Coordenadas esféricas (r, θ, φ)

Gradiente, divergencia, rotor y laplaciano $(0 \leq \theta \leq \pi \;,\; 0 \leq \varphi < 2\pi)$

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial f}{\partial \varphi}\,\hat{\varphi} \tag{1}$$

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2}\,\frac{\partial (r^2 f_r)}{\partial r} + \frac{1}{r\,\mathrm{sen}\,\theta}\,\frac{\partial (\mathrm{sen}\,\theta f_\theta)}{\partial \theta} + \frac{1}{r\,\mathrm{sen}\,\theta}\,\frac{\partial f_\varphi}{\partial \varphi} \tag{2}$$

$$\nabla \times \mathbf{f} = \frac{1}{r\,\mathrm{sen}\,\theta}\Big(\frac{\partial(\mathrm{sen}\,\theta\, f_\varphi)}{\partial \theta} - \frac{\partial f_\theta}{\partial \varphi}\Big)\,\hat{r} + \Big(\frac{1}{r\,\mathrm{sen}\,\theta}\,\frac{\partial f_r}{\partial \varphi} - \frac{1}{r}\,\frac{\partial (r f_\varphi)}{\partial r}\Big)\,\hat{\theta} + \frac{1}{r}\,\Big(\frac{\partial (r f_\theta)}{\partial r} - \frac{\partial f_r}{\partial \theta}\Big)\,\hat{\varphi} \tag{3}$$

$$\nabla^2 f = \frac{1}{r^2}\,\frac{\partial}{\partial r}\Big(r^2\frac{\partial f}{\partial r}\Big) + \frac{1}{r^2 \mathrm{sen}\,\theta}\,\frac{\partial}{\partial \theta}\Big(\mathrm{sen}\,\theta\,\frac{\partial f}{\partial \theta}\Big) + \frac{1}{r^2 \mathrm{sen}^2\theta}\,\frac{\partial^2 f}{\partial \varphi^2} \tag{4}$$

$$\tag{5}$$

$$\begin{aligned}\nabla^2 \mathbf{f} &= \Big(\nabla^2 f_r - \frac{2 f_r}{r^2} - \frac{2}{r^2}\,\frac{\partial f_\theta}{\partial \theta} - \frac{2 f_\theta \cot\theta}{r^2} - \frac{2}{r^2 \mathrm{sen}\,\theta}\frac{\partial f_\varphi}{\partial \varphi}\Big)\,\hat{r} + \\ &+ \Big(\nabla^2 f_\theta + \frac{2}{r^2}\frac{\partial f_r}{\partial \theta} - \frac{f_\theta}{r^2 \mathrm{sen}^2\theta} - \frac{2}{r^2 \mathrm{sen}\,\theta}\frac{\partial f_\varphi}{\partial \varphi}\Big)\,\hat{\theta} + \\ &+ \Big(\nabla^2 f_\varphi - \frac{f_\varphi}{r^2 \mathrm{sen}^2\theta} + \frac{2}{r^2 \mathrm{sen}^2\theta}\frac{\partial f_r}{\partial \varphi} + \frac{2\,\cos\theta}{r^2\,\mathrm{sen}^2\theta}\frac{\partial f_\theta}{\partial \varphi}\Big)\,\hat{\varphi}\end{aligned} \tag{6}$$

Velocidad de deformación, rotación y vorticidad

$$\mathbf{S} = \begin{pmatrix} S_{rr} & S_{r\theta} & S_{r\varphi} \\ S_{r\theta} & S_{\theta\theta} & S_{\theta\varphi} \\ S_{r\varphi} & S_{\theta\varphi} & S_{\varphi\varphi} \end{pmatrix} \qquad \boldsymbol{\Omega} = \begin{pmatrix} \Omega_{rr} & \Omega_{r\theta} & \Omega_{r\varphi} \\ -\Omega_{r\theta} & \Omega_{\theta\theta} & \Omega_{\theta\varphi} \\ -\Omega_{r\varphi} & -\Omega_{\theta\varphi} & \Omega_{\varphi\varphi} \end{pmatrix} \qquad \omega = 2\begin{pmatrix} -\Omega_{\theta\varphi} \\ \Omega_{r\varphi} \\ -\Omega_{r\theta} \end{pmatrix} \tag{7}$$

$$\begin{aligned} S_{rr} &= \frac{\partial u_r}{\partial r} & S_{r\theta} &= \frac{1}{2}\Big[\frac{1}{r}\,\frac{\partial u_r}{\partial \theta} + r\,\frac{\partial}{\partial r}\Big(\frac{u_\theta}{r}\Big)\Big] \\ S_{\theta\theta} &= \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & S_{r\varphi} &= \frac{1}{2}\Big[\frac{1}{r\,\mathrm{sen}\,\theta}\,\frac{\partial u_r}{\partial \varphi} + r\frac{\partial}{\partial r}\Big(\frac{u_\varphi}{r}\Big)\Big] \\ S_{\varphi\varphi} &= \frac{1}{r\,\mathrm{sen}\,\theta}\,\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r} & S_{\theta\varphi} &= \frac{1}{2}\Big[\frac{1}{r\,\mathrm{sen}\,\theta}\,\frac{\partial u_\theta}{\partial \varphi} + \frac{\mathrm{sen}\,\theta}{r}\,\frac{\partial}{\partial \theta}\Big(\frac{u_\varphi}{\mathrm{sen}\,\theta}\Big)\Big] \end{aligned} \tag{8}$$

$$\begin{aligned} \Omega_{rr} &= 0 & \Omega_{r\theta} &= \frac{1}{2r}\Big[\frac{\partial u_r}{\partial \theta} - \frac{\partial (r u_\theta)}{\partial r}\Big] \\ \Omega_{\theta\theta} &= 0 & \Omega_{r\varphi} &= \frac{1}{2r\,\mathrm{sen}\,\theta}\Big[\frac{\partial u_r}{\partial \varphi} - \frac{\partial (r\,\mathrm{sen}\,\theta\, u_\varphi)}{\partial r}\Big] \\ \Omega_{\varphi\varphi} &= 0 & \Omega_{\theta\varphi} &= \frac{1}{2r^2 \mathrm{sen}\,\theta}\Big[\frac{\partial (r\,u_\theta)}{\partial \varphi} - \frac{\partial (r\,\mathrm{sen}\,\theta\, u_\varphi)}{\partial \theta}\Big] \end{aligned}$$

Operadores sobre la velocidad y el esfuerzo[1]

$$\mathbf{u} = \begin{pmatrix} u_r \\ u_\theta \\ u_\varphi \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} T_{rr} & T_{r\theta} & T_{r\varphi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\varphi} \\ T_{\varphi r} & T_{\varphi\theta} & T_{\varphi\varphi} \end{pmatrix} \tag{9}$$

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u} = \begin{pmatrix} u_r\dfrac{\partial u_r}{\partial r} + \dfrac{u_\theta}{r}\,\dfrac{\partial u_r}{\partial \theta} + \dfrac{u_\varphi}{r\,\mathrm{sen}\,\theta}\,\dfrac{\partial u_r}{\partial \varphi} - \dfrac{u_\theta^2 + u_\varphi^2}{r} \\[2ex] u_r\dfrac{\partial u_\theta}{\partial r} + \dfrac{u_\theta}{r}\,\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_\varphi}{r\,\mathrm{sen}\,\theta}\,\dfrac{\partial u_\theta}{\partial \varphi} + \dfrac{u_r\,u_\theta}{r} - \dfrac{u_\varphi^2\cot\theta}{r} \\[2ex] u_r\dfrac{\partial u_\varphi}{\partial r} + \dfrac{u_\theta}{r}\,\dfrac{\partial u_\varphi}{\partial \theta} + \dfrac{u_\varphi}{r\,\mathrm{sen}\,\theta}\,\dfrac{\partial u_\varphi}{\partial \varphi} + \dfrac{u_\varphi\,u_r}{r} + \dfrac{u_\theta\,u_\varphi\cot\theta}{r} \end{pmatrix} \tag{10}$$

$$\nabla\cdot\mathbf{T} = \begin{pmatrix} \dfrac{\partial T_{rr}}{\partial r} + 2\dfrac{T_{rr}}{r} + \dfrac{1}{r}\,\dfrac{\partial T_{\theta r}}{\partial \theta} + \dfrac{T_{\theta r}}{r}\cot\theta + \dfrac{1}{r\,\mathrm{sen}\,\theta}\,\dfrac{\partial T_{\varphi r}}{\partial \varphi} - \dfrac{T_{\theta\theta} + T_{\varphi\varphi}}{r} \\[2ex] \dfrac{\partial T_{r\theta}}{\partial r} + 2\dfrac{T_{r\theta}}{r} + \dfrac{1}{r}\,\dfrac{\partial T_{\theta\theta}}{\partial \theta} + \dfrac{T_{\theta\theta}}{r}\cot\theta + \dfrac{1}{r\,\mathrm{sen}\,\theta}\,\dfrac{\partial T_{\varphi\theta}}{\partial \varphi} + \dfrac{T_{\theta r}}{r} - \dfrac{T_{\varphi\varphi}}{r}\cot\theta \\[2ex] \dfrac{\partial T_{r\varphi}}{\partial r} + 2\dfrac{T_{r\varphi}}{r} + \dfrac{1}{r}\,\dfrac{\partial T_{\theta\varphi}}{\partial \theta} + \dfrac{T_{\theta\varphi}}{r}\cot\theta + \dfrac{1}{r\,\mathrm{sen}\,\theta}\,\dfrac{\partial T_{\varphi\varphi}}{\partial \varphi} + \dfrac{T_{\varphi r}}{r} + \dfrac{T_{\varphi\theta}}{r}\cot\theta \end{pmatrix} \tag{11}$$

[1] No se supone ningún tipo de simetría para **T**. Por lo tanto, prestar atención al orden de los subíndices.