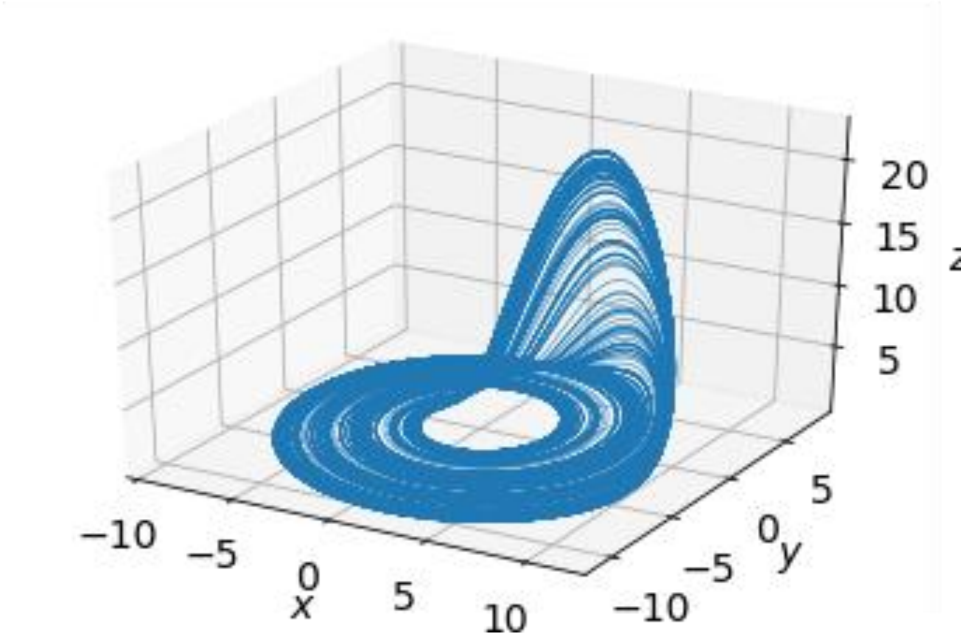
Mapas

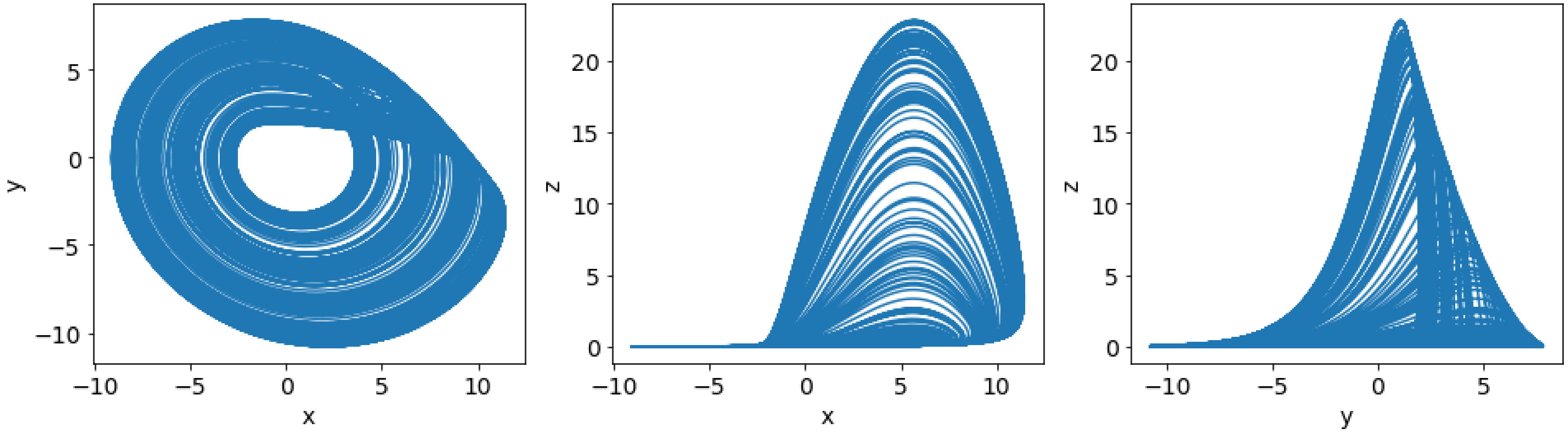
Dinámica no lineal

Cátedra G. Mindlin

Viernes 19 de Junio de 2020

¿Por qué mapas?





5
0
−5
−10
y
−10
−5
0
5
10
x
20
15
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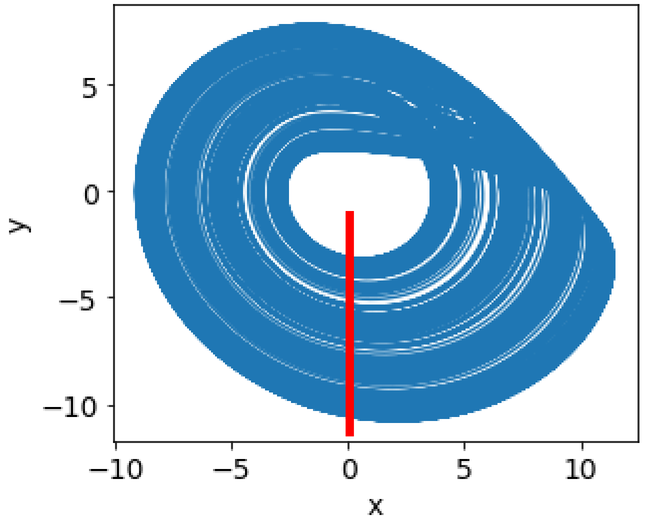
y



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0
−5
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y
−10
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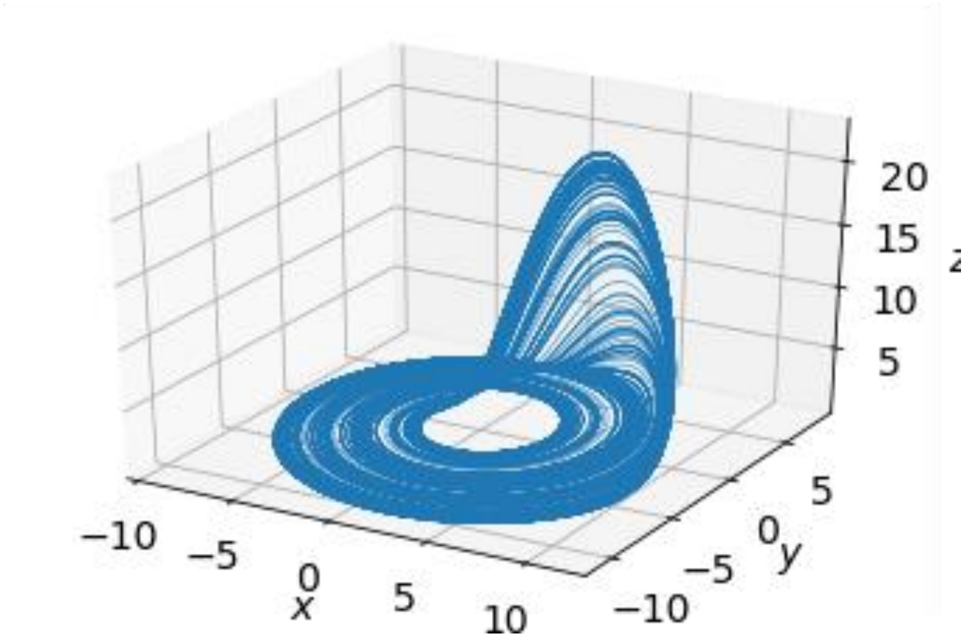
0
5
10
x



20
15
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10
x
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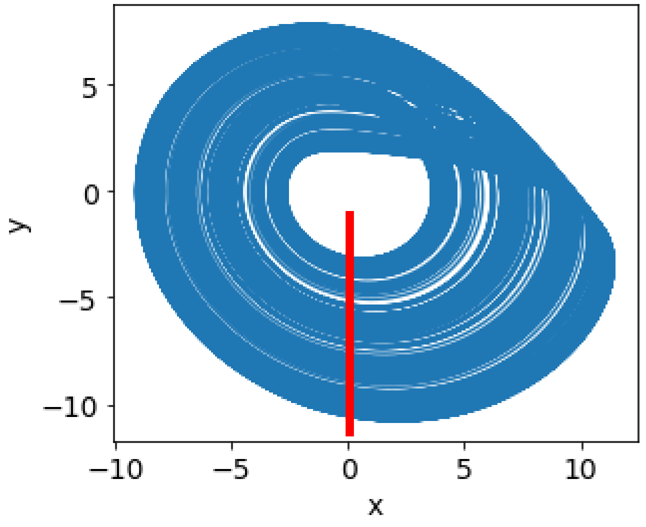
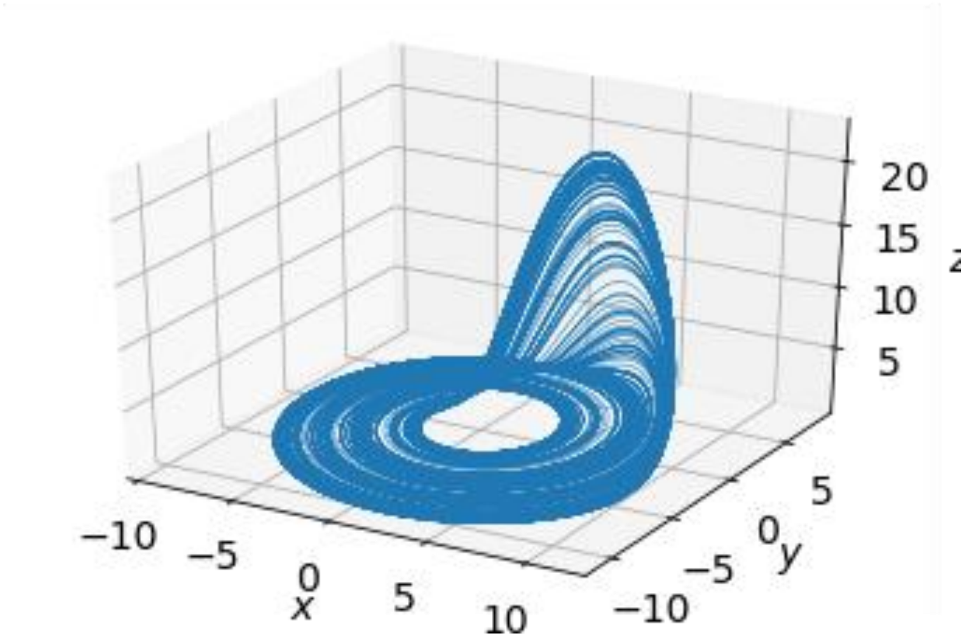
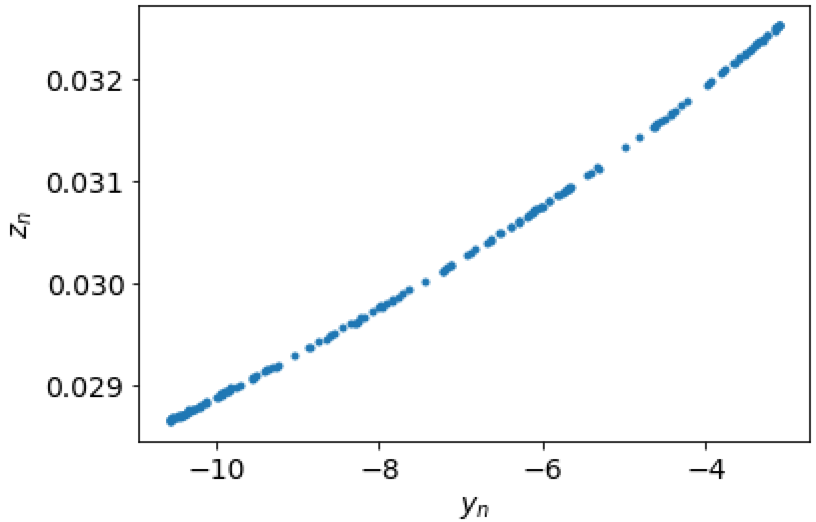
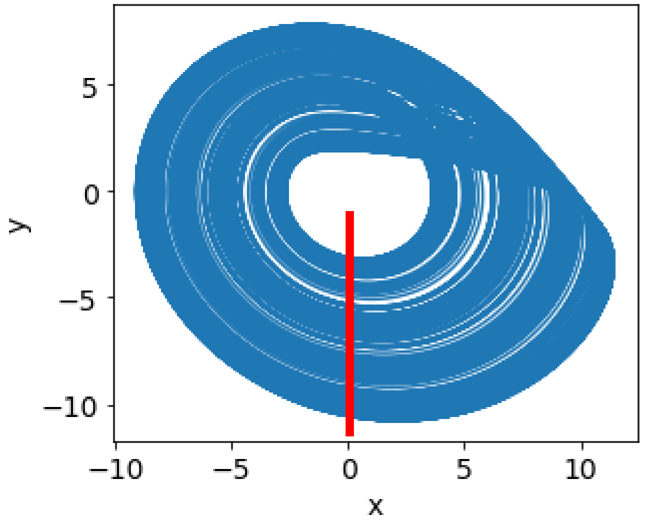
−5
−10
y

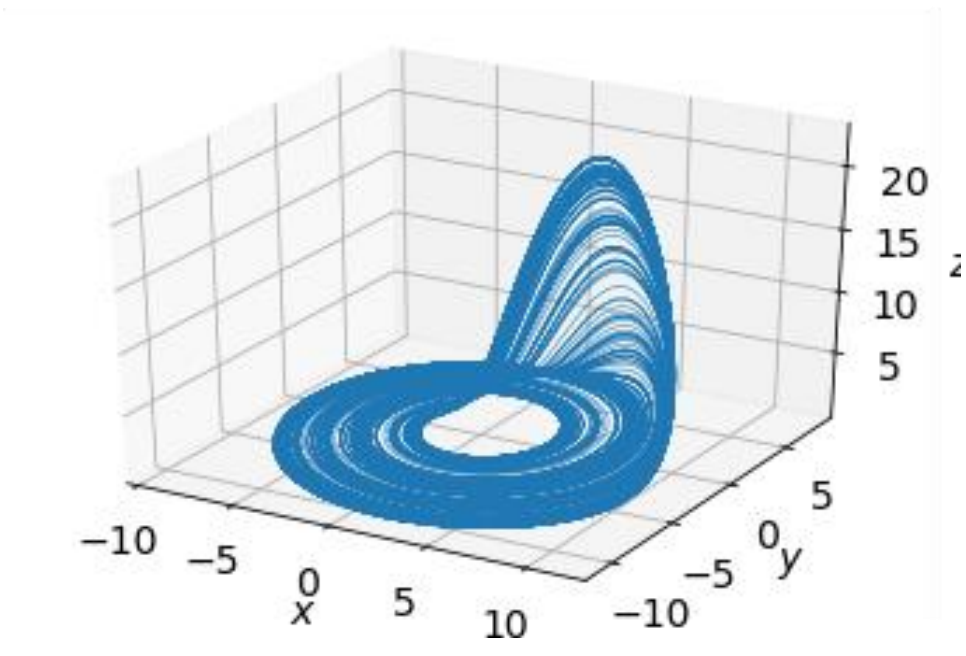


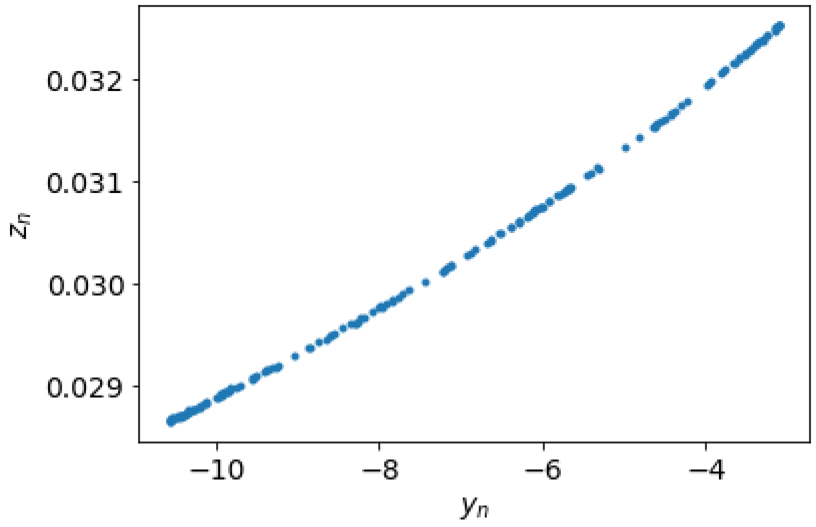


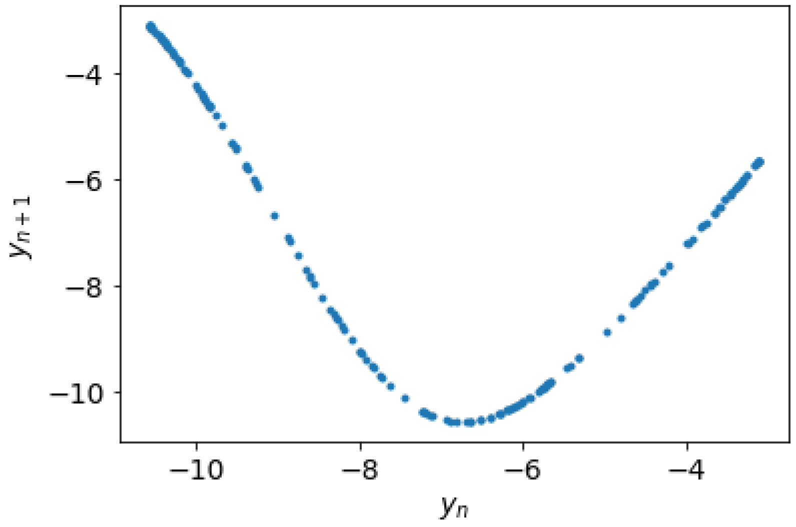


Sección de Poincaré

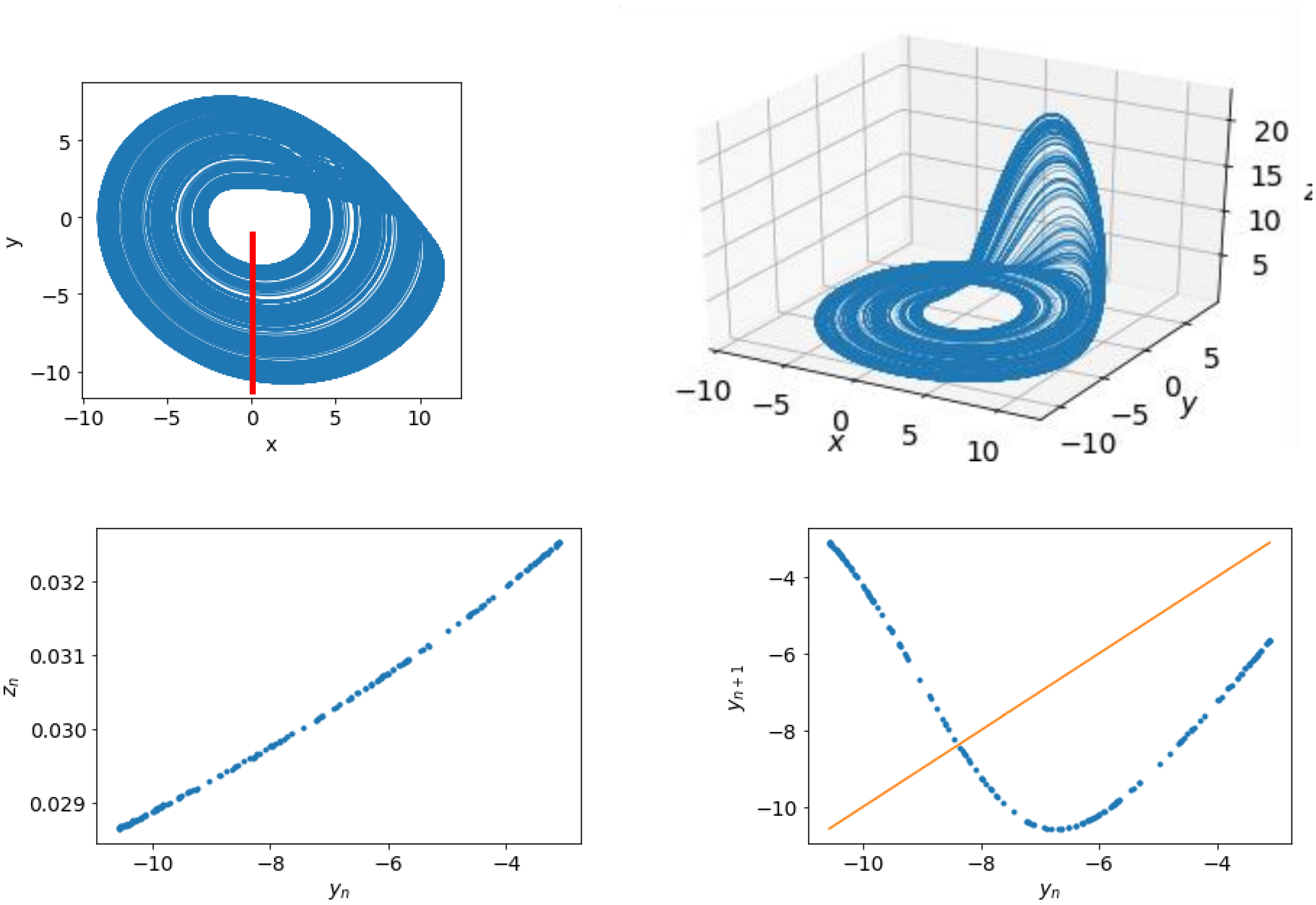




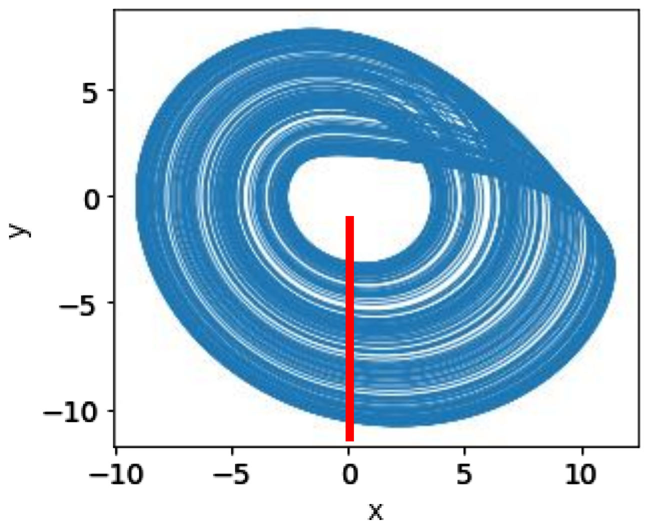


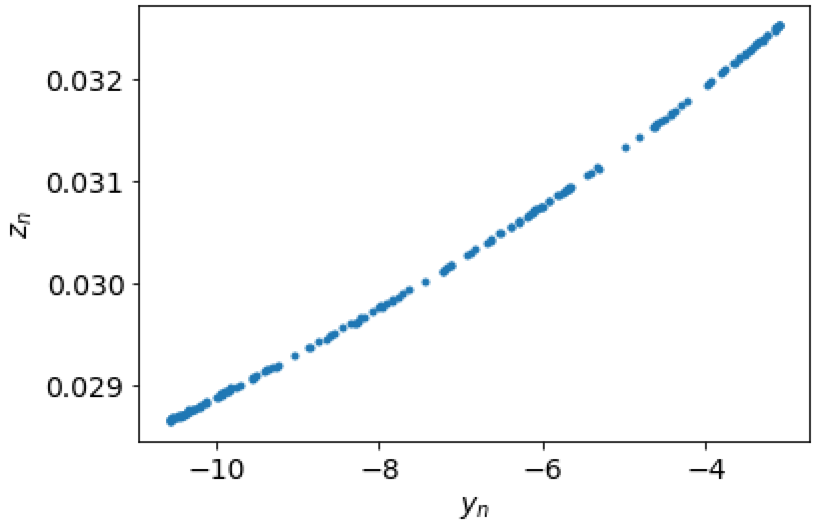


Sección de Poincaré

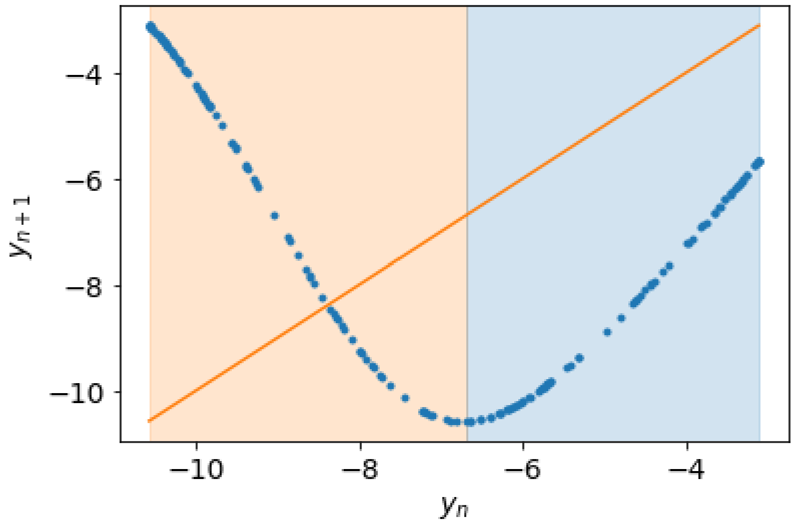


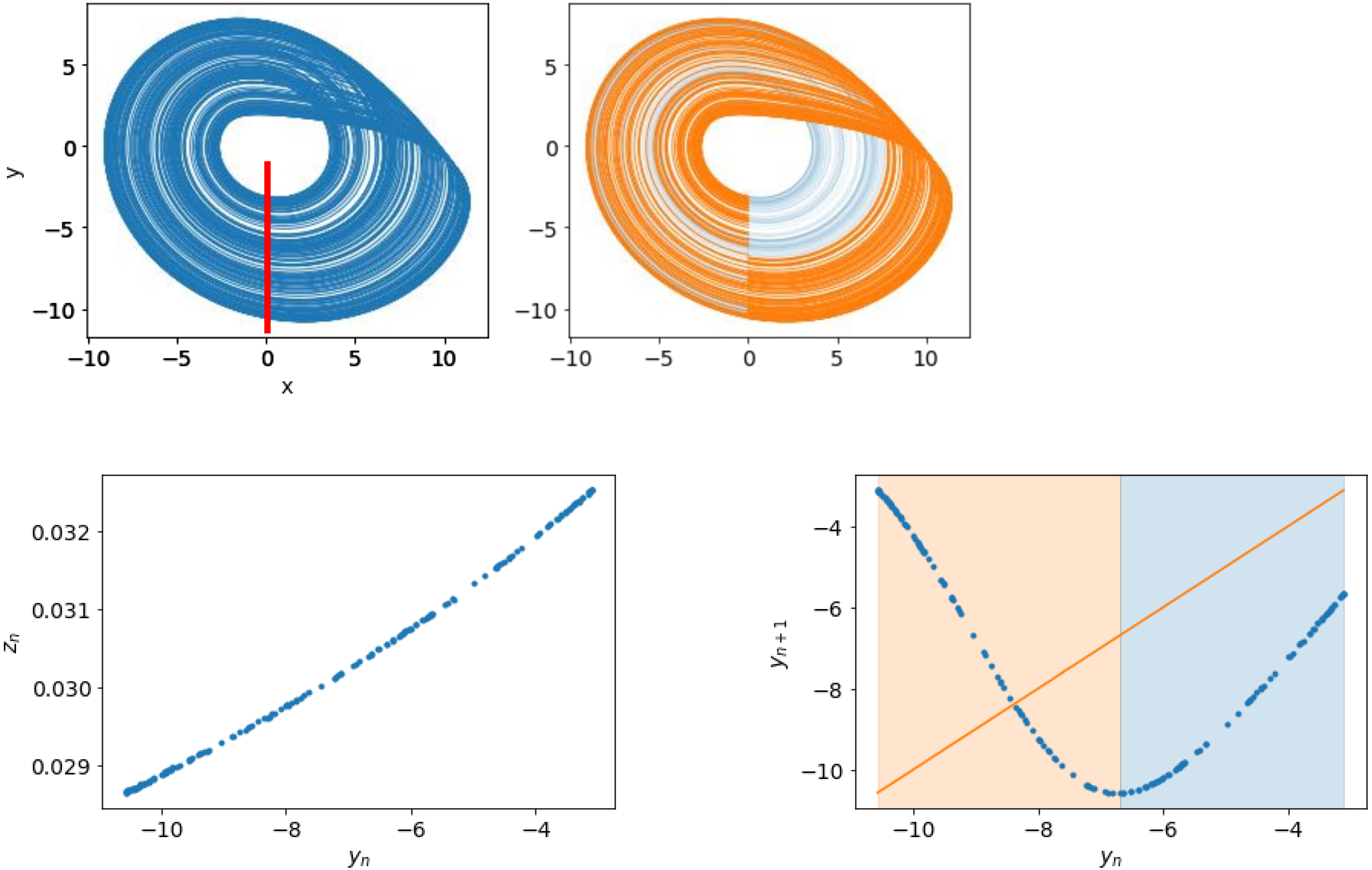
Sección de Poincaré



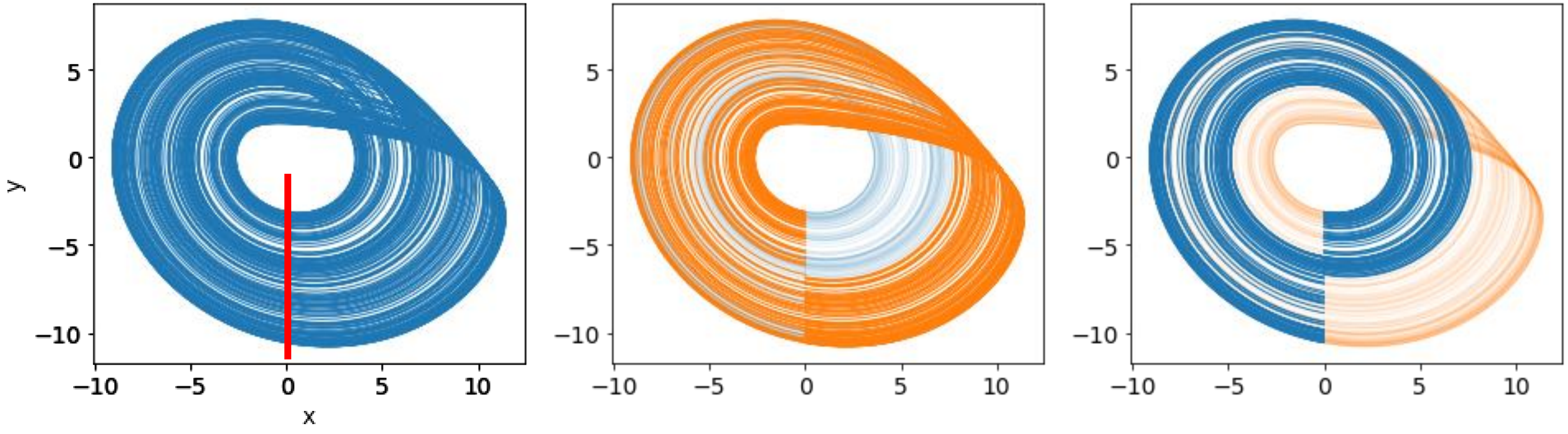


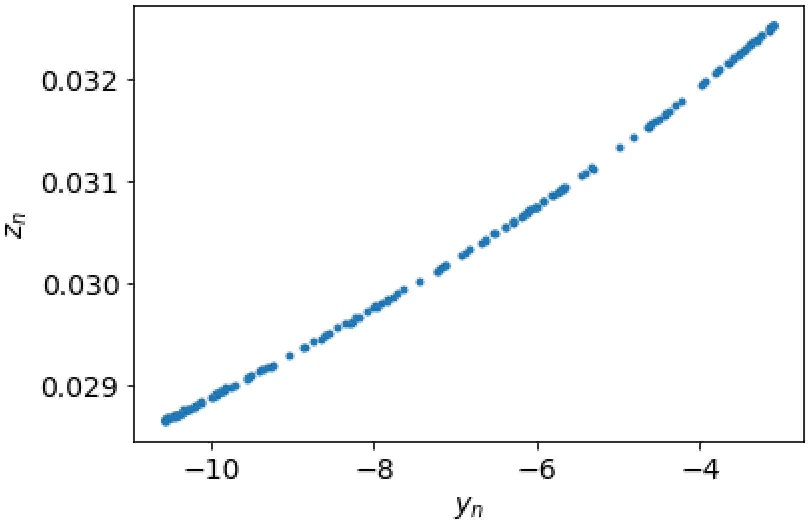
Sección de Poincaré

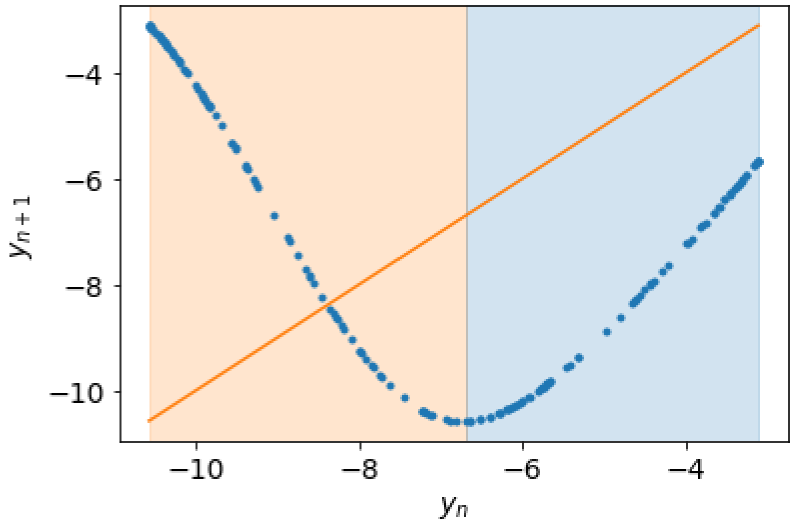




Sección de Poincaré







Sección de Poincaré

Un ejemplo: oscilador forzado periódicamente

1. (*) **Construcción de un Mapa de Poincaré:** Considere la siguiente ecuación diferencial de un oscilador forzado periódicamente con disipación $\delta > 0$:

$$\ddot{x} + \delta \dot{x} + \omega_0^2 x = cos(\omega t)$$

Un ejemplo: oscilador forzado periódicamente

Heagy, J. F. (1992). A physical interpretation of the Hénon map. *Physica D: Nonlinear Phenomena*, *57*(3-4), 436-446.

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 \sum_{n=-\infty}^{+\infty} \delta(t - nT) \rightarrow \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -x - x^2 \sum \delta(t - nT) \end{pmatrix}$$

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Ejercicio:
Llamando (x_n, p_n) a la posición y momento en el instante previo a la patada n-ésima, escribir el mapa

$$\begin{pmatrix} x_n \\ p_n \end{pmatrix} \rightarrow \begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix}$$

Para ello:

i) Escriba (x'_n, p'_n) (posición y momento en el instante posterior a la patada n-ésima) como función de (x_n, p_n)
ii) Notando que entre patadas el sistema evoluciona como un oscilador armónico, integre las ecuaciones con condiciones iniciales (x'_n, p'_n)
iii) Evalúe estas ecuaciones $x(t), p(t)$ en el instante previo a la patada n+1-ésima

$$\begin{pmatrix}\dot{x}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}p\\ -x - x^2\sum\delta(t-nT)\end{pmatrix}$$

Miramos al sistema estroboscópicamente, cada tiempo T

$$\begin{pmatrix}x_n\\ p_n\end{pmatrix} \overset{?}{\rightarrow} \begin{pmatrix}x_{n+1}\\ p_{n+1}\end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -x - x^2 \sum \delta(t - nT) \end{pmatrix}$$

Miramos al sistema estroboscópicamente,
cada tiempo T

$$\begin{pmatrix} x_n \\ p_n \end{pmatrix} \overset{?}{\rightarrow} \begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix}$$

$$x(t = nT - \epsilon) = x(t_n) = x_n = x(antes\ de\ patada\ n)$$

$$x(t = nT + \epsilon) = x'_n = x(después\ de\ patada\ n)$$

$$\begin{pmatrix}\dot{x}\\ \dot{p}\end{pmatrix} = \begin{pmatrix} p \\ -x - x^2 \sum \delta(t - nT)\end{pmatrix}$$

Miramos al sistema estroboscópicamente, cada tiempo T

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$$x(t = nT + \epsilon) = x'_n = x(después\ de\ patada\ n)$$

$$x'_n = x_n$$

$$p'_n = p_n - x_n^2$$

$$x'_n = x_n$$

$$p'_n = p_n - x_n^2$$

Entre patadas evoluciona como un OA

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -x - x^2 \sum \delta(t - nT) \end{pmatrix} \rightarrow \ddot{x} = -x$$

$$x'_n = x_n$$

$$p'_n = p_n - x_n^2$$

Entre patadas evoluciona como un OA

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -x - x^2 \sum \delta(t - nT) \end{pmatrix} \rightarrow \ddot{x} = -x \rightarrow x(t) = C_1 \cos(t - nT) + C_2 \sin(t - nT)$$

$$p(t) = -C_1 \sin(t - nT) + C_2 \cos(t - nT)$$

$$x'_n = x_n$$

$$p'_n = p_n - x_n^2$$

Entre patadas evoluciona como un OA

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$$p(t) = -C_1 \sin(t - nT) + C_2 \cos(t - nT)$$

$$x(nT) = x' = x_n = C_1$$

$$p(nT) = p'_n = p_n - x_n^2 = C_2$$

$$x'_n = x_n$$
$$p'_n = p_n - x_n^2$$

Entre patadas evoluciona como un OA

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -x - x^2 \sum \delta(t - nT) \end{pmatrix} \rightarrow \ddot{x} = -x \rightarrow x(t) = C_1 \cos(t - nT) + C_2 \sin(t - nT)$$
$$p(t) = -C_1 \sin(t - nT) + C_2 \cos(t - nT)$$

$$x(nT) = x' = x_n = C_1$$
$$p(nT) = p'_n = p_n - x_n^2 = C_2$$

$$x\big((n+1)T\big) = x_{n+1} = x_n \cos(T) + (p_n - x_n^2)\sin(T)$$
$$p\big((n+1)T\big) = p_{n+1} = -x_n \sin(T) + (p_n - x_n^2)\cos(T)$$

$$x_{n+1} = x_n \cos(T) + (p_n - x_n^2)\sin(T)$$

$$p_{n+1} = -x_n \sin(T) + (p_n - x_n^2)\cos(T)$$

$$x_{n+1} = x_n \cos(T) + (p_n - x_n^2) \sin(T)$$

$$p_{n+1} = -x_n \sin(T) + (p_n - x_n^2) \cos(T)$$

Existe una transformación que lleva el mapa a la forma

$$X_{n+1} = 1 - aX_n^2 + Y_n$$

$$Y_{n+1} = bX_n$$

Mapa de Hénon

Mapa de Hénon

$X_{n+1} = 1 - aX_n^2 + Y_n$

$Y_{n+1} = bX_n$

Mapa de Hénon

$$X_{n+1} = 1 - aX_n^2 + Y_n$$

$$Y_{n+1} = bX_n$$

$$\binom{X_n}{Y_n}$$

Mapa de Hénon

$$X_{n+1} = 1 - aX_n^2 + Y_n$$

$$Y_{n+1} = bX_n$$

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} X_n \\ 1 - aX_n^2 + Y_n \end{pmatrix}$$

Mapa de Hénon

$$X_{n+1} = 1 - aX_n^2 + Y_n$$

$$Y_{n+1} = bX_n$$

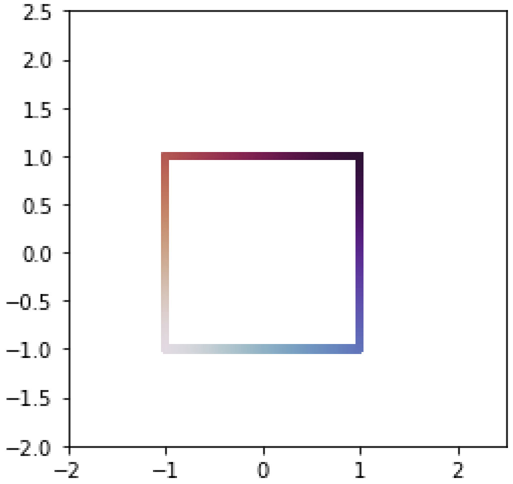
$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} X_n \\ 1 - aX_n^2 + Y_n \end{pmatrix} \rightarrow \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} bX_1 \\ Y_1 \end{pmatrix} \rightarrow \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} = \begin{pmatrix} Y_2 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix}$$

Mapa de Hénon

$$X_{n+1} = 1 - aX_n^2 + Y_n \qquad a = 1.4$$

$$Y_{n+1} = bX_n \qquad b = 0.3$$

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \to \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} X_n \\ 1 - aX_n^2 + Y_n \end{pmatrix} \to \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} bX_1 \\ Y_1 \end{pmatrix} \to \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} = \begin{pmatrix} Y_2 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix}$$



Mapa de Hénon

$X_{n+1} = 1 - aX_n^2 + Y_n$ $\qquad a = 1.4$

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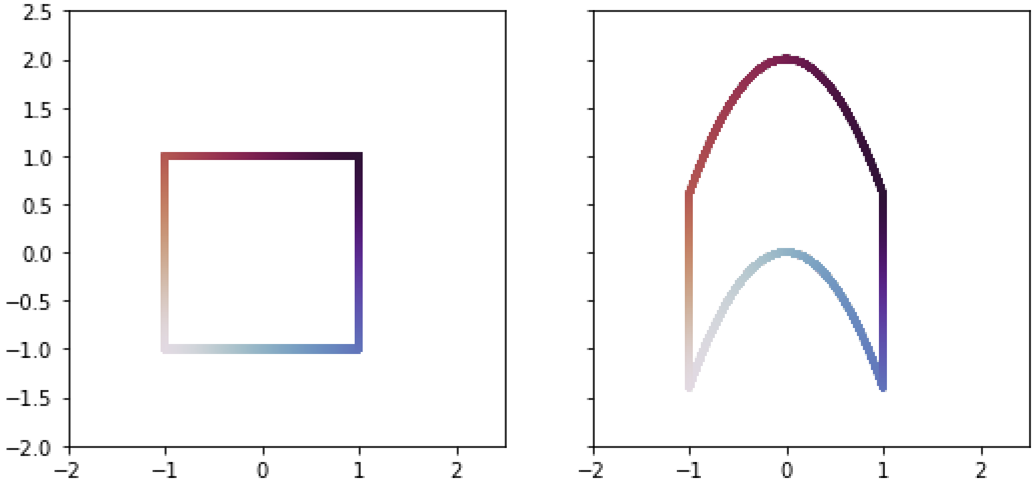
$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} X_n \\ 1 - aX_n^2 + Y_n \end{pmatrix} \rightarrow \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} bX_1 \\ Y_1 \end{pmatrix} \rightarrow \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} = \begin{pmatrix} Y_2 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix}$$



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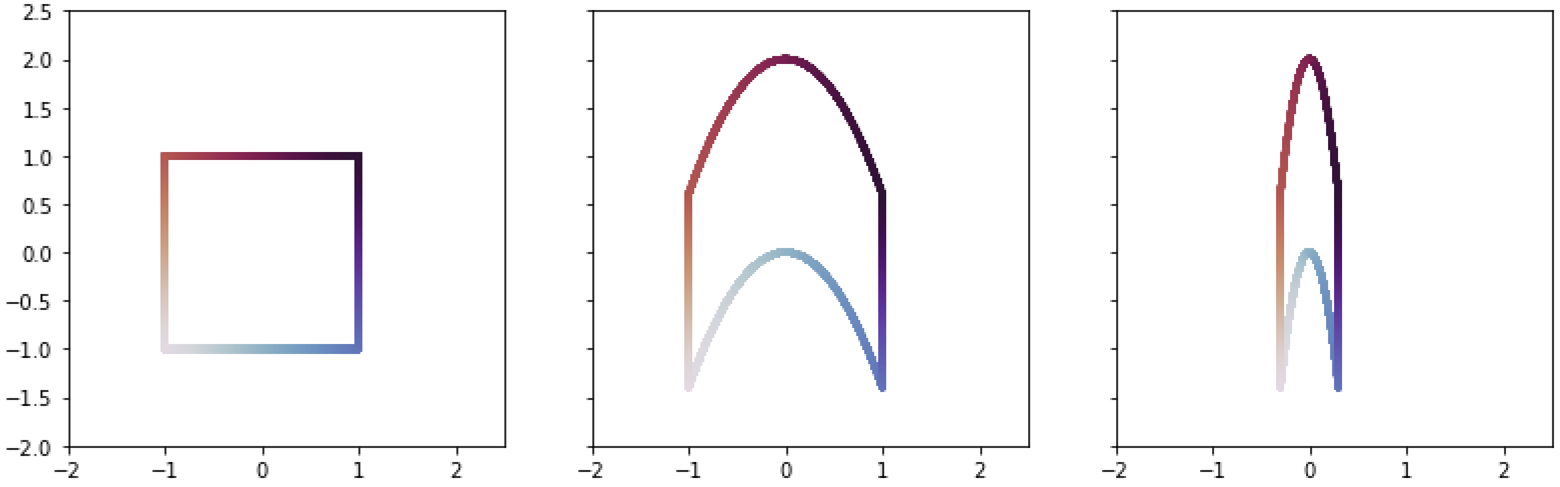

Mapa de Hénon

$$X_{n+1} = 1 - aX_n^2 + Y_n \qquad a = 1.4$$

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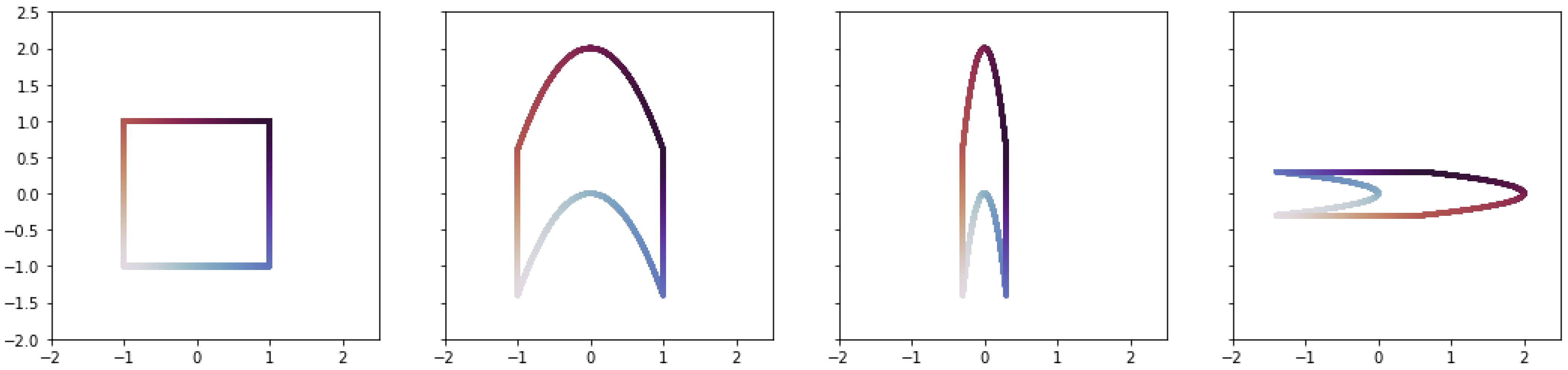
$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} X_n \\ 1 - aX_n^2 + Y_n \end{pmatrix} \rightarrow \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} bX_1 \\ Y_1 \end{pmatrix} \rightarrow \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} = \begin{pmatrix} Y_2 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix}$$



Mapa de Hénon

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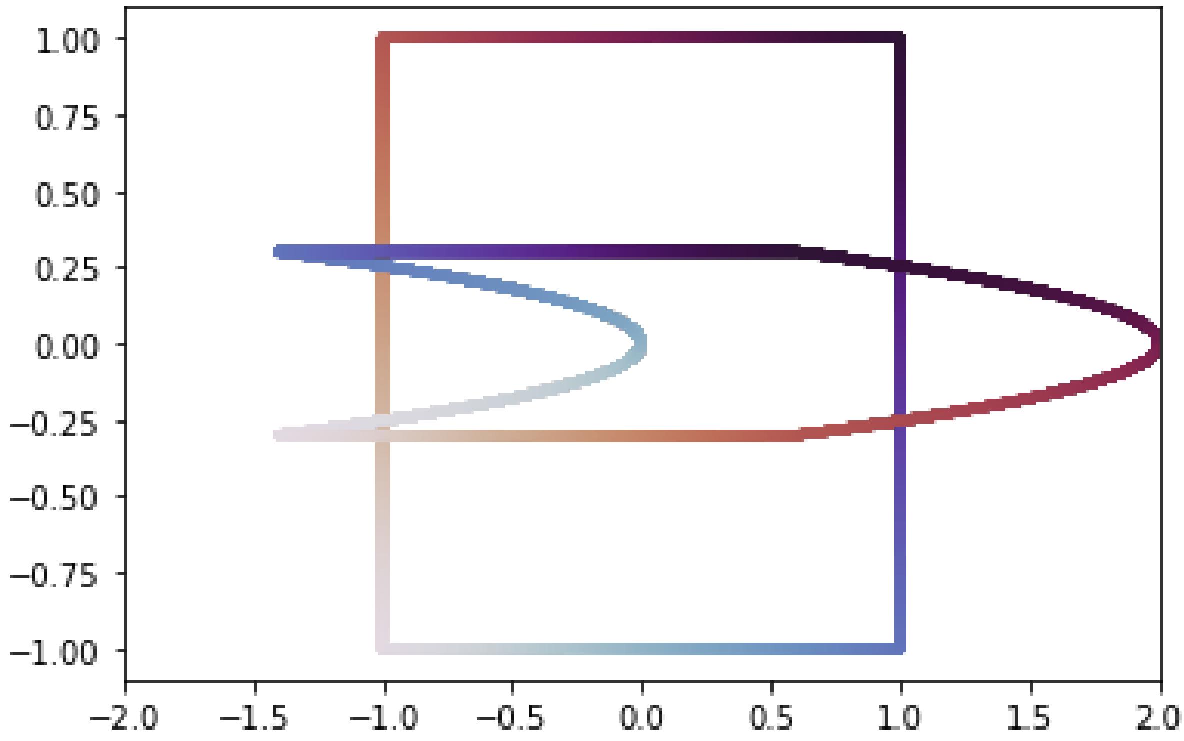
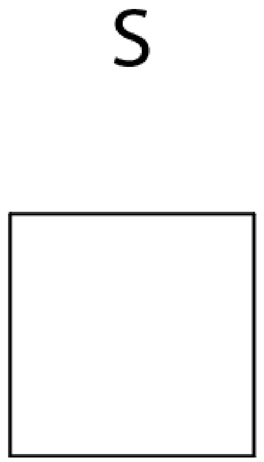


Smale



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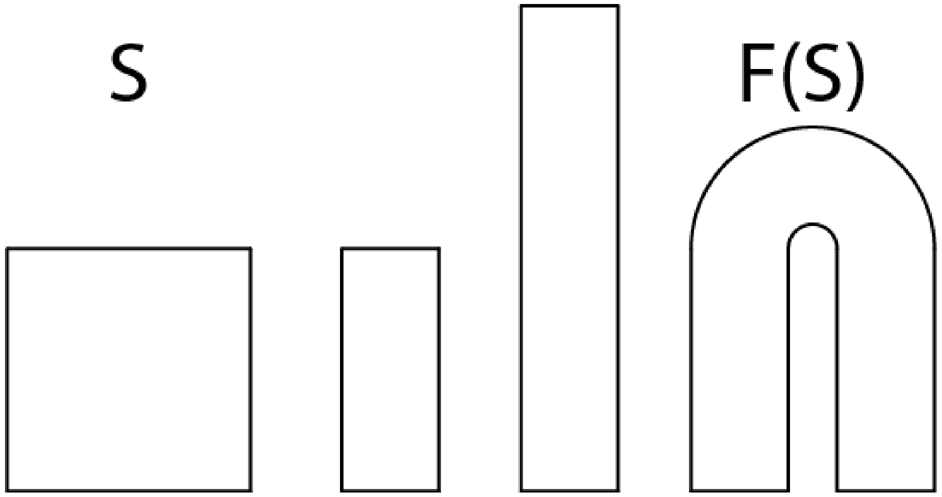
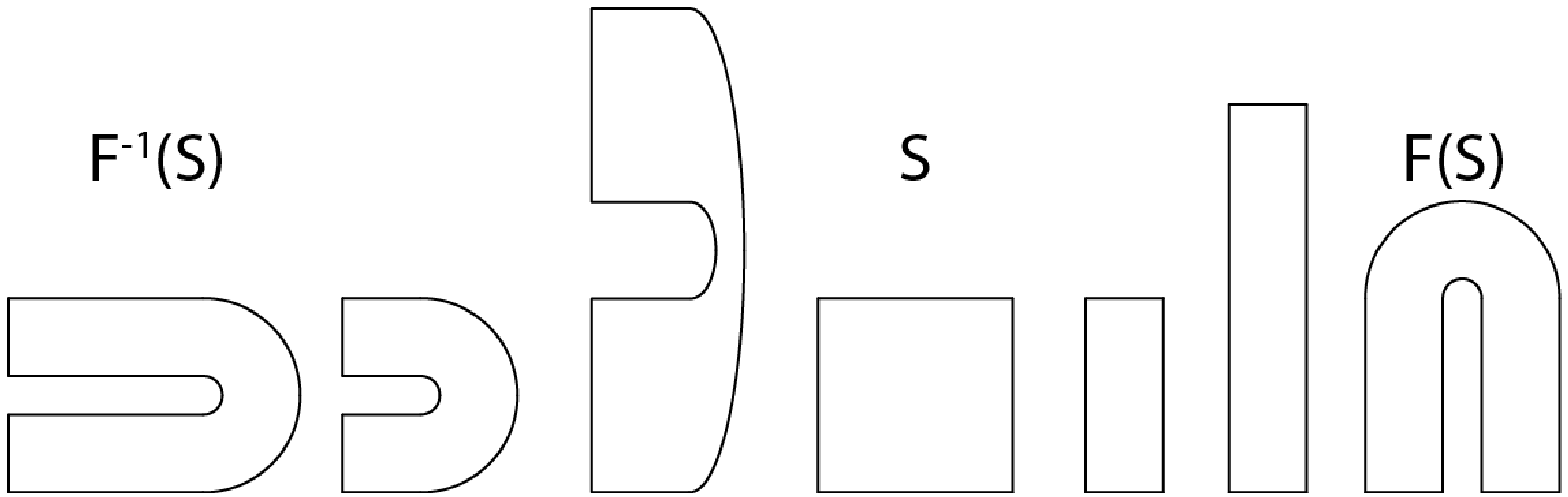
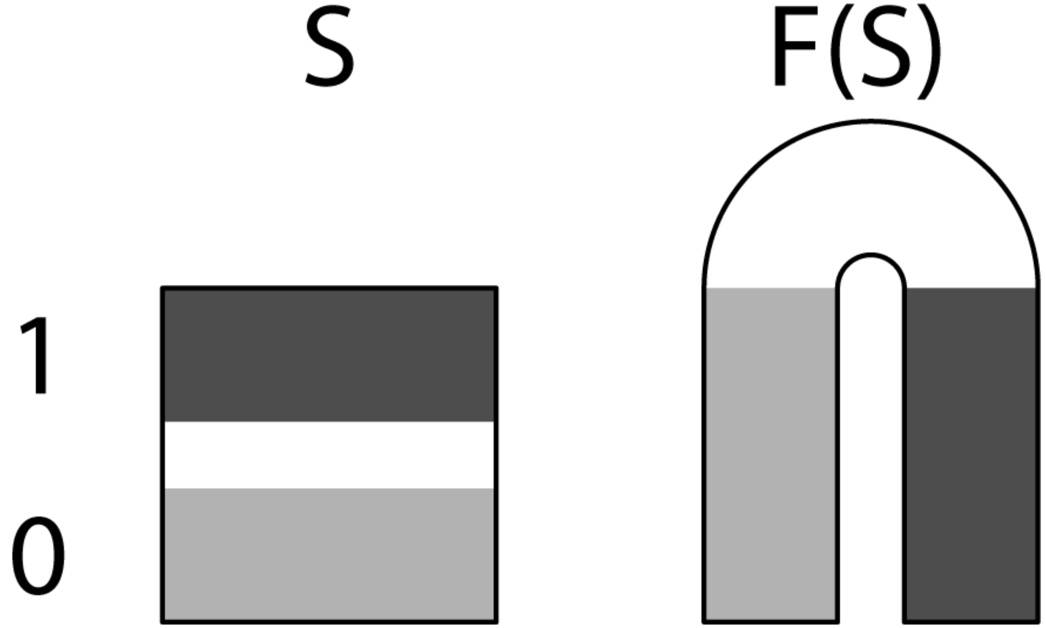


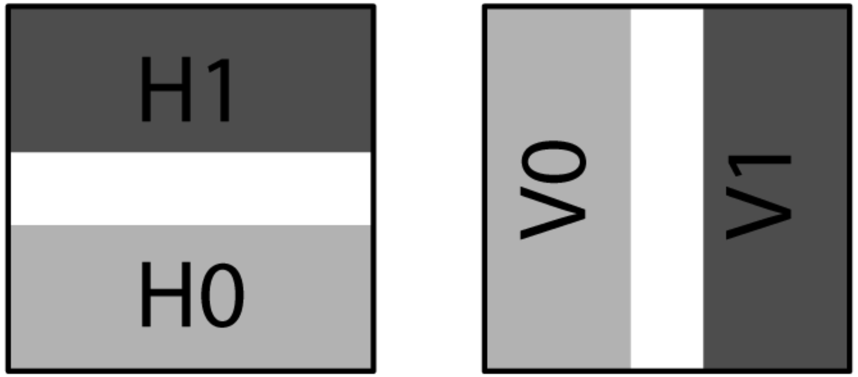
Smale



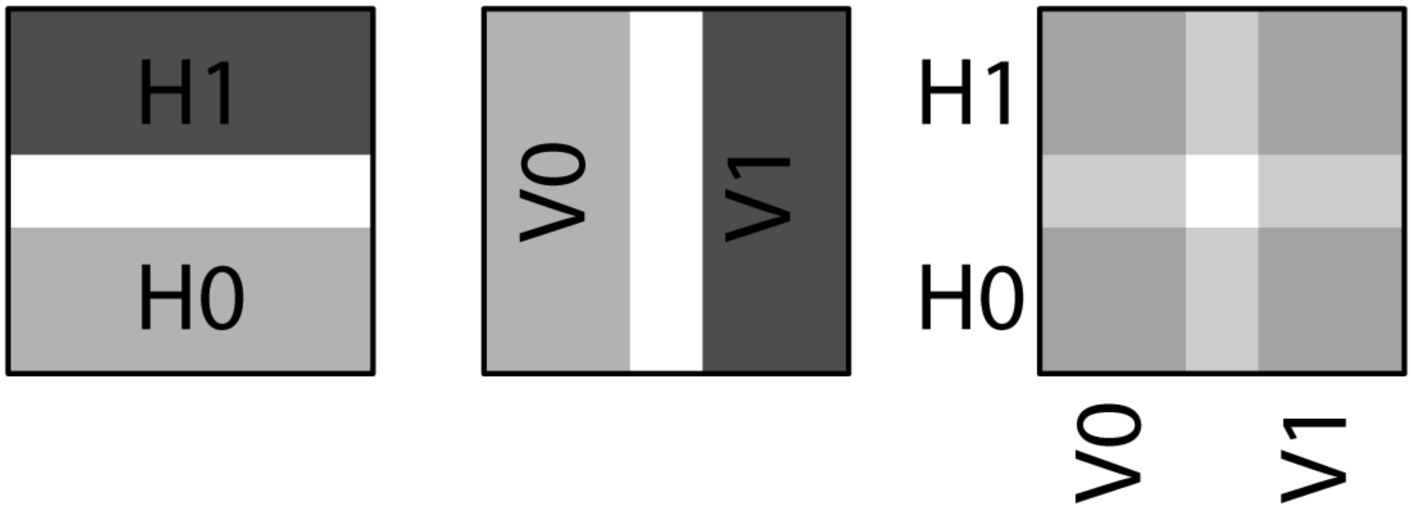
Aproximación del conjunto invariante



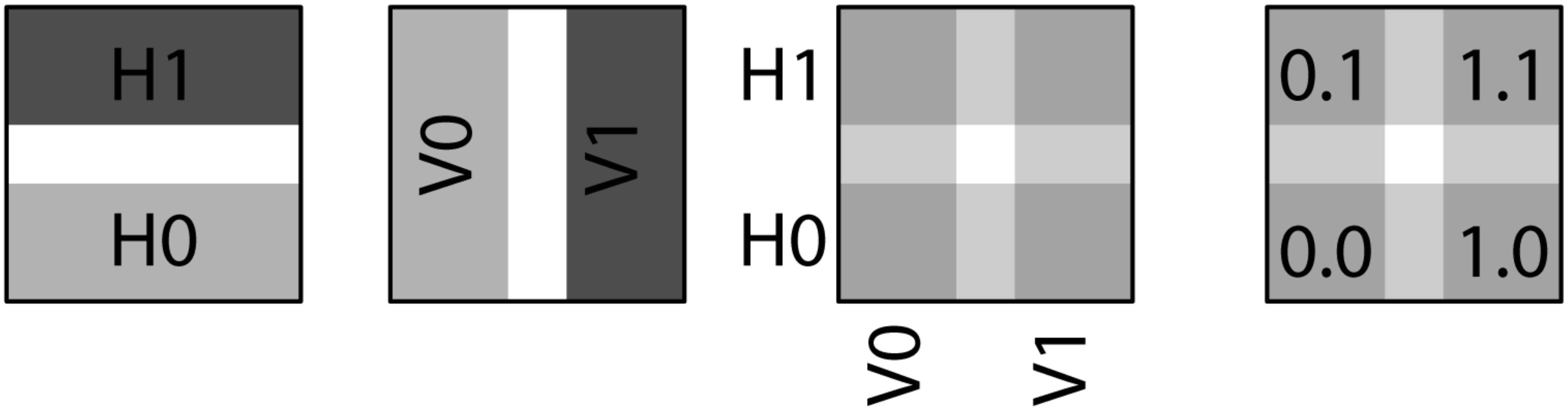
Aproximación del conjunto invariante

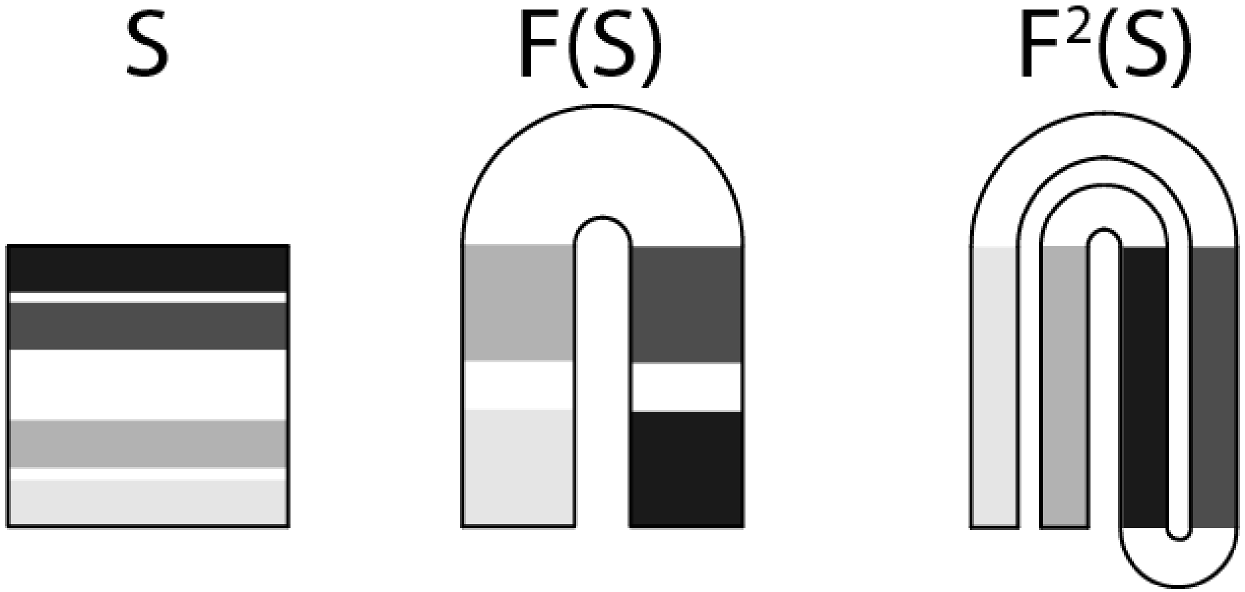


Aproximación del conjunto invariante



Aproximación del conjunto invariante

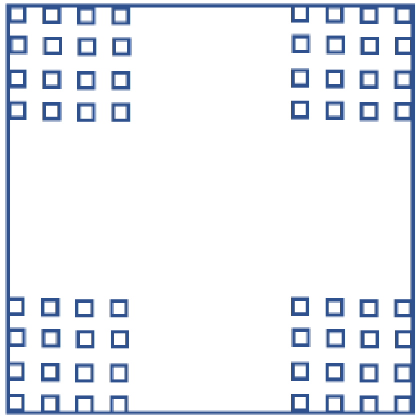



S
F(S)
$F^2(S)$

6. **Mapa de la herradura (Mapa de Smale)** Trabajando en la aproximación al conjunto invariante aplicando tres veces el mapa

$$f^{-3}(s) \cap f^{-2}(s) \cap f^{-1}(s) \cap s \cap f^{1}(s) \cap f^{2}(s) \cap f^{3}(s)$$

(a) Nombre todos los sectores en la aproximación (ayúdese con la figura)



(b) Ubique en qué casilleros caen:

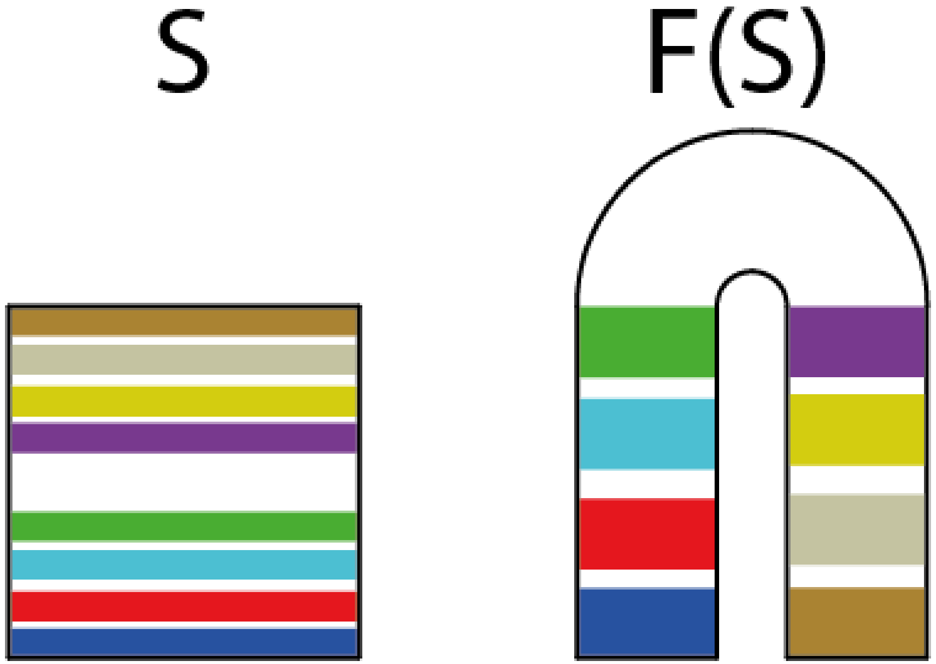
i. [001, 100, 010]

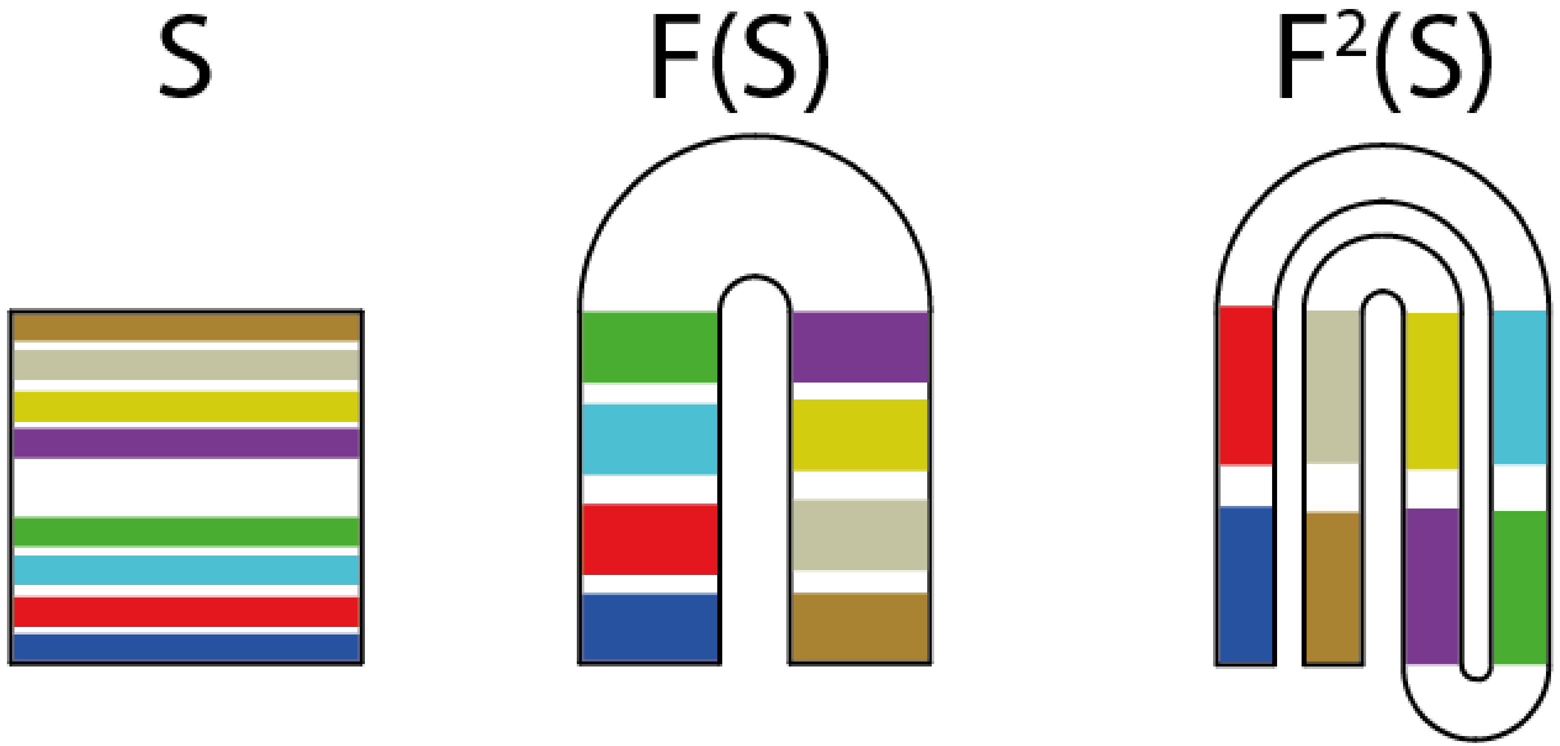
ii. [01, 10]

iii. [011, 101, 110]

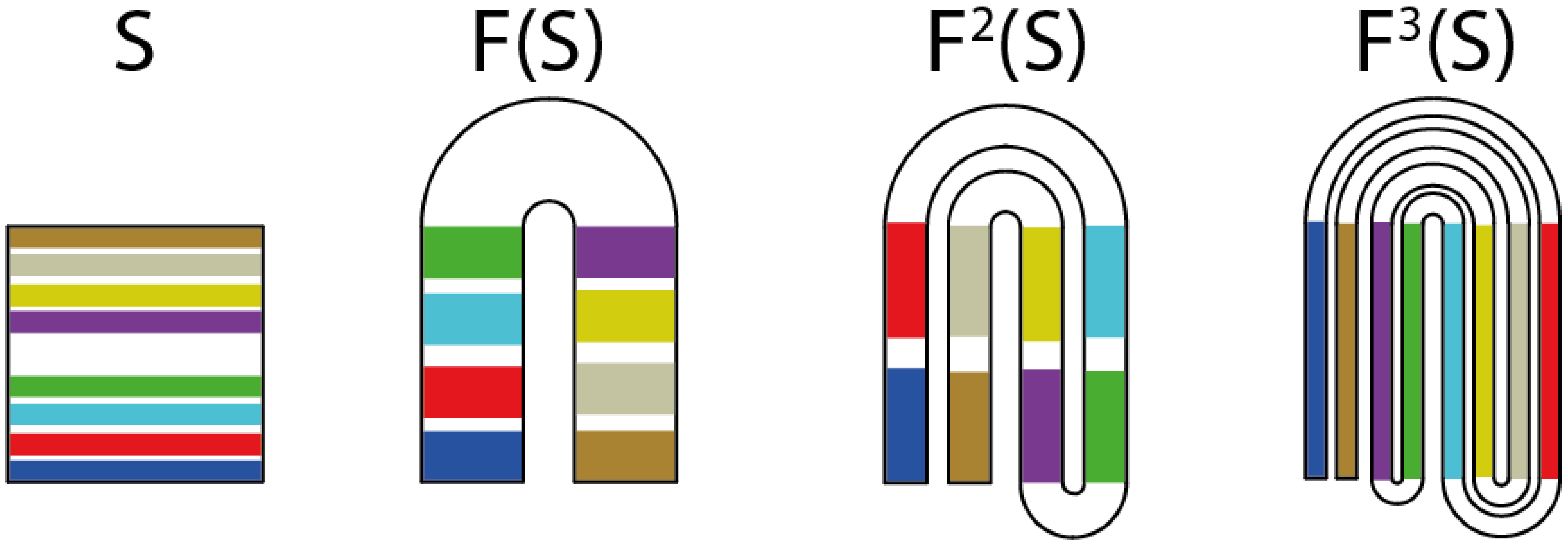
iv. [1] y [0]

S

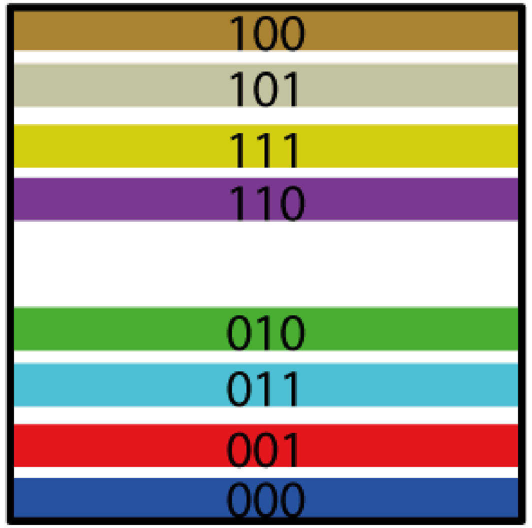

S
F(S)



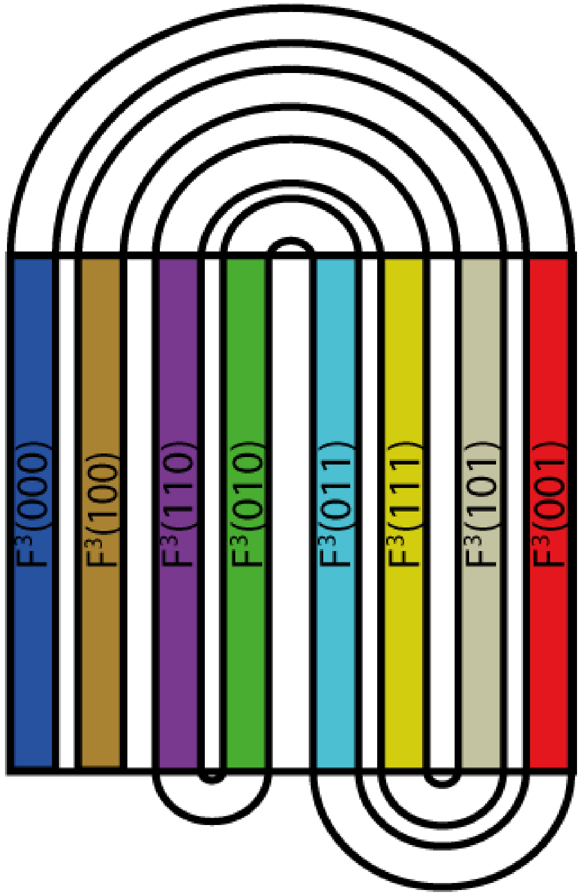
S
F(S)
F²(S)

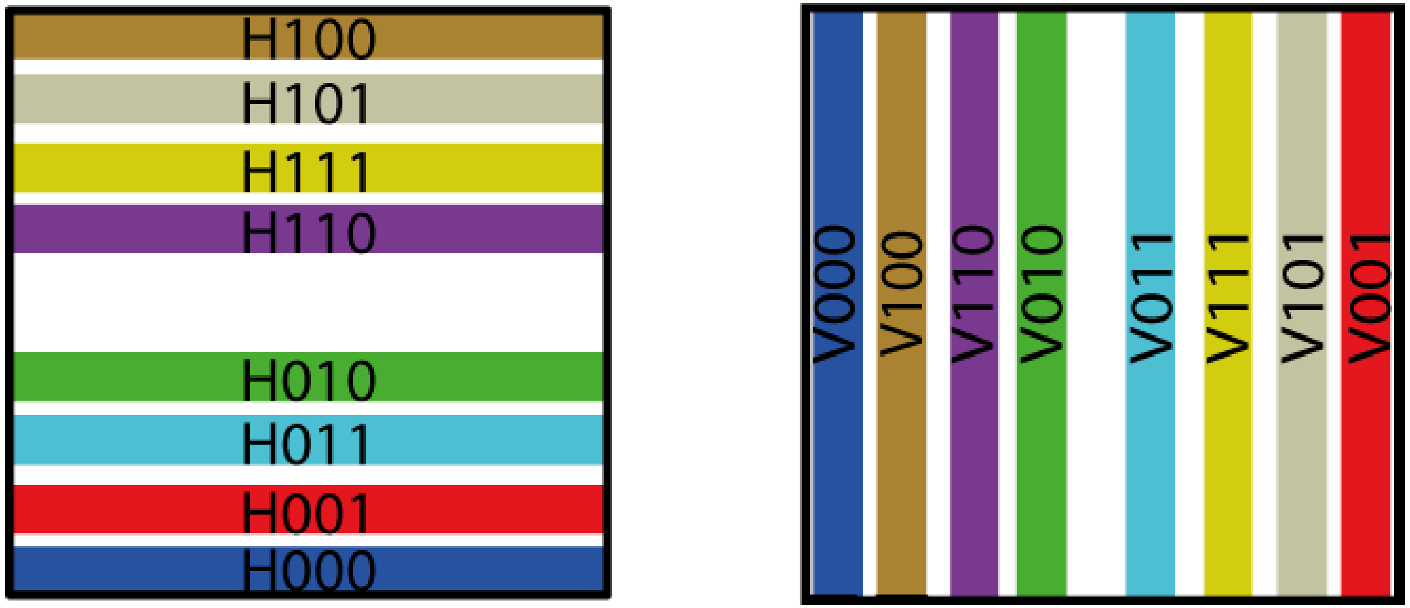

S
F(S)
F2(S)
F3(S)

S

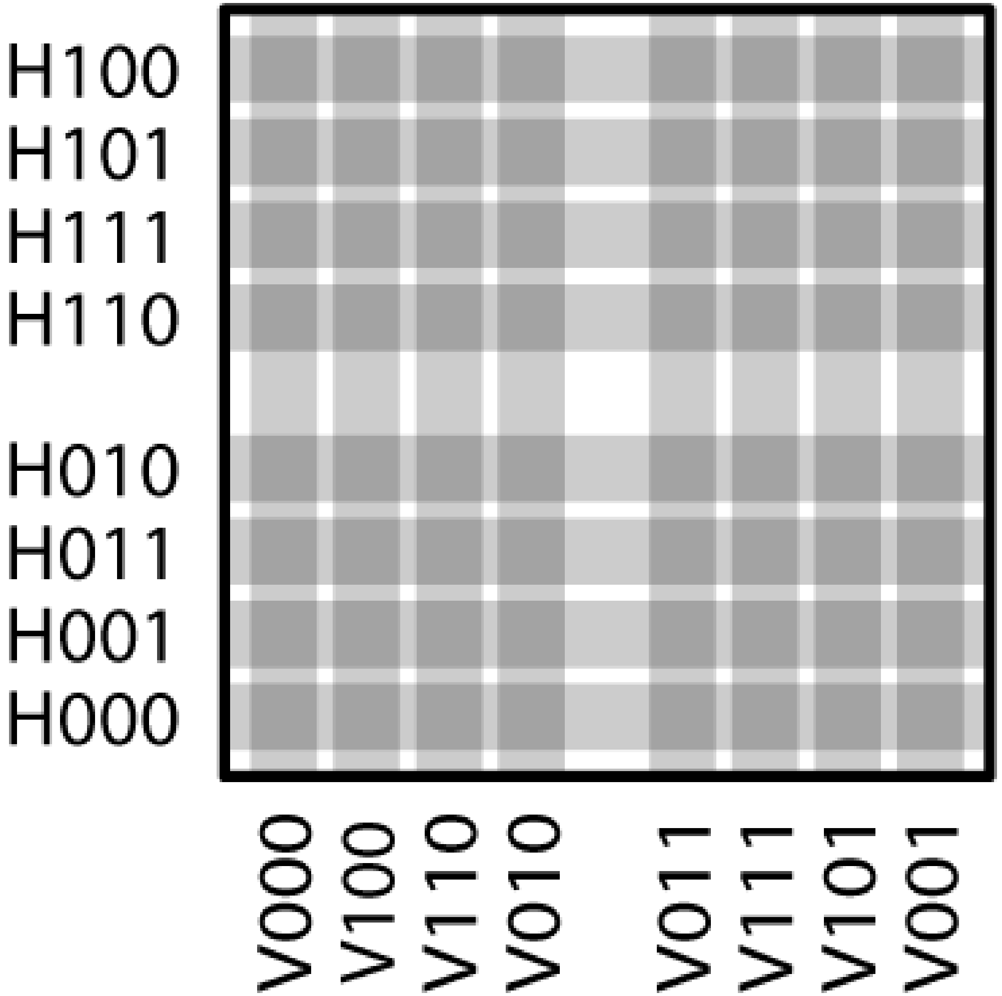

100
101
111
110
010
011
001
000

$F^3(S) \cap S$


F3(000)
F3(100)
F3(110)
F3(010)
F3(011)
F3(111)
F3(101)
F3(001)


H100
H101
H111
H110
H010
H011
H001
H000
V000
V100
V110
V010
V011
V111
V101
V001

(a) Nombre todos los sectores en la aproximación (ayúdese con la figura)



(b) Ubique en qué casilleros caen:

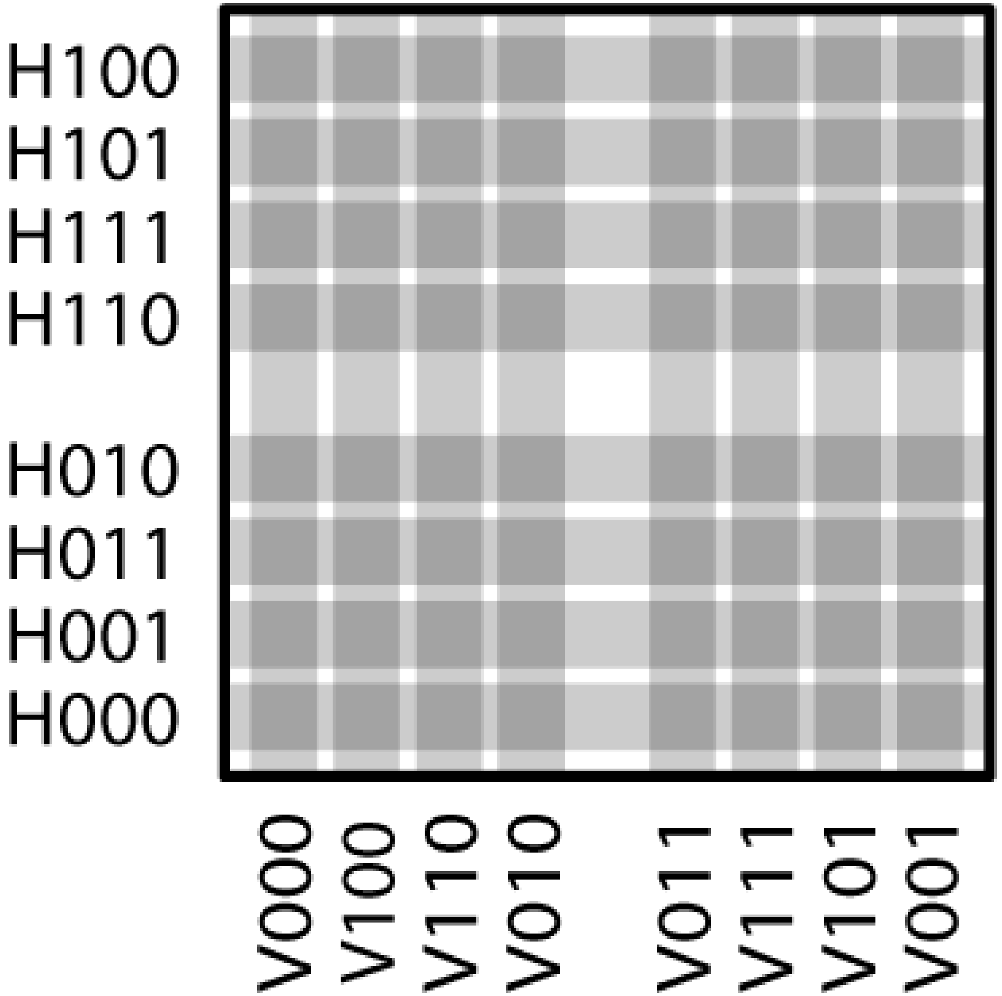
i. $[001, 100, 010]$

ii. $[01, 10]$

iii. $[011, 101, 110]$

iv. $[1]$ y $[0]$

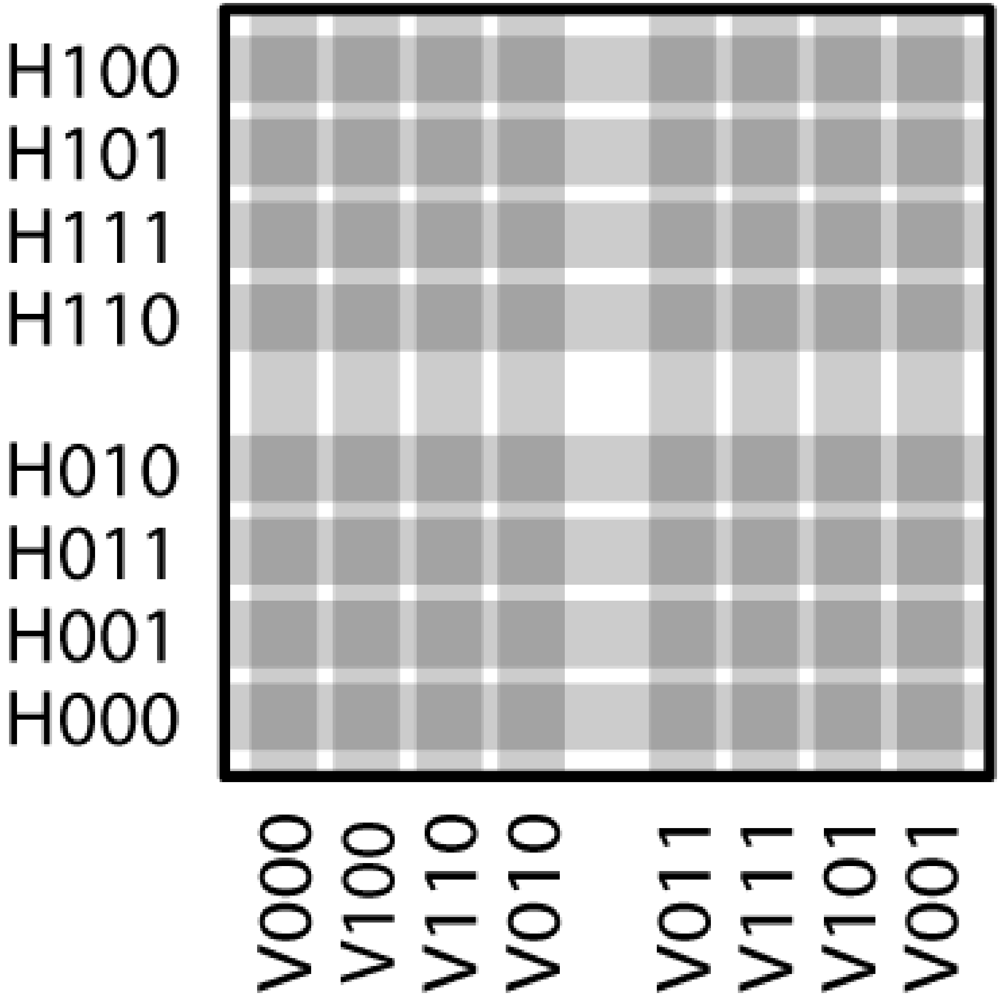
$[100, 010, 001] \rightarrow \qquad 100 \cdot 100$



(b) Ubique en qué casilleros caen:

   i. $[001, 100, 010]$
   ii. $[01, 10]$
   iii. $[011, 101, 110]$
   iv. $[1]$ y $[0]$

$$[100, 010, 001] \rightarrow \ \dots 100100100 \cdot 100100100 \dots$$



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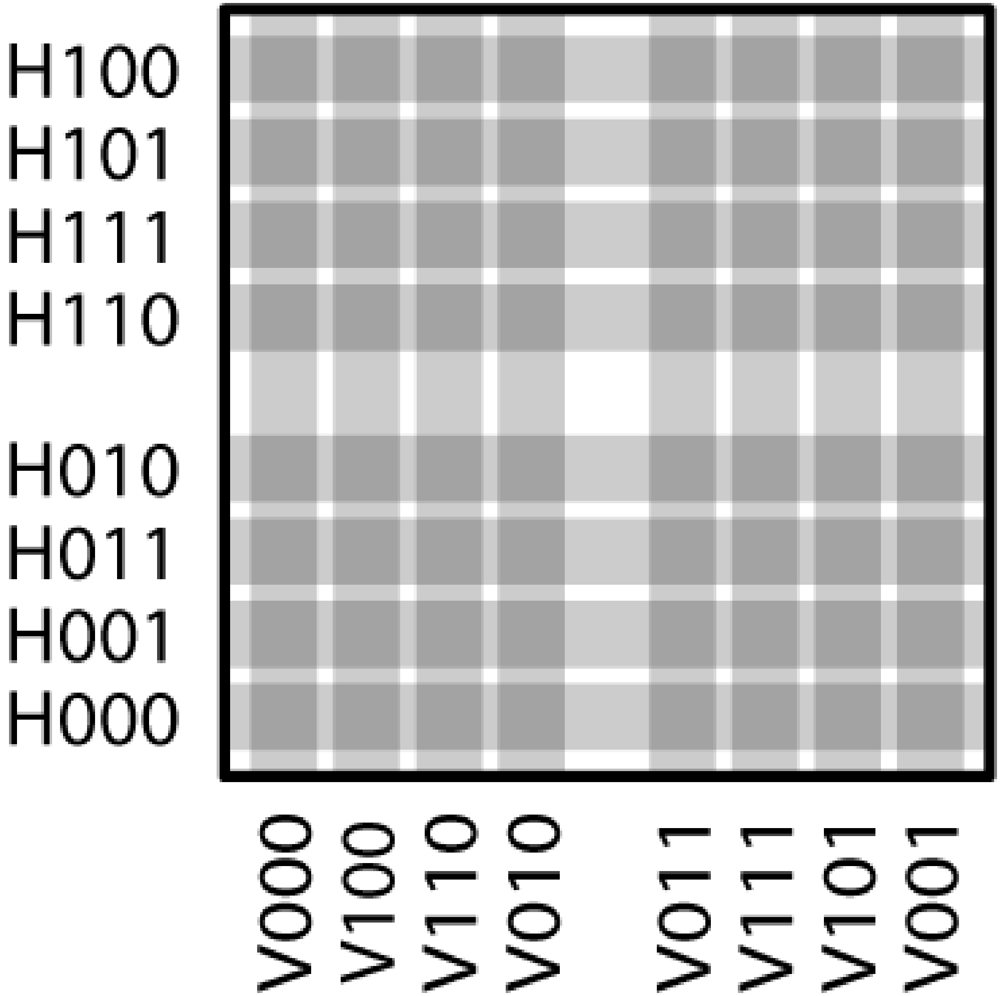
iv. [1] y [0]

$[100, 010, 001] \rightarrow \; \ldots 100100100 \cdot 100100100 \ldots$



$\ldots 100100100 \cdot 100100100 \ldots$

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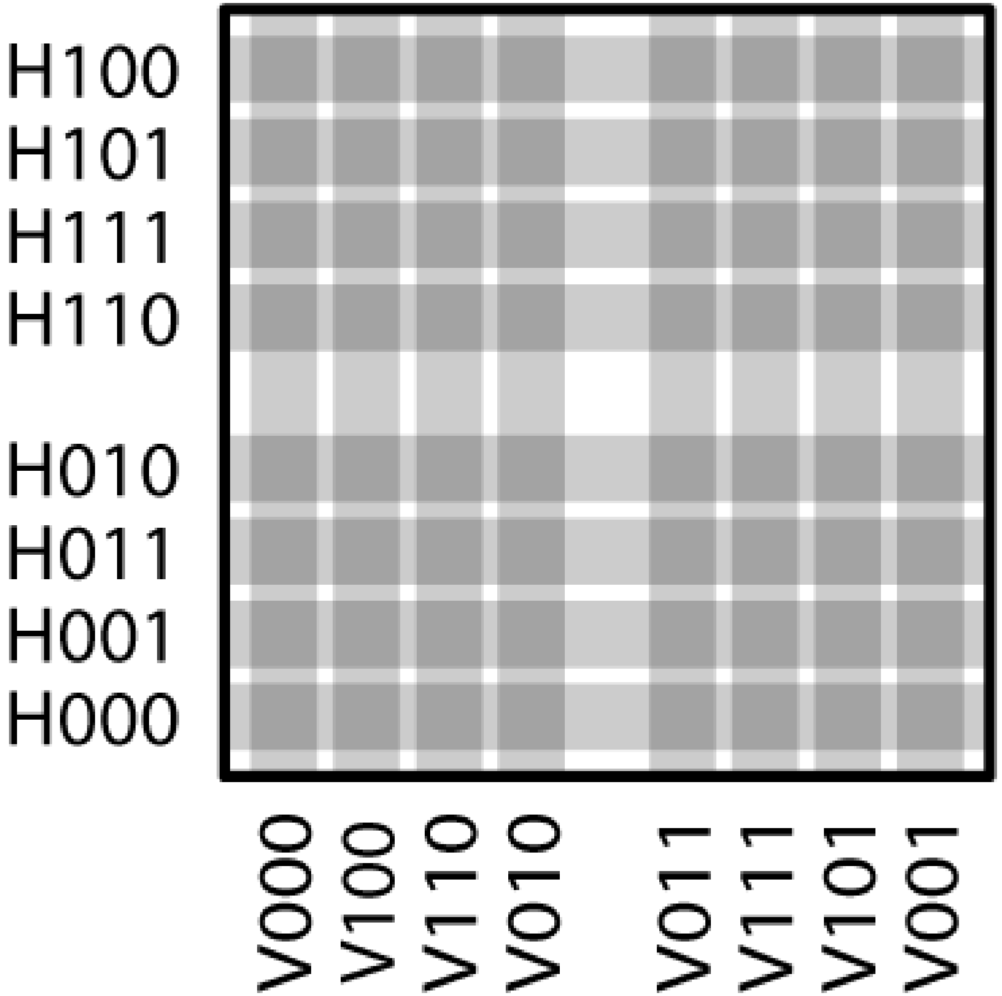

$\ldots 100100100 \cdot 100100100 \ldots$

$\ldots 1001001001 \cdot 00100100 \ldots$

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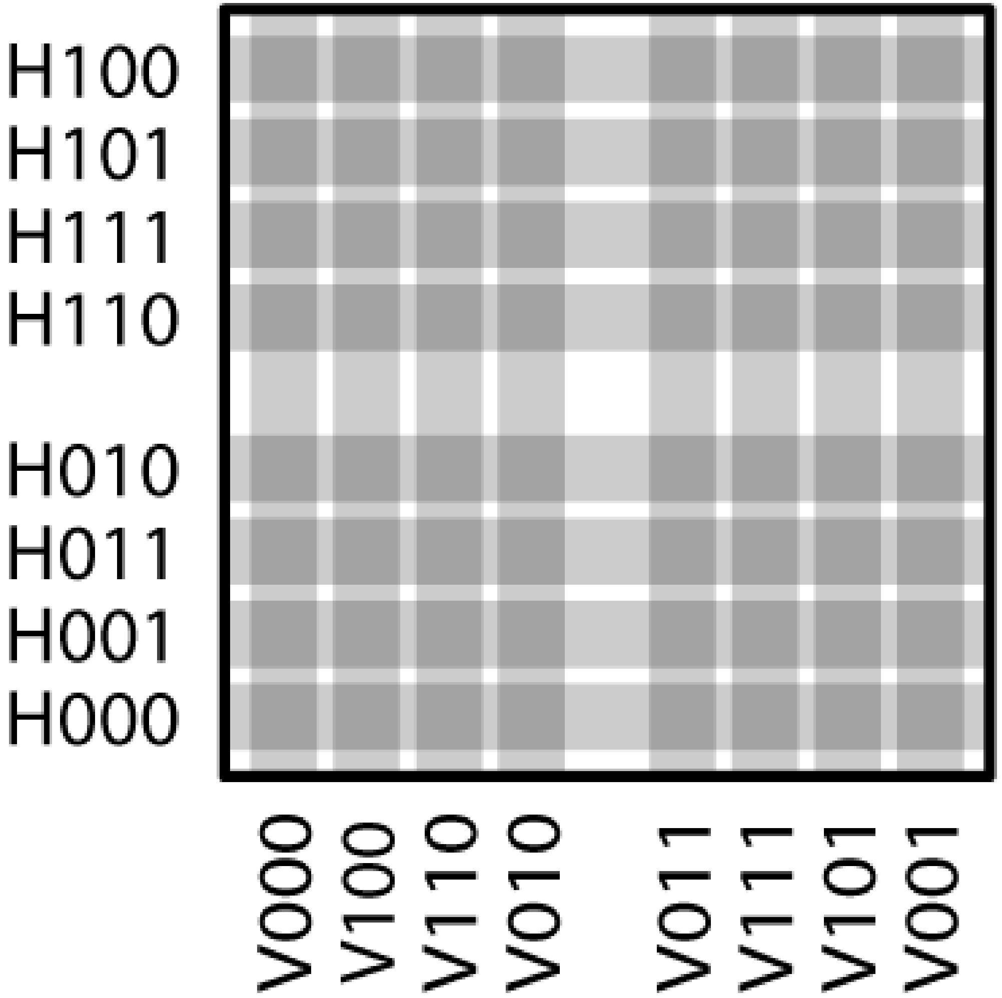
iv. $[1]$ y $[0]$

$$[100, 010, 001] \rightarrow \ \ldots 100100100 \cdot 100100100 \ldots$$



$$\ldots 100100100 \cdot 100100100 \ldots$$

$$\ldots 1001001001 \cdot 00100100 \ldots$$

$$\ldots 10010010010 \cdot 0100100 \ldots$$

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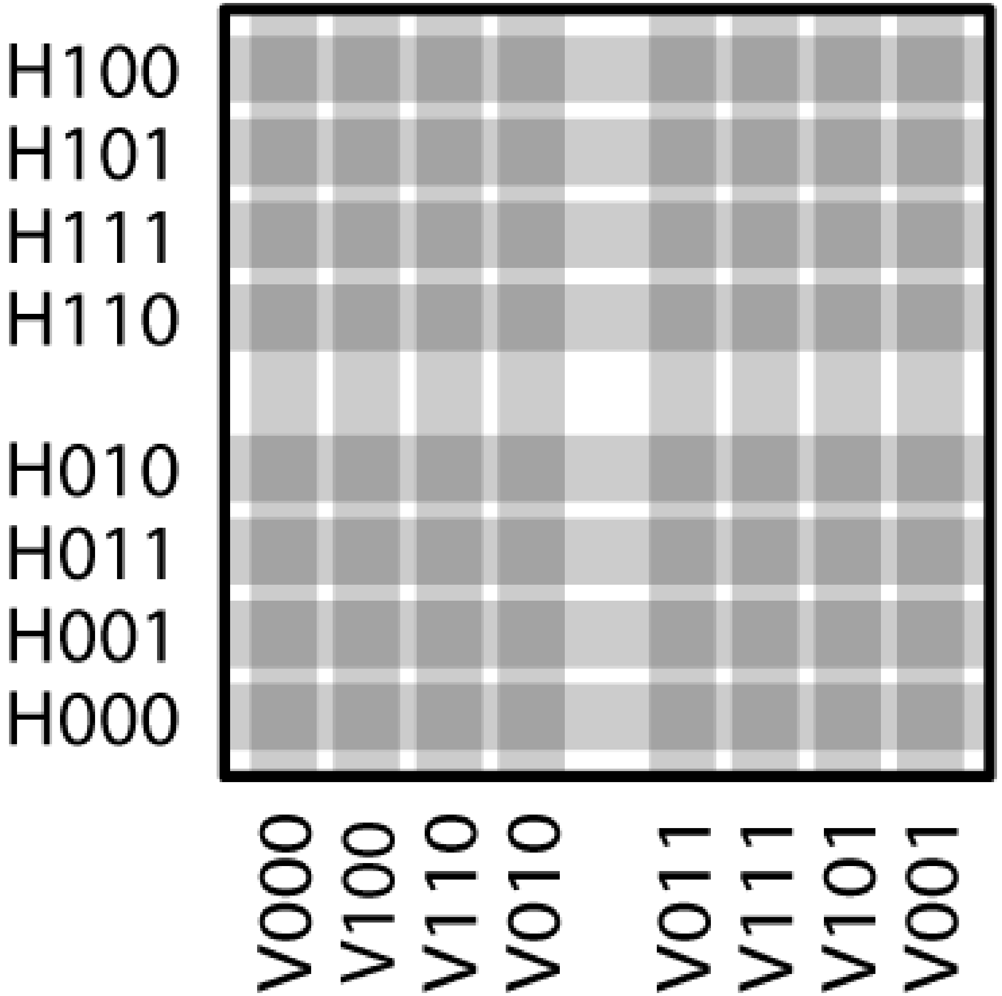
iii. $[011, 101, 110]$
iv. $[1]$ y $[0]$

$$[100, 010, 001] \to \ldots 100100100 \cdot 100100100 \ldots$$



$$\ldots 100100100 \cdot 100100100 \ldots$$

$$\ldots 1001001001 \cdot 00100100 \ldots$$

$$\ldots 10010010010 \cdot 0100100 \ldots$$

$$\ldots 100100100100 \cdot 100100 \ldots$$

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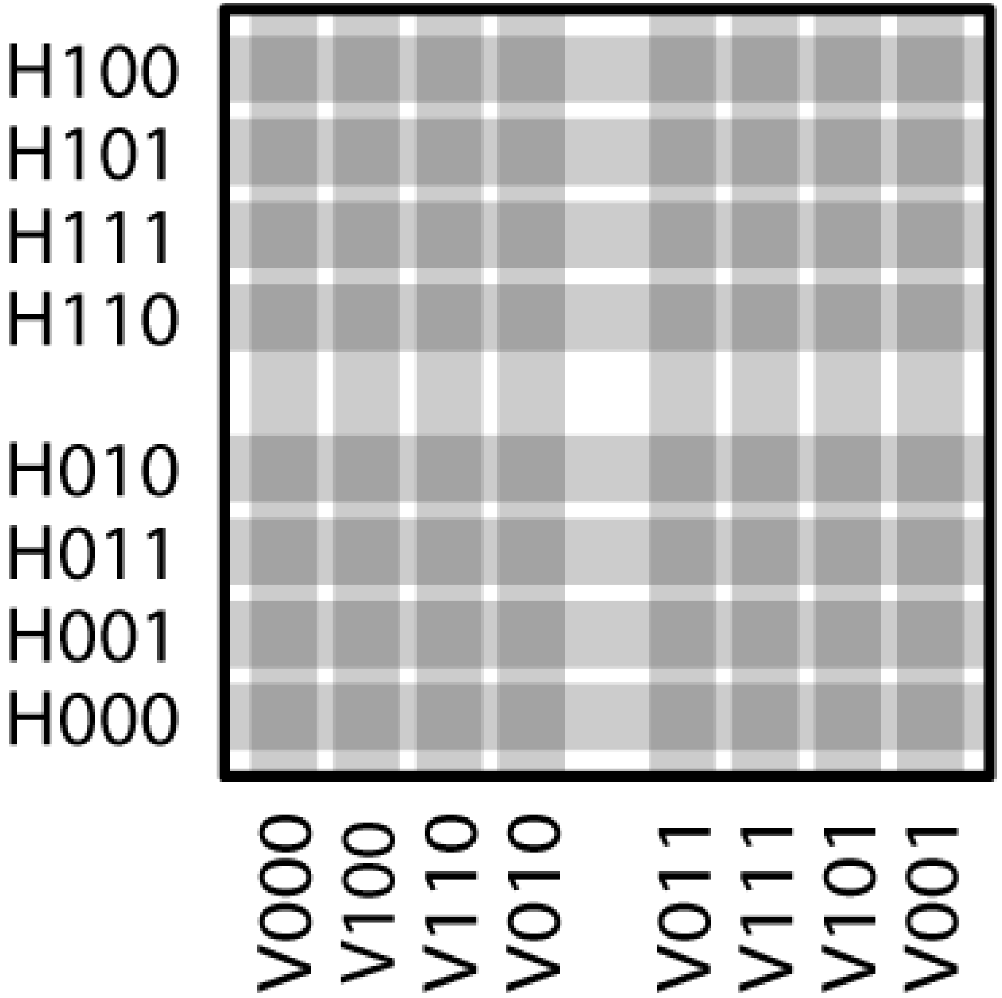

$\ldots 100100100 \cdot 100100100 \ldots$

$\ldots 1001001001 \cdot 00100100 \ldots$

$\ldots 10010010010 \cdot 0100100 \ldots$

$\ldots 100100100100 \cdot 100100 \ldots$