If we have a surface of separation between two immiscible fluids, the conditions at the surface are that the velocities of the fluids must be equal and the forces which they exert on each other must be equal and opposite. The latter condition is written

$$n_{1,k}\sigma_{1,ik} + n_{2,k}\sigma_{2,ik} = 0,$$

where the suffixes 1 and 2 refer to the two fluids. The normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are in opposite directions, i.e. $\mathbf{n}_1 = -\mathbf{n}_2 \equiv \mathbf{n}$, so that we can write

$$n_i\sigma_{1,ik} = n_i\sigma_{2,ik}. \tag{15.15}$$

At a free surface of the fluid the condition

$$\sigma_{ik}n_k \equiv \sigma'_{ik}n_k - pn_i = 0 \tag{15.16}$$

must hold.

EQUATIONS OF MOTION IN CURVILINEAR COORDINATES

We give below, for reference, the equations of motion for a viscous incompressible fluid in frequently used curvilinear coordinates. In cylindrical polar coordinates r, ϕ, z the components of the stress tensor are

$$\sigma_{rr} = -p + 2\eta\frac{\partial v_r}{\partial r}, \qquad \sigma_{r\phi} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right),$$

$$\sigma_{\phi\phi} = -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right), \qquad \sigma_{\phi z} = \eta\left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \phi}\right),$$

$$\sigma_{zz} = -p + 2\eta\frac{\partial v_z}{\partial z}, \qquad \sigma_{zr} = \eta\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right). \tag{15.17}$$

The three components of the Navier–Stokes equation are

$$\frac{\partial v_r}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_r - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\triangle v_r - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2}\right),$$

$$\frac{\partial v_\phi}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_\phi + \frac{v_r v_\phi}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu\left(\triangle v_\phi + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2}\right),$$

$$\frac{\partial v_z}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\triangle v_z, \tag{15.18}$$

where

$$(\mathbf{v}\cdot\mathbf{grad})f = v_r\frac{\partial f}{\partial r} + \frac{v_\phi}{r}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z},$$

$$\triangle f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

The equation of continuity is

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0. \tag{15.19}$$

In spherical polar coordinates r, ϕ, θ we have for the stress tensor

$$
\begin{aligned}
\sigma_{rr} &= -p + 2\eta\frac{\partial v_r}{\partial r},\\
\sigma_{\phi\phi} &= -p + 2\eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right),\\
\sigma_{\theta\theta} &= -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right),\\
\sigma_{r\theta} &= \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right),\\
\sigma_{\theta\phi} &= \eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\phi\cot\theta}{r}\right),\\
\sigma_{\phi r} &= \eta\left(\frac{\partial v_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r}\right),
\end{aligned}
\qquad (15.20)
$$

while the Navier–Stokes equations are

$$
\begin{aligned}
&\frac{\partial v_r}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_r - \frac{v_\theta^2 + v_\phi^2}{r}\\
&\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\triangle v_r - \frac{2}{r^2\sin^2\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi} - \frac{2v_r}{r^2}\right],\\
&\frac{\partial v_\theta}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_\theta + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\\
&\qquad = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\triangle v_\theta - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta}\right],\\
&\frac{\partial v_\phi}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})v_\phi + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi\cot\theta}{r}\\
&\qquad = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi} + \nu\left[\triangle v_\phi + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{v_\phi}{r^2\sin^2\theta}\right],
\end{aligned}
\qquad (15.21)
$$

where

$$
(\mathbf{v}\cdot\mathbf{grad})f = v_r\frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial f}{\partial\phi},
$$

$$
\triangle f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = 0.
$$

The equation of continuity is

$$
\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} = 0. \qquad (15.22)
$$