Appendix B

Curvilinear Coordinates

B1. Cylindrical Polar Coordinates

The coordinates are (R, θ, x), where θ is the azimuthal angle (see Figure 3.1b, where φ is used instead of θ). The equations are presented assuming ψ is a scalar, and

$$\mathbf{u} = \mathbf{i}_R u_R + \mathbf{i}_\theta u_\theta + \mathbf{i}_x u_x,$$

where $\mathbf{i}_R$, $\mathbf{i}_\theta$, and $\mathbf{i}_x$ are the local unit vectors at a point.

Gradient of a scalar

$$\nabla\psi = \mathbf{i}_R \frac{\partial\psi}{\partial R} + \frac{\mathbf{i}_\theta}{R}\frac{\partial\psi}{\partial\theta} + \mathbf{i}_x \frac{\partial\psi}{\partial x}.$$

Laplacian of a scalar

$$\nabla^2\psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial x^2}.$$

Divergence of a vector

$$\nabla\cdot\mathbf{u} = \frac{1}{R}\frac{\partial(Ru_R)}{\partial R} + \frac{1}{R}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_x}{\partial x}.$$

Curl of a vector

$$\nabla\times\mathbf{u} = \mathbf{i}_R\left(\frac{1}{R}\frac{\partial u_x}{\partial\theta} - \frac{\partial u_\theta}{\partial x}\right) + \mathbf{i}_\theta\left(\frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R}\right) + \mathbf{i}_x\left[\frac{1}{R}\frac{\partial(Ru_\theta)}{\partial R} - \frac{1}{R}\frac{\partial u_R}{\partial\theta}\right].$$

Laplacian of a vector

$$\nabla^2\mathbf{u} = \mathbf{i}_R\left(\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\theta}{\partial\theta}\right) + \mathbf{i}_\theta\left(\nabla^2 u_\theta + \frac{2}{R^2}\frac{\partial u_R}{\partial\theta} - \frac{u_\theta}{R^2}\right) + \mathbf{i}_x\nabla^2 u_x.$$

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$\begin{aligned}
e_{RR} &= \frac{\partial u_R}{\partial R} = \frac{1}{2\mu}\sigma_{RR}, \\
e_{\theta\theta} &= \frac{1}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{u_R}{R} = \frac{1}{2\mu}\sigma_{\theta\theta}, \\
e_{xx} &= \frac{\partial u_x}{\partial x} = \frac{1}{2\mu}\sigma_{xx}, \\
e_{R\theta} &= \frac{R}{2}\frac{\partial}{\partial R}\left(\frac{u_\theta}{R}\right) + \frac{1}{2R}\frac{\partial u_R}{\partial \theta} = \frac{1}{2\mu}\sigma_{R\theta}, \\
e_{\theta x} &= \frac{1}{2R}\frac{\partial u_x}{\partial \theta} + \frac{1}{2}\frac{\partial u_\theta}{\partial x} = \frac{1}{2\mu}\sigma_{\theta x}, \\
e_{xR} &= \frac{1}{2}\frac{\partial u_R}{\partial x} + \frac{1}{2}\frac{\partial u_x}{\partial R} = \frac{1}{2\mu}\sigma_{xR}.
\end{aligned}$$

Vorticity ($\boldsymbol{\omega} = \nabla \times \mathbf{u}$)

$$\begin{aligned}
\omega_R &= \frac{1}{R}\frac{\partial u_x}{\partial \theta} - \frac{\partial u_\theta}{\partial x}, \\
\omega_\theta &= \frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R}, \\
\omega_x &= \frac{1}{R}\frac{\partial}{\partial R}(R u_\theta) - \frac{1}{R}\frac{\partial u_R}{\partial \theta}.
\end{aligned}$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(\rho R u_R) + \frac{1}{R}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial x}(\rho u_x) = 0.$$

Navier–Stokes equations with constant ρ and ν, and no body force

$$\begin{aligned}
\frac{\partial u_R}{\partial t} + (\mathbf{u}\cdot\nabla)u_R - \frac{u_\theta^2}{R} &= -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\theta}{\partial \theta}\right), \\
\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_R u_\theta}{R} &= -\frac{1}{\rho R}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{R^2}\frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R^2}\right), \\
\frac{\partial u_x}{\partial t} + (\mathbf{u}\cdot\nabla)u_x &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u_x,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{u}\cdot\nabla &= u_R\frac{\partial}{\partial R} + \frac{u_\theta}{R}\frac{\partial}{\partial \theta} + u_x\frac{\partial}{\partial x}, \\
\nabla^2 &= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}.
\end{aligned}$$

B2. Plane Polar Coordinates

The plane polar coordinates are (r, θ), where r is the distance from the origin (Figure 3.1a). The equations for plane polar coordinates can be obtained from those of the cylindrical coordinates presented in Section B1, replacing R by r and suppressing all components and derivatives in the axial direction x. Some of the expressions are repeated here because of their frequent occurrence.

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$
\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr}, \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta}, \\
e_{r\theta} &= \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta} = \frac{1}{2\mu}\sigma_{r\theta}.
\end{aligned}
$$

Vorticity

$$
\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}.
$$

Equation of continuity

$$
\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) = 0.
$$

Navier–Stokes equations with constant ρ and ν, and no body force

$$
\begin{aligned}
\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right), \\
\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right),
\end{aligned}
$$

where

$$
\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.
$$

B3. Spherical Polar Coordinates

The spherical polar coordinates used are (r, θ, φ), where φ is the azimuthal angle (Figure 3.1c). Equations are presented assuming ψ is a scalar, and

$$
\mathbf{u} = \mathbf{i}_r u_r + \mathbf{i}_\theta u_\theta + \mathbf{i}_\varphi u_\varphi,
$$

where $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\varphi$ are the local unit vectors at a point.

Gradient of a scalar

$$\nabla\psi = \mathbf{i}_r \frac{\partial\psi}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{i}_\varphi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}.$$

Laplacian of a scalar

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}.$$

Divergence of a vector

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}.$$

Curl of a vector

$$\begin{aligned}\nabla\times\mathbf{u} = {}& \frac{\mathbf{i}_r}{r\sin\theta}\left[\frac{\partial(u_\varphi\sin\theta)}{\partial\theta} - \frac{\partial u_\theta}{\partial\varphi}\right] + \frac{\mathbf{i}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(ru_\varphi)}{\partial r}\right]\\ &+ \frac{\mathbf{i}_\varphi}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right].\end{aligned}$$

Laplacian of a vector

$$\begin{aligned}\nabla^2\mathbf{u} = {}& \mathbf{i}_r\left[\nabla^2u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right]\\ &+ \mathbf{i}_\theta\left[\nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right]\\ &+ \mathbf{i}_\varphi\left[\nabla^2u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right].\end{aligned}$$

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$\begin{aligned}e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr},\\ e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta},\\ e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r} = \frac{1}{2\mu}\sigma_{\varphi\varphi},\\ e_{\theta\varphi} &= \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial\varphi} = \frac{1}{2\mu}\sigma_{\theta\varphi},\\ e_{\varphi r} &= \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right) = \frac{1}{2\mu}\sigma_{\varphi r},\\ e_{r\theta} &= \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta} = \frac{1}{2\mu}\sigma_{r\theta}.\end{aligned}$$

Vorticity

$$\omega_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\varphi \sin\theta) - \frac{\partial u_\theta}{\partial\varphi}\right],$$

$$\omega_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(ru_\varphi)}{\partial r}\right],$$

$$\omega_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right].$$

Equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho u_\varphi) = 0.$$

Navier–Stokes equations with constant ρ and ν, and no body force

$$\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2\cot\theta}{r}$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi\cot\theta}{r}$$
$$= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left[\nabla^2 u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right],$$

where

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi},$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$