

Introduction to the Second Quantization

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Identical particles

- Examples of quantum many body problems:
 - ▶ Photons in a block-body cavity
 - ▶ Electrons in a solid
 - ▶ Nucleons in a nucleus
 - ▶ Neutrons in a neutron star
 - ▶ Helium atoms in a superfluid
- The concept of identical particles:
 - ▶ Intrinsic: Those do not depend on the dynamical state of the particle, such as the *rest mass*, the *charge*, or the *total spin*.
 - ▶ Extrinsic: Those depend on the state of the particle or on its preparation: the position, and velocity, spin orientation and other internal parameters.
- Here two particles are considered identical if they possess the same intrinsic properties.
- More mathematically speaking, we say n particles are identical if all $\mathcal{O}(\mathbf{x}_1, \mathbf{p}_1; \dots, \mathbf{x}_n, \mathbf{p}_n)$ are invariant under the exchange of their coordinates $(\mathbf{x}_i, \mathbf{p}_i)$, $i = 1, \dots, n$.

$$H(n) = H_0(n) + V(n) = \sum_{i=1}^n \frac{|\mathbf{p}_i|^2}{2m} + \sum_{i < j} V(|\mathbf{x}_i - \mathbf{x}_j|).$$

One-particle system: states

The collection of states of a single particle form a Hilbert space

$$\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3) = \{\varphi(\mathbf{x}); \int d\mathbf{x} |\varphi(\mathbf{x})|^2 < \infty\}$$

with the inner product

$$\langle \varphi | \psi \rangle = \int d\mathbf{x} \varphi^*(\mathbf{x}) \psi(\mathbf{x}).$$

If a particle posses a spin s , its wavefunction will have $2s + 1$ components $\varphi(\mathbf{x}, \sigma)$, $\sigma = -s, -s + 1, \dots, s - 1, s + 1$. i.e.

$$\varphi(\mathbf{x}) = \begin{pmatrix} \varphi(\mathbf{x}, -s) \\ \varphi(\mathbf{x}, -s + 1) \\ \vdots \\ \varphi(\mathbf{x}, s - 1) \\ \varphi(\mathbf{x}, s) \end{pmatrix} \in \mathcal{H} \otimes \mathbb{C}^{2s+1}.$$

Now, I will focus on the case $s = 0$.

One-particle system: Observables

- Position \mathbf{q} and momentum \mathbf{p} are linear operators on \mathcal{H} :

$$(\mathbf{q}\varphi)(\mathbf{x}) = \mathbf{x}\varphi(\mathbf{x})$$

$$(\mathbf{p}\varphi)(\mathbf{x}) = -i\hbar\nabla\varphi(\mathbf{x}), \quad \nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right)$$

- The canonical commutation relations

$$[q^\alpha, p^\beta] = i\hbar\delta_{\alpha,\beta}, \quad \alpha, \beta = 1, 2, 3.$$

- In the Fourier representation

$$\tilde{\varphi}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \varphi(\mathbf{x})$$

the position operator \mathbf{q} and the momentum operator \mathbf{p} becomes

$$(\mathbf{q}\tilde{\varphi})(\mathbf{k}) = -i\nabla_{\mathbf{k}}\tilde{\varphi}(\mathbf{k})$$

$$(\mathbf{p}\tilde{\varphi})(\mathbf{k}) = \hbar\mathbf{k}\tilde{\varphi}(\mathbf{k})$$

One-particle system: Evolution equations

- The evolution of the particle is governed by Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \varphi(\mathbf{x}, t) = (H\varphi)(\mathbf{x}, t)$$

where H is the given Hamiltonian. The formal solution is given by

$$\varphi(\mathbf{x}, t) = (U(t)\varphi)(\mathbf{x}, 0), \quad U(0) = I$$

where $U(t)$ is the evolution operator (semi-group on \mathcal{H}).

- For example, if the external fields are independent of time, then

$$U(t) = \exp\left(-\frac{itH}{\hbar}\right)$$

$$U^*(t) = U^{-1}(t), \quad U(t_1)U(t_2) = U(t_1 + t_2).$$

n -particle system: States

- The collection of the states of n non-relativistic spinless particles form a Hilbert space

$$\mathcal{H}^n = \mathcal{L}^2(\mathbb{R}^{3n}) = \left\{ \Phi(\mathbf{x}_1, \dots, \mathbf{x}_n); \int d\mathbf{x}_1 \dots d\mathbf{x}_n |\Phi(\mathbf{x}_1, \dots, \mathbf{x}_n)|^2 < \infty \right\}.$$

If the particles carry a spin s , the Hilbert space of states is $\mathcal{H}^n \otimes \mathbb{C}^{2s+1}$.

- Now we are going to construct the n -particle states in \mathcal{H}^n from the one-particle states.

$$\langle \mathbf{x}_1, \dots, \mathbf{x}_n | \varphi_1 \otimes \dots \otimes \varphi_n \rangle = \varphi_1(\mathbf{x}_1) \dots \varphi_n(\mathbf{x}_n)$$

Here $|\varphi_i\rangle$, $i = 1, \dots, n$ are the single-particle state in \mathcal{H} . We interpret it as the joint probability of the state $|\varphi_1 \otimes \dots \otimes \varphi_n\rangle$ to find particle 1 in \mathbf{x}_1 , \dots , and particle n in \mathbf{x}_n .

- The inner product of $|\varphi_1 \otimes \dots \otimes \varphi_n\rangle$ and $|\psi_1 \otimes \dots \otimes \psi_n\rangle$ is

$$\langle \psi_1 \otimes \dots \otimes \psi_n | \varphi_1 \otimes \dots \otimes \varphi_n \rangle = \langle \varphi_1 | \psi_1 \rangle \langle \varphi_2 | \psi_2 \rangle \dots \langle \varphi_n | \psi_n \rangle$$

n -particle system: Basis

- Suppose $\{|\varphi_i\rangle, i = 1, 2, \dots\}$ form a basis in the Hilbert space of a single-particle system. We know

$$\{|\varphi_{j_1} \otimes \dots \otimes \varphi_{j_n}\rangle; j_1, j_2, \dots = 1, 2, \dots\}$$

form a basis of the space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \dots \otimes \mathcal{H}$.

- For all wavefunction $\Phi(\mathbf{x}_1, \dots, \mathbf{x}_n)$, we can express as

$$\Phi(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{j_1 \dots j_n} c_{j_1 \dots j_n} \varphi_{j_1}(\mathbf{x}_1) \dots \varphi_{j_n}(\mathbf{x}_n).$$

- We could also write as

$$\langle \mathbf{x}_1, \dots, \mathbf{x}_n | \Phi \rangle = \sum_{j_1 \dots j_n} c_{j_1 \dots j_n} \langle \mathbf{x}_1, \dots, \mathbf{x}_n | \varphi_{j_1} \otimes \dots \otimes \varphi_{j_n} \rangle.$$

$$|\Phi\rangle = \sum_{j_1 \dots j_n} c_{j_1 \dots j_n} |\varphi_{j_1} \otimes \dots \otimes \varphi_{j_n}\rangle.$$

- This means we can identify n -particle space \mathcal{H}^n with the tensor product $\mathcal{H}^{\otimes n}$.

n -particle system: One-body observables

- Observables of a n -identical-particle system can be constructed from observables in a single-particle system.
- If A is a single-particle observable in \mathcal{H} , we define the action A on the j th factor of the tensor product $\mathcal{H}^{\otimes n}$ as

$$(I \otimes \cdots \otimes A \otimes \cdots \otimes I) |\varphi_1 \otimes \cdots \otimes \varphi_j \otimes \cdots \otimes \varphi_n\rangle = |\varphi_1 \otimes \cdots \otimes A\varphi_j \otimes \cdots \otimes \varphi_n\rangle$$

and denote it as A_j .

- The corresponding extensive observable for the n -particle system is defined by

$$A(n) = \sum_{j=1}^n A_j.$$

n -particle system: Two-body observables

- Let V be an observable in two-particle space $\mathcal{H} \otimes \mathcal{H}$, such as interaction potential.
- Define V_{ij} on $\mathcal{H}^{\otimes n}$ as the operator acts on i th and j th factors of $\mathcal{H}^{\otimes n}$ and trivially on the others.
- The corresponding observable for a n -particle system is

$$V(n) = \sum_{i < j}^n V_{ij}.$$

- If V is acting on two identical particles, we have $V_{ij} = V_{ji}$. Therefore we have

$$V(n) = \frac{1}{2} \sum_{i \neq j}^n V_{ij}.$$

Symmetrization: Permutation operator

- Permutation operator P_π actin on $\mathcal{H}^{\otimes n}$ is

$$P_\pi |\varphi_1 \otimes \cdots \otimes \varphi_n\rangle = |\varphi_{\pi(1)} \otimes \cdots \otimes \varphi_{\pi(n)}\rangle$$

where $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$.

- Some properties:

$$P_{\pi_1\pi_2} = P_{\pi_1}P_{\pi_2}, \quad P_\pi^* = P_\pi^{-1}$$

- Let A be an observable on a single-particle space \mathcal{H} . We have

$$[A(n), P_\pi] = 0 \quad \text{for all permutation } \pi$$

where $A(n) = \sum_{j=1}^n A_j$.

Symmetrization: Exchange degeneracy

- Suppose $|\Phi\rangle$ is an eigenstate of a n -identical-particle system. i.e.

$$H(n)|\Phi\rangle = E|\Phi\rangle$$

We will find

$$H(n)P_\pi|\Phi\rangle = EP_\pi|\Phi\rangle \quad \text{for all permutation } \pi$$

- In a n -identical-particle system, $P_\pi|\Phi\rangle$ and their linear combinations describe the same physical state.
- **Principle of symmetrization:** the state of n identical particles are either completely symmetric or completely antisymmetric.
- **Bosons** are particles in the symmetric case.
- **Fermions** are particles in the antisymmetric case.

Symmetrization: Symmetric and Antisymmetric States

- For any $|\Phi\rangle$ in $\mathcal{H}^{\otimes n}$, we can construct corresponding one which is totally symmetric or totally antisymmetric:

$$|\Phi\rangle_+ = S_+ |\Phi\rangle$$

$$|\Phi\rangle_- = S_- |\Phi\rangle$$

where two operators S_+ and S_- defined by

$$S_+ = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} P_\pi, \quad S_- = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} (-1)^\pi P_\pi.$$

- Properties:

$$S_+^2 = S_+, \quad S_-^2 = S_- = S_-^*, \quad S_+ S_- = S_- S_+ = 0.$$

$$P_\pi S_+ = S_+ P_\pi = S_+, \quad P_\pi S_- = S_- P_\pi = (-1)^\pi S_-.$$

- According to the principle of symmetrization, the space of state of a system of n identical particle particles is not entire $\mathcal{H}^{\otimes n}$.
 - Space for bosons (symmetric): $\mathcal{H}_+^{\otimes n} \equiv S_+ \mathcal{H}^{\otimes n}$.
 - Space for fermions (anti-symmetric): $\mathcal{H}_-^{\otimes n} \equiv S_- \mathcal{H}^{\otimes n}$.

Hilbert spaces for bosons or fermions

- Denote the symmetric or antisymmetric state of $\mathcal{H}_+^{\otimes n}$ or $\mathcal{H}_-^{\otimes n}$ by

$$|\varphi_1, \varphi_2, \dots, \varphi_n\rangle_\nu \equiv S_\nu |\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n\rangle$$

where $\nu = 1$ for bosons and $\nu = -1$ for *fermions*.

- Exchanging the i th and j th components, we find

$$\begin{aligned} |\varphi_1, \dots, \varphi_i, \dots, \varphi_j, \dots, \varphi_n\rangle_+ &= |\varphi_1, \dots, \varphi_j, \dots, \varphi_i, \dots, \varphi_n\rangle_+ \\ |\varphi_1, \dots, \varphi_i, \dots, \varphi_j, \dots, \varphi_n\rangle_- &= -|\varphi_1, \dots, \varphi_j, \dots, \varphi_i, \dots, \varphi_n\rangle_- \end{aligned}$$

- The inner product is

$$\nu \langle \varphi_1, \dots, \varphi_n | \psi_1, \dots, \psi_n \rangle_\nu = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} (\nu)^\pi \langle \varphi_1 | \psi_{\pi(1)} \rangle \dots \langle \varphi_n | \psi_{\pi(n)} \rangle.$$

- In particular for the fermions case,

$$-\langle \varphi_1, \dots, \varphi_n | \psi_1, \dots, \psi_n \rangle_- = \frac{1}{n!} \det\{\langle \varphi_i | \psi_j \rangle\}.$$

Occupation Number Representation

Let $\{|\psi_r\rangle\}$ be a basis of a single-particle space \mathcal{H} .

- The occupation number normalized to 1 are defined by

$$|n_1, n_2, \dots, n_r, \dots\rangle_+ = \frac{\sqrt{n!}}{\sqrt{n_1! \dots n_r! \dots}} S_+ |\psi_{r_1} \otimes \dots \otimes \psi_{r_n}\rangle$$

$$|n_1, n_2, \dots, n_r, \dots\rangle_- = \sqrt{n!} S_- |\psi_{r_1} \otimes \dots \otimes \psi_{r_n}\rangle$$

where n_r is the number of indices r_1, \dots, r_n which are equal to r .

- For bosons, $n_r = 0, 1, 2, 3, \dots$
For fermions, $n_r = 0, 1$.
- The system has exactly $n = \sum_r n_r$ particles.

Occupation Number Representation: Observables

- Let $|\psi_r\rangle$ are the eigenvector of a single-particle observable A such that

$$A|\psi_r\rangle = \alpha_r|\psi_r\rangle.$$

Then $|n_1, n_2, \dots, n_t, \dots\rangle_\nu$ are also eigenvectors of the one-body observable $A(n)$ with eigenvalue $\sum_r n_r \alpha_r$.

$$A(n)|\psi_{r_1} \otimes \dots \otimes \psi_{r_n}\rangle = \sum_{j=1}^n A_j |\psi_{r_1} \otimes \dots \otimes \psi_{r_n}\rangle = \left(\sum_r n_r \alpha_r \right) |\psi_{r_1} \otimes \dots \otimes \psi_{r_n}\rangle$$

Therefore if $[S_\nu, A(n)] = 0$, we have

$$A(n)|n_1, n_2, \dots, n_r, \dots\rangle_\nu = \left(\sum_r n_r \alpha_r \right) |n_1, n_2, \dots, n_r, \dots\rangle_\nu.$$

- Orthogonality:

$${}_\nu \langle m_1, m_2, \dots, m_r, \dots | n_1, n_2, \dots, n_r, \dots \rangle_\nu = \delta_{m_1, n_1} \delta_{m_2, n_2} \dots \delta_{m_r, n_r} \dots$$

- Basis: $\{|n_1, n_2, \dots\rangle_\nu\}$ form a basis of $\mathcal{H}_\nu^{\otimes n}$.

Since $|\Phi\rangle_\nu = \sum_{r_1, \dots, r_n} c_{c_1, \dots, r_n} S_\nu |\psi_{r_1} \otimes \dots \otimes \psi_{r_n}\rangle$.

Systems with Variable Particle Numbers

In many physical systems, the number of particles of a given type is not exactly known or may vary during reactions.

- A photon gas
- Phonon in condensed-matter
- In high-energy physics, particles may be created or annihilated.

Because of these reasons, it is better to construct a Hilbert space of many-particle system which allowing undetermined number of particles. Such construction is called the **second quantization**.

Note that the second quantization does not introduce any new theory. Conceptually, it is just an useful formulation of quantum mechanics.

Fock Space

- Let \mathcal{H} be a Hilbert space of single-particle states and $\mathcal{H}_\nu^{\otimes n}$ be the corresponding Hilbert space of n identical particles. ($\nu = 1$ for bosons, $\nu = -1$ for fermions)
- Vacuum state:** $\mathcal{H}^0 = \{\lambda|\Phi(0)\rangle; \lambda \in \mathbb{C}\}$ where $|\Phi(0)\rangle = |0\rangle$ is a unit vector.

•

$$\mathcal{V} = \mathcal{H}^0 \oplus \mathcal{H}_\nu^{\otimes 1} \oplus \dots \oplus \mathcal{H}_\nu^{\otimes n} \oplus \dots$$

- The elements $|\Phi\rangle$ in the space \mathcal{V} take the form

$$|\Phi\rangle = \{|\Phi(0)\rangle, |\Phi(1)\rangle, |\Phi(2)\rangle, \dots, |\Phi(n)\rangle, \dots\} = \{|\Phi(n)\rangle\}_n$$

- Addition and scalar multiplication:

$$|\Phi\rangle + |\Psi\rangle \equiv \{|\Phi(n)\rangle + |\Psi(n)\rangle\}_n$$

$$\lambda|\Phi\rangle \equiv \{\lambda|\Phi(n)\rangle\}_n$$

- Inner product on the space \mathcal{V} :

$$\langle\Phi|\Psi\rangle \equiv \sum_{n=0}^{\infty} \langle\Phi(n)|\Psi(n)\rangle$$

where $\langle\Phi(n)|\Psi(n)\rangle$ is an inner product in $\mathcal{H}_\nu^{\otimes n}$.

- Fock space:**

$$\mathcal{F}_\nu(\mathcal{H}) = \{|\Phi\rangle \in \mathcal{V}; \langle\Phi|\Phi\rangle < \infty\}$$

Observables in Fock space

- Let $A(n)$ be an observable for n particle. The corresponding observable A acting on Fock space is given by

$$A|\Phi\rangle = \sum_{n=0}^{\infty} A(n)|\Phi(n)\rangle.$$

- The vacuum state is an eigenvector of all extensive observables with eigenvalue zero.
- The particle number operator N is defined by $N|\Phi\rangle \equiv \sum_{n=0}^{\infty} n|\Phi(n)\rangle$.
- Expectation of A :

$$\langle\Phi|A|\Phi\rangle = \sum_{n=0}^{\infty} \langle\Phi(n)|\Phi(n)\rangle \frac{\langle\Phi(n)|A(n)|\Phi(n)\rangle}{\langle\Phi(n)|\Phi(n)\rangle}$$

Basis in Fock space

- The collection of $\{|\varphi_1, \dots, \varphi_n\rangle_\nu\}$ generates \mathcal{H} and the vacuum state $|0\rangle$ generate the Fock space $\mathcal{F}_\nu(\mathcal{H})$ when $\{|\varphi_i\rangle\}$ form a basis of a single-particle space \mathcal{H} .
- Occupation number representation:
Given a basis $\{|\varphi_i\rangle\}$ of one-particle space \mathcal{H} , the collection of

$$|n_1, n_2, \dots, n_r, \dots\rangle_\nu, \quad \sum_r n_r < \infty$$

form a basis in the Fock space.

- ▶ for bosons, $n_r = 0, 1, 2, \dots$
- ▶ for fermions, $n_r = 0, 1$.

Creation and Annihilation Operators

- The **creation operator** $a^*(\varphi)$ of one particle in the state $|\varphi\rangle_\nu$ is defined by

$$a^*(\varphi)|\varphi_1, \dots, \varphi_n\rangle_\nu \equiv \sqrt{n+1}|\varphi, \varphi_1, \dots, \varphi_n\rangle_\nu$$

- The **annihilation operator** $a(\varphi)$ of a particle in the state $|\varphi\rangle$ is defined by

$$a(\varphi) \equiv (a^*(\varphi))^*$$

where the second $*$ means the Hermitian conjugate.

Properties of creation operators and annihilation operators

- $a^*(\varphi)$ is linear but $a(\varphi)$ is anti-linear:

$$a^*(\lambda_1|\varphi_1\rangle + \lambda_2|\varphi_2\rangle) = \lambda_1 a^*(\varphi_1) + \lambda_2 a^*(\varphi_2)$$

$$a(\lambda_1|\varphi_1\rangle + \lambda_2|\varphi_2\rangle) = \lambda_1^* a(\varphi_1) + \lambda_2^* a(\varphi_2)$$

- Inner product:

$$\begin{aligned} & \nu \langle \psi_1, \dots, \psi_m | a(\varphi) | \varphi_1, \dots, \varphi_n \rangle_\nu \\ &= \sqrt{n+1} \nu \langle \varphi, \psi_1, \dots, \psi_m | \varphi_1, \dots, \varphi_n \rangle_\nu \\ &= \begin{cases} 0 & \text{for } m \neq n-1, \\ \frac{1}{\sqrt{n}} \sum_{i=1}^n (\nu)^{i-1} \times \\ \quad \nu \langle \psi_1, \dots, \psi_{n-1} | \varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_n \rangle_\nu \langle \varphi | \varphi_i \rangle. & \text{for } m = n-1 \end{cases} \end{aligned}$$

- Therefore, $a(\varphi)$ decreases the number of particles by one unit, while preserving the symmetry of the state

$$a(\varphi) | \varphi_1, \dots, \varphi_n \rangle_\nu = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\nu)^{i-1} | \varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_n \rangle_\nu \langle \varphi | \varphi_i \rangle$$

Commutation and Anti-commutation Relations

- $$a^*(\varphi)a^*(\psi) - a^*(\psi)a^*(\varphi) = 0 \quad (\text{bosons})$$
- $$a^*(\varphi)a^*(\psi) + a^*(\psi)a^*(\varphi) = 0 \quad (\text{fermions})$$
- $$a(\psi)a^*(\varphi) - a^*(\varphi)a(\psi) = \langle \psi | \varphi \rangle I \quad (\text{bosons})$$
- $$a(\psi)a^*(\varphi) + a^*(\varphi)a(\psi) = \langle \psi | \varphi \rangle I \quad (\text{fermions})$$

Here I is the identity operator in $\mathcal{F}_\nu(\mathcal{H})$.

- Let $\{|\psi_r\rangle\}$ be a given fixed basis of one-particle space. Set $a_r^* = a^*(\psi_r)$ and $a_r = a(\psi_r)$. The commutation and anti-commutation relation become

$$a_r^* a_s^* \mp a_s^* a_r^* = 0$$

$$a_r a_s \mp a_s a_r = 0$$

$$a_r a_s^* \mp a_s^* a_r = \delta_{r,s}$$

Formation of States from the Vacuum (I)

All states of n independent particles can be obtained by successive application of the creation operators on the vacuum.

$$|\varphi_1, \dots, \varphi_n\rangle_\nu = \frac{1}{\sqrt{n!}} a^*(\varphi_1) \dots a^*(\varphi_n) |0\rangle.$$

- The state is symmetric or antisymmetric depending on whether $a^*(\varphi_i)$ is commute or anti-commute.
- All of states in Fock space is their linear combination of such states or their limit.

Occupation number representation relative to the single-particle basis $\{|\psi_r\rangle\}$ and $a_r^* = a^*(|\psi_r\rangle)$:

$$|n_1, \dots, n_r, \dots\rangle = \frac{(a_1^*)^{n_1}}{\sqrt{n_1!}} \dots \frac{(a_r^*)^{n_r}}{\sqrt{n_r!}} \dots |0\rangle \quad \text{for } \begin{cases} n_r = 0, 1, 2, \dots & (\text{bosons}) \\ n_r = 0, 1 & (\text{fermions}) \end{cases}$$

Formation of States from the Vacuum (II)

- For bosons,

$$a_r^* |n_1, \dots, n_r, \dots\rangle_+ = \sqrt{n_r + 1} |n_1, \dots, n_r + 1, \dots\rangle_+$$

$$a_r |n_1, \dots, n_r, \dots\rangle_+ = \sqrt{n_r} |n_1, \dots, n_r - 1, \dots\rangle_+$$

- For fermions,

$$a_r^* |n_1, \dots, n_r, \dots\rangle_- = (1 - n_r)(-1)^{\gamma_r} |n_1, \dots, n_r + 1, \dots\rangle_-$$

$$a_r |n_1, \dots, n_r, \dots\rangle_+ = n_r(-1)^{\gamma_r} |n_1, \dots, n_r - 1, \dots\rangle_-$$

where $\gamma_r = \sum_{s=1}^{r-1} n_s$.

- For both bosons and fermions, the occupation number states are eigenstates for all of the operations $a_r^* a_r$:

$$a_r^* a_r |n_1, \dots, n_r, \dots\rangle_\nu = n_r |n_1, \dots, n_r, \dots\rangle_\nu$$

Coherent States of Bosons

- States of type

$$|\varphi_1, \dots, \varphi_n\rangle_\nu = \frac{1}{\sqrt{n!}} a^*(\varphi_1) \dots a^*(\varphi_n) |0\rangle$$

have a determined number of particles. This is typical for the case of a scattering process in which the incoming particles are specified and the particles produced by the reaction are observed.

- States of type consist of superposition of n -particle state with $n = 0, 1, 2, \dots$

$$|\varphi\rangle_{coh} \equiv \exp\left(-\frac{1}{2}\langle\varphi|\varphi\rangle\right) \sum_{n=0}^{\infty} \frac{[a^*(\varphi)]^n}{n!} |0\rangle = D(\varphi) |0\rangle$$

where

$$D(\varphi) \equiv \exp[a^*(\varphi) - a(\varphi)] = \exp\left(-\frac{1}{2}\langle\varphi|\varphi\rangle\right) \exp[a^*(\varphi)] \exp[-a(\varphi)].$$

- A coherent state represents a state of photons (or phonons) associated with a classical electromagnetic (or acoustic) wave.
- One has $a(\psi) |\varphi\rangle_{coh} = (\langle\psi|\varphi\rangle) |\varphi\rangle_{coh}$.

One-Body Operators in Fock Space

- Consider a single-particle observable A . The operator $A(n) = \sum_{j=1}^n A_j$ is the associated observable of n -particle system. The corresponding observable A in the Fock space is

$$A |\varphi_1, \dots, \varphi_n\rangle_\nu = \sum_{j=1}^n |\varphi_1, \dots, \varphi_{j-1}, A\varphi_j, \varphi_{j+1}, \dots, \varphi_n\rangle_\nu$$

- Let $\{|\psi_r\rangle\}$ be some basis of single-particle space.

$$A = \sum_r a^*(A\psi_r)a(\psi_r) = \sum_{r,s} \langle r|A|s\rangle a_s^* a_r.$$

- If $|\psi_r\rangle$ are chosen as eigenvector of A with eigenvalue α_r , then we have

$$A = \sum_r \alpha_r a_r^* a_r.$$

Some typical one-body operators in Fock space (I)

- Particle-number operator

$$N = \sum_r a^*(\psi_r)a(\psi_r)$$

If we express the operator N in terms of the basis $|\mathbf{x}\sigma\rangle$ of the configuration representation

$$N = \int_{\Lambda} d\mathbf{x} \sum_{\sigma} a^*(\mathbf{x}, \sigma)a(\mathbf{x}, \sigma) = \int_{\Lambda} d\mathbf{x} n(\mathbf{x})$$

where the particle density in \mathbf{x} is $n(\mathbf{x}) = \sum_{\sigma} n(\mathbf{x}, \sigma)$ and the particle density of particles with spin σ is $n(\mathbf{x}, \sigma) = a^*(\mathbf{x}, \sigma)a(\mathbf{x}, \sigma)$.

Some typical one-body operators in Fock space (II)

- Kinetic energy

- ▶ The Hamiltonian of free particles

$$H_0 = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} a_{\mathbf{k}\sigma}^* a_{\mathbf{k}\sigma}$$

where $\epsilon_{\mathbf{k}}$ is the kinetic energy of the particle.

- ▶ The momentum of free particles

$$P = \sum_{\mathbf{k}\sigma} \hbar \mathbf{k} a_{\mathbf{k}\sigma}^* a_{\mathbf{k}\sigma}$$

Some typical one-body operators in Fock space (III)

Consider a system of particles under an external potential $V(\mathbf{x})$ which is independent of spin.

- In the basis $|\mathbf{x}\sigma\rangle$

$$\langle \mathbf{x}\sigma | V | \mathbf{x}'\sigma' \rangle = \delta(\mathbf{x} - \mathbf{x}') \delta_{\sigma,\sigma'} V(\mathbf{x})$$

The representation of the external potential V is

$$V = \int_{\Lambda} d\mathbf{x} V(\mathbf{x}) n(\mathbf{x}).$$

- In the basis of plane wave $|\mathbf{k}\sigma\rangle$

$$\langle \mathbf{k}\sigma | V | \mathbf{k}'\sigma' \rangle = \delta_{\sigma,\sigma'} \frac{1}{L^3} \tilde{V}(\mathbf{k} - \mathbf{k}')$$

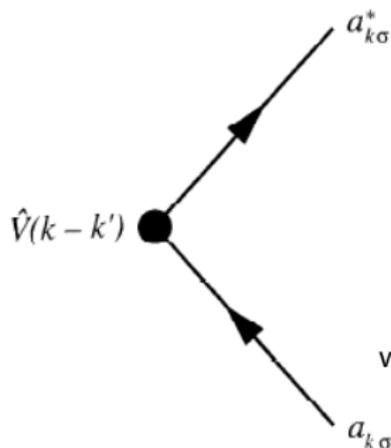
where $\tilde{V}(\mathbf{k}) = \int_{\Lambda} d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} V(\mathbf{x})$.

Some typical one-body operators in Fock space (IV)

In the basis of plane wave $|\mathbf{k}\sigma\rangle$

$$\langle \mathbf{k}\sigma | V | \mathbf{k}'\sigma' \rangle = \delta_{\sigma,\sigma'} \frac{1}{L^3} \tilde{V}(\mathbf{k} - \mathbf{k}')$$

where $\tilde{V}(\mathbf{k}) = \int_{\Lambda} d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} V(\mathbf{x})$.



$$\begin{aligned} V &= \frac{1}{L^3} \sum_{\mathbf{k}\mathbf{k}'} \sum_{\sigma} \tilde{V}(\mathbf{k} - \mathbf{k}') a_{\mathbf{k}\sigma}^* a_{\mathbf{k}'\sigma} \\ &= \frac{1}{L^3} \sum_{\mathbf{k}\mathbf{k}'} \sum_{\sigma} \tilde{V}(\mathbf{k}) a_{(\mathbf{k}+\mathbf{k}')\sigma}^* a_{\mathbf{k}'\sigma} \\ &= \frac{1}{L^3} \sum_{\mathbf{k}} \tilde{n}^*(\mathbf{k}) \tilde{V}(\mathbf{k}) \end{aligned}$$

where

$$\begin{aligned} \tilde{n}(\mathbf{k}) &= \int_{\Lambda} d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \sum_{\sigma} a^*(\mathbf{x}, \sigma) a(\mathbf{x}, \sigma) = \sum_{\mathbf{k}'\sigma} a_{\mathbf{k}'\sigma}^* a_{(\mathbf{k}'+\mathbf{k})\sigma} \\ &= \sum_{\mathbf{k}'\sigma} a_{(\mathbf{k}'-\mathbf{k})\sigma}^* a_{\mathbf{k}'\sigma} = \tilde{n}^*(-\mathbf{k}). \end{aligned}$$

Two-Body Operators

Consider an operator V acting on two-particle space $\mathcal{H} \otimes \mathcal{H}$.

$$V|\varphi_1, \dots, \varphi_n\rangle_\nu = \sum_{i < j}^n V_{ij}|\varphi_1, \dots, \varphi_n\rangle_\nu.$$

We express V in terms of the creation and annihilation operators

$$V = \frac{1}{2} \sum_{r_1 r_2 s_1 s_2} \langle r_1 r_2 | V | s_1 s_2 \rangle a_{r_1}^* a_{r_2}^* a_{s_1} a_{s_2}.$$

Two-Body Operator: Configuration Representation

Consider a two-body potential $V(\mathbf{x}_1 - \mathbf{x}_2)$ which is invariant under translations and independent of spin.

$$\langle \mathbf{x}_1\sigma_1, \mathbf{x}_2\sigma_2 | V | \mathbf{x}'_1\sigma_1, \mathbf{x}'_2\sigma_2 \rangle = V(\mathbf{x}_1 - \mathbf{x}_2) \delta(\mathbf{x}_1 - \mathbf{x}'_1) \delta(\mathbf{x}_2 - \mathbf{x}'_2) \delta_{\sigma_1, \sigma'_1} \delta_{\sigma_2, \sigma'_2}.$$

Therefore we have

$$V = \frac{1}{2} \left(\int_{\Lambda} d\mathbf{x}_1 \int_{\Lambda} d\mathbf{x}_2 \sum_{\sigma_1\sigma_2} V(\mathbf{x}_1 - \mathbf{x}_2) \right) \times a^*(\mathbf{x}_1, \sigma_1) a^*(\mathbf{x}_2, \sigma_2) a(\mathbf{x}_2, \sigma_2) a(\mathbf{x}_1, \sigma_1).$$

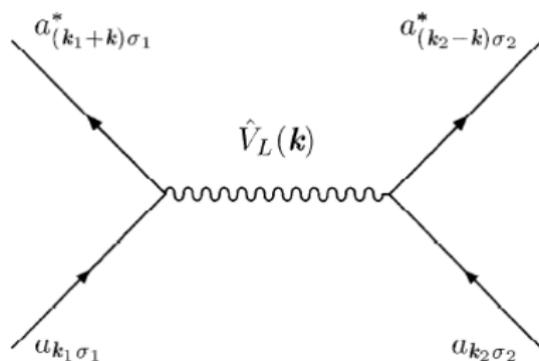
Two-Body Operator: Momentum Representation

Here we express V in the plane-wave basis $|\mathbf{k}\sigma\rangle$ with periodic boundary conditions.

$$\langle \mathbf{k}_1\sigma_1, \mathbf{k}_2\sigma_2 | V | \mathbf{k}'_1\sigma'_1, \mathbf{k}'_2\sigma'_2 \rangle = \delta_{\sigma_1, \sigma'_1} \delta_{\sigma_2, \sigma'_2} \frac{1}{L^3} \sum_{\mathbf{k}} \tilde{V}_L(\mathbf{k}) \delta_{\mathbf{k}_1, \mathbf{k}'_1 + \mathbf{k}} \delta_{\mathbf{k}_2, \mathbf{k}'_2 + \mathbf{k}}$$

where $\tilde{V}_L = \int_{\Lambda} d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} V(\mathbf{x})$. Therefore we can write

$$V = \frac{1}{2L^3} \sum_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}} \sum_{\sigma_1 \sigma_2} \tilde{V}_L(\mathbf{k}) a_{(\mathbf{k}_1 + \mathbf{k})\sigma_1}^* a_{(\mathbf{k}_2 - \mathbf{k})\sigma_2}^* a_{\mathbf{k}_2\sigma_2} a_{\mathbf{k}_1\sigma_1}$$



Reference

- Ph. A. Martin and F. Rothen, Many-Body Problems and Quantum Field Theory — An Introduction

Thank you for your attention!