is the result of an idealization. One is then naturally interested in the effects of small symmetry-breaking imperfections that are inevitably present. In some cases, bifurcation theory can be used to determine the effect of all such imperfections. The best known example is provided by the perturbation of a pitchfork bifurcation. Here it has been rigorously shown that the pitchfork can be embedded in a two-parameter family of bifurcation problems  $h(x, \lambda; \alpha, \beta) = 0$ ,  $h(x, \lambda; 0, 0) = \lambda x - x^3$ , with the property that all perturbations of the pitchfork are “equivalent” to the *universal unfolding* (Golubitsky & Schaeffer 1985)

$$h(x, \lambda; \alpha, \beta) = \lambda x - x^3 + \alpha + \beta x^2. \quad (6.1)$$

Thus,  $\alpha$  and  $\beta$  capture all the possible imperfections, and one does not have to worry that some new hitherto undiscovered imperfection will change the pitchfork into a bifurcation diagram not described by (6.1). For steady-state bifurcations there exist constructive techniques for deter-

mining such universal unfoldings. For dynamical problems the techniques are less systematic.

Equation (6.1) was first suggested in the context of fluid mechanics by Benjamin (1978), and explicit derivations (using asymptotics) have been carried out (e.g. Hall & Walton 1977). Such derivations do not, however, establish that (6.1) is the appropriate universal unfolding in the sense described. For the pitchfork of revolution the corresponding analysis is due to Golubitsky & Schaeffer (1983).

For the Hopf bifurcation with  $O(2)$  symmetry, the normal-form symmetry  $O(2) \times S^1$  can be broken in a variety of ways. Even in a system with perfect  $O(2)$  symmetry, the  $S^1$  normal-form symmetry is typically broken by terms beyond all orders. Nevertheless, the modulated traveling waves revealed by analyzing appropriate degeneracies continue to exist, and no frequency locking takes place (Chossat 1986). In other problems leading to global bifurcations in the normal form, the breaking of the  $S^1$  symmetry will lead to chaotic behavior (Guckenheimer 1981, Langford 1983). The  $S^1$  phase-shift symmetry may also be broken by time-dependent forcing; in this case, the phase difference  $\theta_1 - \theta_2$  no longer decouples from the amplitudes. Riecke et al (1988) and Walgraef (1988) studied the effects of periodic forcing near the 2:1 resonance on a translation- and reflection-symmetric system undergoing a Hopf bifurcation. They concluded that the parametric forcing stabilizes SW, whereas in the absence of forcing these would be unstable to TW; this conclusion has been verified by Rehberg et al (1988).

If the group  $O(2)$  is broken, then the pair of eigenvalues of multiplicity two splits into two pairs that cross the imaginary axis in succession. The case  $O(2) \times S^1 \rightarrow SO(2) \times S^1$  is easiest to analyze (van Gils & Mallet-Paret 1986) because the translation and phase-shift symmetries continue to be responsible for decoupling the phases from the amplitudes. Here SW are no longer solutions. Instead, two successive primary bifurcations to left- and right-traveling waves take place. The role of the SW is taken by a two-frequency wave that bifurcates in a secondary bifurcation from one of the TW, and that looks more and more like SW with increasing amplitude. The effects on the various codimension-two degeneracies were analyzed by Crawford & Knobloch (1988b).

The case  $O(2) \rightarrow Z_2 \times S^1$  is more complicated because the total phase no longer decouples. Dangelmayr & Knobloch (1987b, 1990) have shown that when the translation symmetry is broken, pure traveling waves are no longer possible. Instead, two successive bifurcations to standing waves with a specified phase  $\theta \equiv \theta_1 + \theta_2 = 0, \pi$  take place. The place of TW is taken by a superposition of left- and right-traveling waves but with identical frequencies. This frequency locking is a consequence of the broken

translation symmetry. Such waves are singly periodic, but there is no comoving frame in which they are time independent. In addition, a new state is found. This is a two-frequency wave in which the new frequency describes slow oscillations between left- and right-traveling waves. These “blinking” states exist only locally in amplitude, and their prediction on the basis of symmetry-breaking considerations preceded their discovery in numerical simulations (Deane et al 1988, Cross 1988) and in experiments (Fineberg et al 1988, Kolodner & Surko 1988). If the translation symmetry is broken, the breakdown of the additional phase-shift symmetry is likely to lead to chaotic dynamics associated with global bifurcations involving the “blinking” states (Dangelmayr & Knobloch 1987b). Nagata (1988) has analyzed the Hopf bifurcation under  $O(2) \times S^1 \rightarrow Z_n \times S^1$  ( $n = 3, n \geq 5$ ).

The effects of breaking the reflection symmetry in the Takens-Bogdanov bifurcation with  $Z_2$  symmetry have also been worked out (Dangelmayr & Guckenheimer 1987). Recent experiments by Simonelli & Gollub (1989) on the Faraday system with nearly square geometry suggest another problem along these lines. Some aspects of the symmetry-breaking  $D_4 \rightarrow Z_2 \times Z_2$  are analyzed by Crawford & Knobloch (1988b).

In all of the above problems, the effects of the broken symmetry are felt locally near the original primary bifurcation; at larger amplitudes, the symmetric results are essentially recovered.

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