Lagrangian Formulation

As mentioned, the Eulerian formulation provides a field description of a flow. The Lagrange formulation provides a particle description. Suppose a fluid particle has the location $\vec{r} = \vec{r}_o$ at $t = t_o$. In the Lagrangian approach the independent variables are $\vec{r}_o$ and t. Thus, the position $\vec{r}$ of the fluid particle at time t is given by

$$\vec{r} = \vec{r}(\vec{r}_o, t) \tag{1.19}$$

where $\vec{r}_o$ is the particle's position at time t_o

$$\vec{r}_o = \vec{r}(\vec{r}_o, t_o)$$

and $\vec{r}_o$ is a fixed label on the particle as it moves. In this formulation, the velocity and acceleration are

$$\vec{w} = \frac{\partial \vec{r}}{\partial t}, \qquad \vec{a} = \frac{\partial^2 \vec{r}}{\partial t^2} \tag{1.20}$$

where $\vec{r}_o$ is kept fixed in both derivatives.

The two formulations can be related by assuming we know $\vec{w}(\vec{r}, t)$ in the Eulerian description. We then integrate Equation (1.13) subject to the initial condition

$$\vec{r} = \vec{r}_o \quad \text{at} \quad t = t_o \tag{1.21}$$

The solution is then the Lagrangian description, Equation (1.19).

The Lagrangian approach is widely used in mechanics, e.g., consider a marble rolling down an inclined curved plane under the influence of gravity. The problem is solved by first establishing a differential equation for the motion of the marble. The solution of this equation provides the position of the marble as a function of time and its initial position.

The Lagrangian description is seldom used in fluid dynamics. One exception occurs in the unsteady, one-dimensional, anisentropic flow of an inviscid, compressible fluid. This type of flow occurs when a normal shock accelerates or decelerates as it propagates into a nonuniform medium. Another exception is in the modeling of flows dominated by large vortices.

There are several reasons for not utilizing the Lagrangian description. First, we generally are not interested in the actual location of a fluid particle whereas we are interested in the pressure and velocity, since these provide the pressure and shear stress forces on a body. Second, obtaining $\vec{r}(\vec{r}_o, t)$ represents a greater effort than is required for obtaining p and $\vec{w}$. Finally, the Lagrangian approach is cumbersome for a viscous flow. We, therefore, follow a well-established tradition and hereafter focus on the Eulerian description.

Before leaving this topic, recall that the substantial derivative follows a fluid particle. While the concept is Lagrangian the derivative itself is Eulerian, since $\vec{r}$ and t, not $\vec{r}_o$ and t, are the independent variables.

Pathlines and Streamlines

The trajectory of a fluid particle is called a pathline or particle path. These are found by integrating Equation (1.13) subject to the initial condition, Equation (1.21). We shall not discuss a different type of curve called streaklines. Much more important than either pathlines or streaklines are the streamlines. Streamlines are curves, which at a given instant are tangent to the velocity field. In an unsteady flow, pathlines, streaklines, and streamlines are all different. In a steady flow, however, they all coincide.

Let $d\vec{r}$ be tangent to the velocity and therefore tangent to a streamline. Then $d\vec{r}$ satisfies

$$d\vec{r} \times \vec{w} = 0 \tag{1.21a}$$

or with Cartesian coordinates

$$\begin{vmatrix} \hat{\imath}_1 & \hat{\imath}_2 & \hat{\imath}_3 \\ dx_1 & dx_2 & dx_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

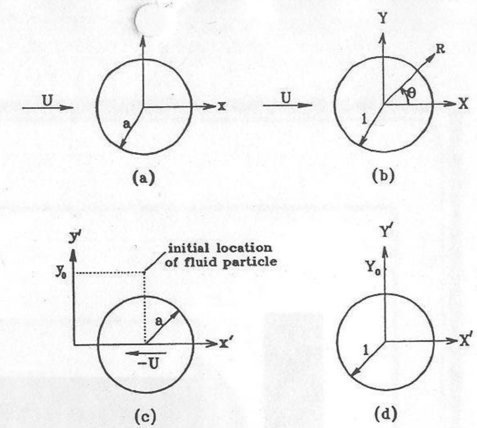

Fig. 1.1 Coordinate systems associated with flow about a circular cylinder; (a) and (b) are for steady flow; (c) and (d) are for unsteady flow.

On expanding this relation we obtain

$$(w_3 dx_2 - w_2 dx_3)\hat{\imath}_1 - (w_3 dx_1 - w_1 dx_3)\hat{\imath}_2 + (w_2 dx_1 - w_1 dx_2)\hat{\imath}_3 = 0$$

or, in scalar form,

$$\frac{dx_1}{w_1} = \frac{dx_2}{w_2} = \frac{dx_3}{w_3} \tag{1.21b}$$

The solution of these two ordinary differential equations provides the streamline curves, subject to a given boundary or initial conditions. Recall that the streamlines are tangent to the velocity field at a given instant of time. Thus, if the w_i are time-dependent, the t variable is treated as a fixed parameter during the integration of Equations (1.21b).

Illustrative Example

As an example, we first determine the streamline equation for steady, two-dimensional cross flow about a circular cylinder of radius a, as sketched in Figure 1.1(a).[1] (Later, the unsteady flow pathlines are found.) In addition, we assume a uniform freestream, with speed U, and an incompressible, inviscid flow without circulation. Hence, the cylinder is not subjected to either a lift or drag force. From elementary aerodynamic theory, we obtain the x and y velocity components as

$$\frac{u}{U} = 1 + \frac{Y^2 - X^2}{(X^2 + Y^2)^2}, \qquad \frac{v}{U} = -\frac{2XY}{(X^2 + Y^2)^2} \tag{1.22}$$

where $X = (x/a)$ and $Y = (y/a)$. Since the flow is two-dimensional, we only need to integrate one of the equations in (1.21b), written as

$$\frac{dx}{u} = \frac{dy}{v}$$

to obtain the equation for the streamlines. The equations in (1.22) are substituted into this differential equation with the result

$$\frac{dX}{dY} = -\frac{Y^2 - X^2 + (X^2 + Y^2)^2}{2XY}$$

[1] The author is indebted to C.-H. Hsu for Figures 1.1 and 1.2.

To separate variables, cylindrical coordinates shown in Figure 1.1(b) [illegible] introduced

$$X = R\cos\theta, \qquad Y = R\sin\theta$$

to obtain

$$\frac{R^2+1}{(R-1)(R+1)R}\,dR = -\cot\theta\, d\theta$$

The method of partial fractions is now used for the left side, with the result

$$\int_{Y_o}^{R}\left(\frac{1}{2}\frac{R^2}{R-1}+\frac{1}{2}\frac{1}{R-1}+\frac{1}{2}\frac{R^2}{R+1}+\frac{1}{2}\frac{1}{R+1}-R-\frac{1}{R}\right)dR = -\int_{\pi/2}^{\theta}\cot\theta\, d\theta$$

where a point Y_o on the Y axis is used for the lower limit and, at this point, $\theta = \pi/2$. As a result of the integration we obtain

$$\frac{Y_o}{Y_o^2-1}\frac{R^2-1}{R} = \frac{1}{\sin\theta}$$

By returning to X, Y coordinates, the streamline equation simplifies to

$$X^2 + Y^2 = \frac{Y}{Y - Y_\infty} \tag{1.23a}$$

where Y_∞ is the streamline ordinate at $X = \pm\infty$ [see Figure 1.2(a)], which shows a typical streamline pattern. The two special Y values are related by

$$Y_\infty = Y_o - \frac{1}{Y_o} \tag{1.23b}$$

where $Y_o \geq 1$ for any streamline outside the cylinder. (There is a related streamline pattern inside the cylinder.)

The solution, Equation (1.23a), also can be obtained, with negligible effort, from the stream function (defined in Chapter 5) equation

$$\psi = Uy\left(1 - \frac{a^2}{x^2+y^2}\right)$$

where $Y_\infty = \psi(\pm\infty, Y_\infty)/(aU)$ and from the fact that a stream function is constant along streamlines in a steady flow. Only in special cases, however, is a stream function available, whereas our purpose is to illustrate how Equations (1.21b) are generally utilized.

UNSTEADY

The determination of the pathlines in an unsteady flow is more difficult. Moreover, the physical interpretation of a pathline solution is far from trivial. As indicated in Figure 1.1(c), the same problem as above is considered, but now the cylinder is moving to the left, with a speed $-U$, into a fluid that is quiescent far from the cylinder. A prime is used to denote unsteady variables, and our goal is to determine the trajectory of a fluid particle. It is analytically convenient to fix the initial condition for the particle directly over the center of the cylinder with $t' = 0$ and $y' = y_o$, as shown in Figure 1.1(c). Consequently, a full trajectory requires the particle's position for both positive and negative time. The "initial condition" phrase therefore does not refer to the particle's position when $t' \to -\infty$.

This flow is essentially the same as the steady flow, only our viewpoint is different. In the steady flow case we move with the cylinder, whereas in the unsteady case we have a fixed (laboratory) coordinate system. It is convenient to again introduce nondimensional variables

$$X' = \frac{x'}{a}, \qquad Y' = \frac{y'}{a}, \qquad T' = \frac{U}{a}t'$$

and use a Galilean transformation

$$x' = x - Ut, \qquad y' = y, \qquad t' = t, \qquad u' = u - U, \qquad v' = v$$

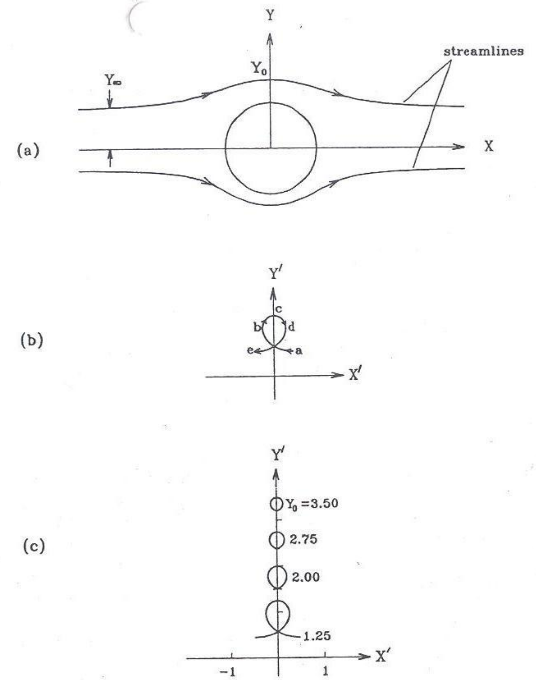

Fig. 1.2 Streamlines (a) and pathlines, (b) and (c), are for flow about a circular cylinder.

to convert the steady flow velocity field into the unsteady one. Equations (1.22) thus become

$$\frac{u'}{U} = \frac{Y'^2 - (X'+T')^2}{[(X'+T')^2 + Y'^2]^2}, \qquad \frac{v'}{U} = -\frac{2(X'+T')Y'}{[(X'+T')^2 + Y'^2]^2}$$

The center of the cylinder is at $x = y = 0$, or

$$X' + T' = 0, \qquad Y' = 0$$

Hence, the initial condition for a fluid particle is

$$X' = 0, \qquad Y' = Y_o \quad \text{when } T' = 0$$

with $Y_o \geq 1$. The X', Y' coordinate system is therefore shifted to the left or right until the position of the particle of interest is located at $X' = 0$ when $T' = 0$. When T' is negative, the particle is upstream of the center of the cylinder, which is at a positive X' value. Remember that when the particle is above the cylinder's center, $T' = X' = 0$. Similarly, when T' is positive, X' is negative. This behavior is illustrated in Figure 1.2(b), where point a is the location of a particle when $T' = -\infty$, while point e is the location when $T' = +\infty$. In this figure, the center of the cylinder moves from $X' = \infty$, $T' = -\infty$ to $X' = -\infty$, $T' = \infty$, whereas the lateral motion of a particle is finite. The one exception is a particle with $Y_\infty = 0$; this particle wets the cylinder.

At its initial location, when $T' = 0$, the velocity components of the particle are

$$\left(\frac{u'}{U}\right)_o = \frac{1}{Y_o^2}, \qquad \left(\frac{v'}{U}\right)_o = 0$$

Thus the particle, at this time, is moving in the positive X' direction. For a particle far upstream of the cylinder we have

$$X' > 0, \qquad T' \ll 0, \qquad Y' \cong Y_\infty, \qquad \frac{u'}{U} < 0, \qquad \frac{v'}{U} > 0$$

When the particle is far downstream of the cylinder, we have

$$X' < 0, \qquad T' \gg 0, \qquad Y' \cong Y_\infty, \qquad \frac{u'}{U} < 0, \qquad \frac{v'}{U} < 0$$

and the cylinder is to the left of the particle. Far from the cylinder, in either X' direction, the particle moves in the negative X' direction. The sign change in u', which occurs when the particle is near the cylinder, is discussed shortly. Note that Y_∞ and Y_o are still related by Equation (1.23b).

We are now ready to utilize Equation (1.13), written as

$$\frac{dx'}{dt'} = u', \qquad \frac{dy'}{dt'} = v'$$

for the particle paths. In contrast to the streamline situation, we have one additional differential equation to solve. In terms of nondimensional variables these equations become

$$\frac{dX'}{dT'} = \frac{Y'^2 - (X' + T')^2}{[(X' + T')^2 + Y'^2]^2}, \qquad \frac{dY'}{dT'} = -\frac{2(X' + T')Y'}{[(X' + T')^2 + Y'^2]^2} \tag{1.24}$$

After Equation (1.23a) is transformed, it also represents a particle path. In other words,

$$(X' + T')^2 + Y'^2 = \frac{Y'}{Y' - Y_\infty} \tag{1.25}$$

is a first integral of Equations (1.24). This can be demonstrated by differentiating this equation with respect to T' and eliminating dX'/dT' and dY'/dT' with the aid of Equations (1.24) to obtain an identity. We next utilize Equation (1.25) to eliminate $X' + T'$ from the dY'/dT' equation, with the result

$$\frac{dY'}{dT'} = \pm\frac{2(1 + Y_\infty Y' - Y'^2)^{1/2}(Y' - Y_\infty)^{3/2}}{Y'^{1/2}}$$

where a $\pm$ sign is introduced when the square root of $(X' + T')^2$ is taken. The plus sign holds when $T' < 0$, while the minus sign holds when $T' > 0$.

The above differential equation is integrated from the initial condition, $Y' = Y_o$ when $T' = 0$, to obtain

$$T' = \pm\frac{1}{2}\int_{Y'}^{Y_o}\left(\frac{Y'}{1 + Y_\infty Y' - Y'^2}\right)^{1/2}\frac{dY'}{(Y' - Y_\infty)^{3/2}}$$

Since

$$1 + Y_\infty Y' - Y'^2 = (Y_o - Y')\left(Y' + \frac{1}{Y_o}\right)$$

the integral can be written as

$$T' = \pm\frac{1}{2}\int_{Y'}^{Y_o}\left[\frac{Y'}{(Y_o - Y')(Y' - Y_\infty)^3\left(Y' + \frac{1}{Y_o}\right)}\right]^{1/2} dY'$$

This quadrature can be evaluated in terms of elliptic integrals of the first, F, and second, E, kinds, defined as

$$F(\phi\backslash\alpha) = \int_0^\phi \frac{d\theta}{(1 - \sin^2\alpha\, \sin^2\theta)^{1/2}}$$

$$E(\phi\backslash\alpha) = \int_0^\phi (1 - \sin^2\alpha\, \sin^2\theta)^{1/2}\, d\theta$$

where θ is a dummy integration variable. With the aid of a table of elliptic integrals, one can show that the final form for T' then is

$$T' = \pm Y_o\left\{F(\phi\backslash\alpha) - E(\phi\backslash\alpha) + \left[\frac{Y_oY'(Y_o - Y')}{(Y' - Y_\infty)(1 + Y_oY')}\right]^{1/2}\right\}$$

where ϕ and α are given by

$$\phi(Y') = \sin^{-1}\left[Y_o^{3/2}\left(\frac{Y_o - Y'}{1 + Y_oY'}\right)^{1/2}\right]$$

$$\alpha = \sin^{-1}\frac{1}{Y_o^2}$$

This relation, in conjunction with Equation (1.25), represents the pathlines in an implicit form. In other words, given Y_o (or Y_∞) and Y', these two equations determine X' and T'.

Figure 1.2 shows, to scale, the expected streamline pattern in (a) and the pathline pattern in (*b*) and (*c*), where all patterns are symmetric about the two axis. The arrows on the streamlines and pathlines indicate increasing time or the direction of the velocity.

Along *a–b–c* in Figure 1.2(b) T' is negative, and the center of the cylinder is at the origin only when the fluid particle is at point *c*, where T' is zero. At point *a*, T' equals $-\infty$, while at point *e*, T' is $+\infty$. (The value of X'_a is the subject of Problem 1.7.) For any other point on $a - b - c$, the center of the cylinder is on the positive X' axis and is to the right of the fluid particle. In this regard, it is useful to note that a particle is upstream of the cylinder's center when $X' + T' < 0$ and downstream otherwise. This result stems from the Galilean transformation, $X = X' + T'$. At points *b* and *d*, u' is zero, while at point *c*, v' is zero. One exception to some of this discussion is a particle with $Y_\infty = 0$ and $X' > 0$, which ultimately wets the cylinder's surface. Otherwise, all other fluid particles have similar trajectories, including the loop.

Along *c–d–e*, $T' \geq 0$ and the particle is downstream of the cylinder's center. Consequently, along *a–b* the particle is being pushed by the cylinder and $u' \leq 0$, while along *d–e* the particle is being pulled by the cylinder, and again $u' \leq 0$. When the particle is close to the cylinder along *b–c–d*, there is a transition region between the pushing and pulling where $u' \geq 0$. In this region, v' changes sign. As evident in Figure 1.2(c), the size of the loop depends on Y_∞ (or Y_o). Particles with a small Y_∞ value, which initially are close to the X' axis, have a relatively large loop. This is caused by the cylinder imparting a large transverse velocity component to the particle as it is shoved aside.

A particle experiences a horizontal displacement as a result of the cylinder's motion, given by

$$\Delta = X'_a - X'_e = 2X'_a$$

As shown in Problem 1.7, Δ becomes infinite when $Y_\infty \to 0$ and goes to zero as $Y_\infty \to \infty$. This displacement also occurs in the steady flow case, since the particles that pass close to the cylinder are retarded more than those that pass at a distance. As shown in Problem 5.22, along a given pathline the change in kinetic energy balances the work done in moving a fluid particle. Because viscosity is not present, the work done on adjacent pathlines or streamlines is not related. Consequently, the change in displacement Δ with Y_∞ does not involve any work.

As you might imagine, the streamlines and pathlines for flow about a sphere are similar to that of a cylinder. Both types of patterns are also considered in Problems 5.23 and 5.24.