Repaso de Clase 7

- Fricción entre elementos de fluido >>> Viscosidad de Newton >>> $\tau = \dfrac{F}{A} = \mu \dfrac{\partial u_x}{\partial y}$

- Tensor de esfuerzos viscosos >>> $\sigma'_{i,j} = \mu \left(\partial_i u_j + \partial_j u_i - \dfrac{2}{3}\partial_k u_k \, \delta_{i,j}\right) + \mu' \partial_k u_k \, \delta_{i,j}$

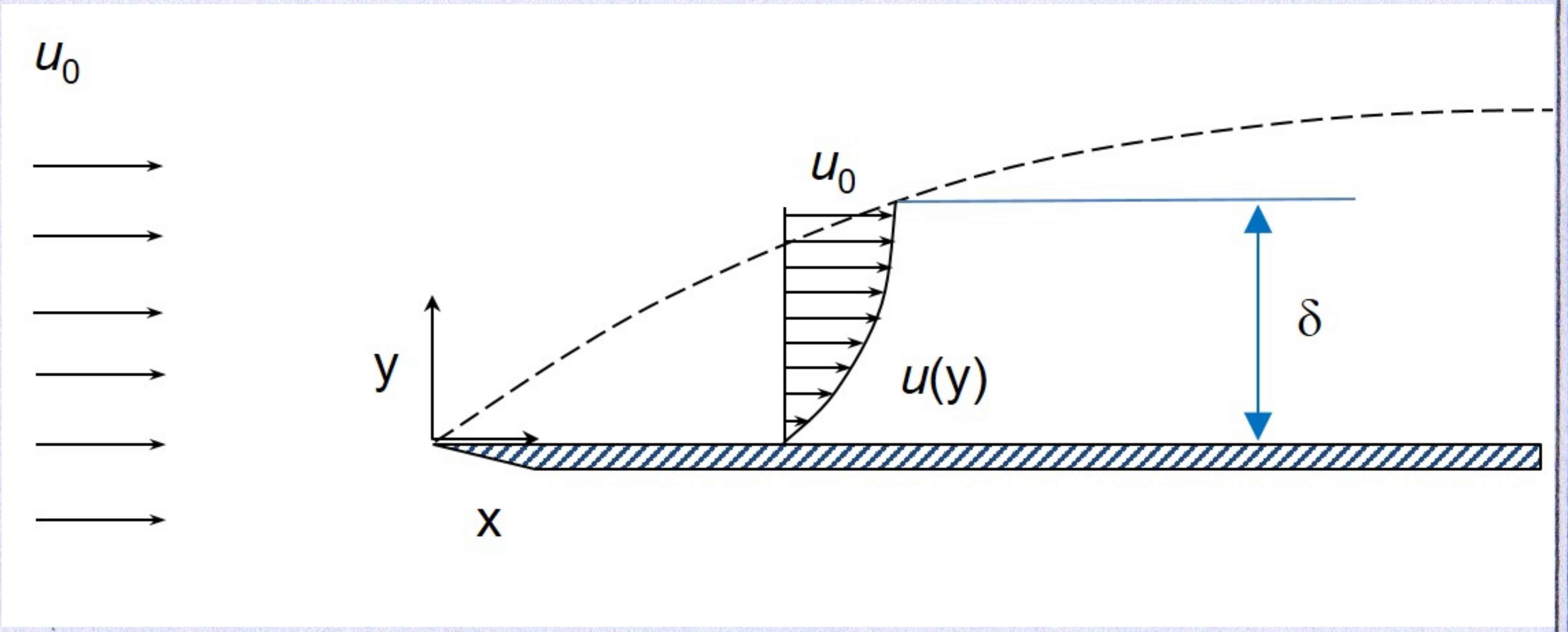
- Ecuación de Navier-Stokes >>> $\rho(\partial_t \underline{u} + (\underline{u} \cdot \underline{\nabla})\underline{u}) = \rho \underline{f} - \underline{\nabla}\, p + \mu \nabla^2 \underline{u} + (\dfrac{\mu}{3} + \mu')\underline{\nabla}(\underline{\nabla} \cdot \underline{u})$

- Condiciones de contorno para fluidos viscosos >>> rigidez (fluido ideal) + no deslizamiento

- Viscosidad >>> Disipación de energía ($\dfrac{dE}{dt} \leq 0$)

- Número de Reynolds >>> $Re = \dfrac{u_0 L_0}{\nu}$

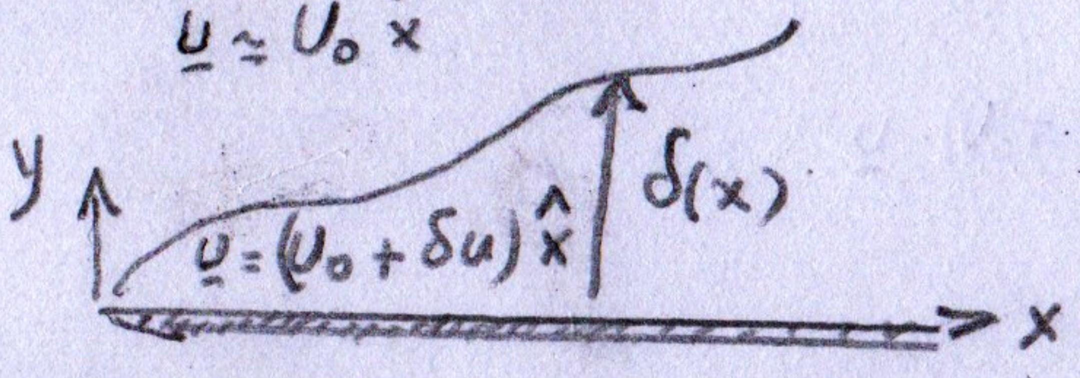
Capa límite



Las condiciones de contorno de un fluido ideal o viscoso, son marcadamente diferentes. Se espera que las respectivas soluciones difieran apreciablemente cerca de los contornos $\longrightarrow$ capas límites

En el problema de la figura suponemos:

- plano: $\underline{u} = U_x(x,y)\,\hat{x}$
- estacionario: $\partial_t = 0$
- incompresible: $\rho = cte$
- Lineal: $Re \ll 1 \longrightarrow U_x \simeq U_0 + \delta u$, $\delta u \ll U_0$

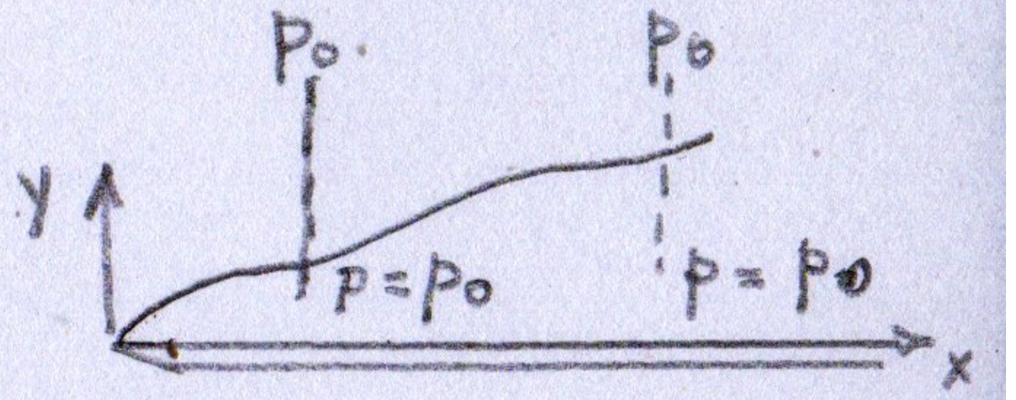


$(\hat{x})$ $$(U_0 + \cancel{\delta u})\,\partial_x(\underbrace{\cancel{U_0}}_{=0} + \delta u) = -\frac{1}{\rho}\partial_x p + \nu(\cancel{\partial_{xx}} + \partial_{yy})(\underbrace{\cancel{U_0}}_{=0} + \delta u)$$

$\delta u \ll U_0$ $\qquad \partial_{xx} \ll \partial_{yy}$

$(\hat{y})$ $$0 = -\frac{1}{\rho}\partial_y p \longrightarrow p = p(x)$$

Pero $p(x,y) = p(x,\infty) = p_0$



$$\therefore\ \partial_x p = 0 \longrightarrow U_0\,\partial_x \delta u \simeq \nu\,\partial_{yy}\delta u$$

Ecuación de difusión $\longrightarrow$ Cond. Contorno $\begin{cases} \delta u(y \to \infty) = 0 \\ \delta u(y=0) = -U_0 \end{cases}$

Buscamos soluciones auto-similares para esta ecuación, donde la incógnita δu dependa solo de

$$\xi = \frac{y}{\sqrt{4\nu x / U_0}} \xrightarrow[\delta u(x,y) = \delta u(\xi(x,y))]{} \begin{cases} \partial_x \delta u = \partial_x \xi \dfrac{d\delta u}{d\xi} \\ \partial_y \delta u = \partial_y \xi \dfrac{d\delta u}{d\xi} \end{cases}$$

Entonces:

$$\partial_x \delta u = -\frac{\xi}{2x}\,\delta u'$$

$$\partial_y \delta u = \frac{\xi}{y}\,\delta u' \longrightarrow \partial_{yy}\delta u = \frac{\xi^2}{y^2}\,\delta u''$$

Capa límite

Reemplazando: $\delta u'' = -2\xi\, \delta u'$

La primera integral: $\delta u' = A\, e^{-\xi^2}$

La segunda integral involucra a la llamada <u>función error</u> definida como:

$$f_{err}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} d\xi\, e^{-\xi^2}$$

tal que $f_{err}(0) = 0$ y $f_{err}(\infty) = 1$

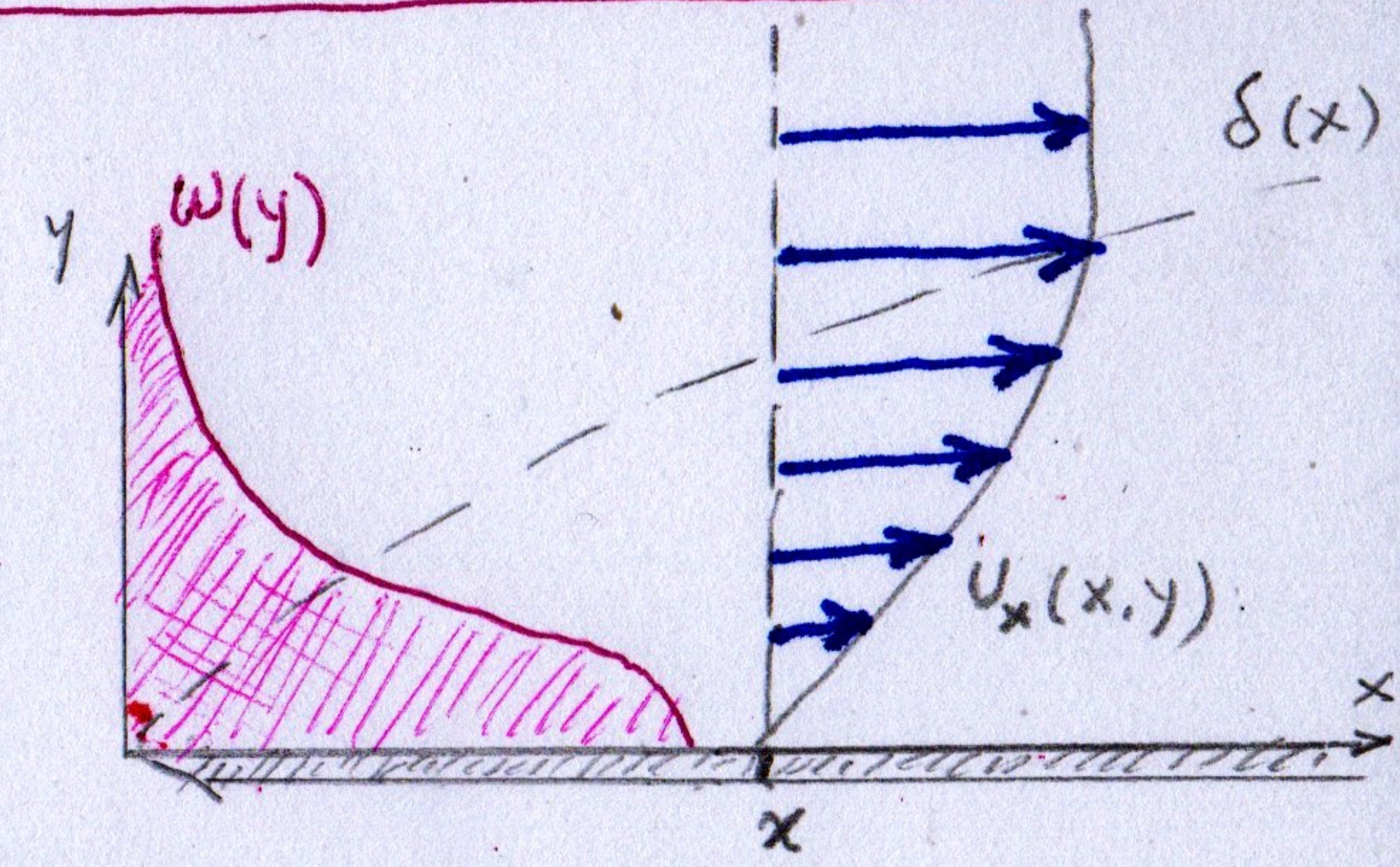
$$\delta u = A \int e^{-\xi^2} d\xi = C + \frac{\sqrt{\pi}\, A}{2} f_{err}(\xi)$$

Con las condiciones de contorno, fijamos los valores de las constantes de integración:

- $\delta u(\xi \to \infty) = 0 \longrightarrow C + \frac{\sqrt{\pi}\, A}{2} = 0$
- $\delta u(\xi = 0) = -U_0 \longrightarrow C = -U_0$

$$\therefore\ u_x(x,y) = U_0 + \delta u_{(x,y)} = U_0\, f_{err}\left(\frac{y}{\sqrt{4\nu x / U_0}}\right)$$

Para interpretar este resultado, definimos el espesor de la <u>capa límite</u> $\delta_{(x)} = \sqrt{\frac{2\nu x}{U_0}}$



de modo que

$$u_x(x,y) = U_0\, f_{err}\left(\frac{y}{\sqrt{2}\,\delta(x)}\right)$$

Calculamos la vorticidad de este flujo:

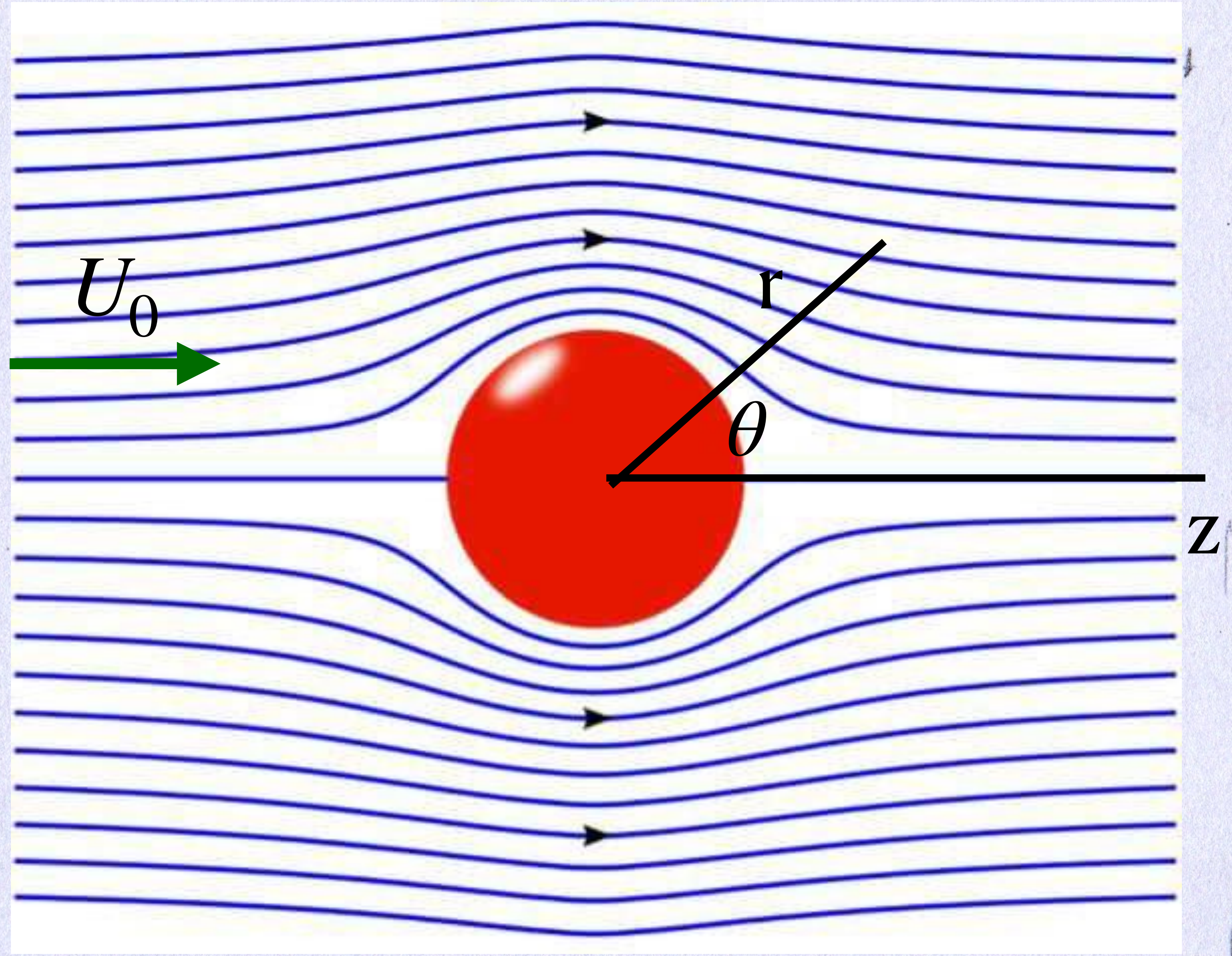
$$\underline{\omega} = \underline{\nabla} \times \underline{u} \longrightarrow \omega_z = -\frac{\partial u_x}{\partial y}$$

$$f'_{err}(\xi) = \frac{2}{\sqrt{\pi}} e^{-\xi^2} \longrightarrow \omega_z = -\sqrt{\frac{2}{\pi}}\, \frac{U_0}{\delta(x)}\, e^{-\frac{y^2}{2\delta^2_{(x)}}}$$

Es decir que en una dada posición x, la vorticidad se distribuye en altura con un perfil <u>gaussiano</u>.

Ley de Stokes

Calculemos la fuerza que ejerce un flujo estacionario, laminar e incompresible sobre una esfera



$$\rho\Big(\underbrace{\cancel{\partial_t \underline{u}}}_{\substack{=0\\(\text{estac.})}} + \underbrace{\cancel{(\underline{u}\cdot\underline{\nabla})\underline{u}}}_{\substack{=0\\(Re\ll 1)}}\Big) = -\underline{\nabla} p + \mu \nabla^2 \underline{u}$$

Por lo tanto:

$$\boxed{\begin{array}{ll} \underline{\nabla} p = \mu \nabla^2 \underline{u} & \\ \underline{\nabla}\cdot\underline{u} = 0 & \underline{u}(r=a) = 0 \end{array}}$$

El problema es axisimétrico. En esféricas:

$$\underline{u} = u_r(r,\theta)\,\hat{r} + u_\theta(r,\theta)\,\hat{\theta} \qquad \frac{\partial}{\partial\varphi} = 0$$

$$\underline{\nabla}\cdot\underline{u} = 0 \longrightarrow \underline{u} = \underline{\nabla}\times(\psi\,\hat{\varphi})$$

$$u_r = \frac{1}{r^2\sin\theta}\,\partial_\theta\psi \qquad u_\theta = -\frac{1}{r\sin\theta}\,\partial_r\psi$$

La vorticidad resulta:

$$\underline{\omega} = \underline{\nabla}\times\underline{u} = -\frac{\hat{\varphi}}{r\sin\theta}\,E^2\psi$$

donde

$$E^2 = \partial^2_{rr} + \frac{\sin\theta}{r^2}\,\partial_\theta\left(\frac{1}{\sin\theta}\,\partial_\theta\right)$$

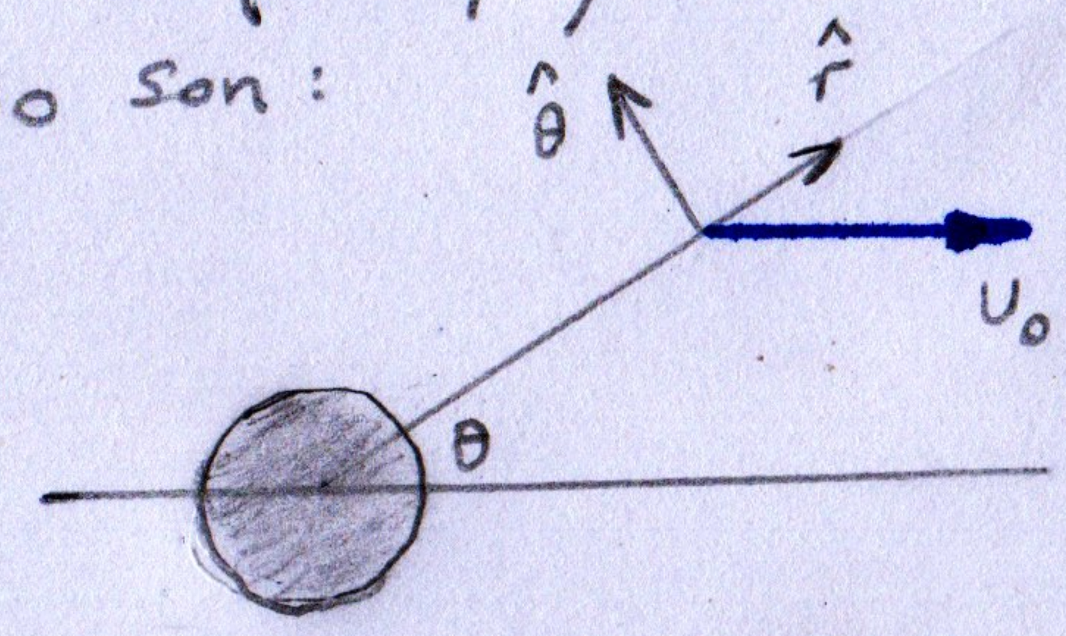
La ecuación N-S resulta:

$$(\hat{r}) \qquad \partial_r p = \frac{\mu}{r^2\sin\theta}\,\partial_\theta(E^2\psi)$$

$$(\hat{\theta}) \qquad \frac{1}{r}\,\partial_\theta p = -\frac{\mu}{r\sin\theta}\,\partial_r(E^2\psi)$$

Como $\partial^2_{r\theta} p = \partial^2_{\theta r} p \longrightarrow E^2(E^2\psi) = 0$

Las condiciones de contorno son:



- $r = a \longrightarrow \partial_r\psi = 0 = \frac{1}{r}\,\partial_\theta\psi$
- $r = \infty \longrightarrow u_r = u_0\cos\theta$
  $u_\theta = -u_0\sin\theta$

Ley de Stokes

$$r=\infty \longrightarrow \left\{\begin{array}{l} u_r = u_0\cos\theta = \dfrac{1}{r^2\sin\theta}\,\partial_\theta\psi \\[2ex] u_\theta = -u_0\sin\theta = -\dfrac{1}{r\sin\theta}\,\partial_r\psi \end{array}\right\} \longrightarrow \psi_{(r\to\infty)} = \frac{u_0 r^2}{2}\sin^2\theta$$

Proponemos soluciones de la forma:

$$\psi(r,\theta) = f(r)\sin^2\theta \xrightarrow[E^2(E^2\psi)=0]{} \left(\partial^2_{rr} - \frac{2}{r^2}\right)^2 f = 0$$

Probamos $f(r)\sim r^\alpha \longrightarrow [(\alpha-2)(\alpha-3)-2][\alpha(\alpha-1)-2]=0$

Resulta $\alpha = -1, 1, 2, 4$

Por lo tanto $f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4$

$$r=\infty \to \psi \to \frac{u_0 r^2}{2}\sin^2\theta \quad \begin{array}{l} \nearrow D = 0 \\ \searrow C = \dfrac{u_0}{2} \end{array}$$

$$r=a \longrightarrow \partial_r\psi\big|_{r=a} = 0 = \left(-\frac{A}{a^2} + B + u_0 a\right)\sin^2\theta$$

$$\searrow \partial_\theta\psi\big|_{r=a} = 0 = \left(\frac{A}{a} + Ba + \frac{u_0}{2}a^2\right)2\sin\theta\cos\theta$$

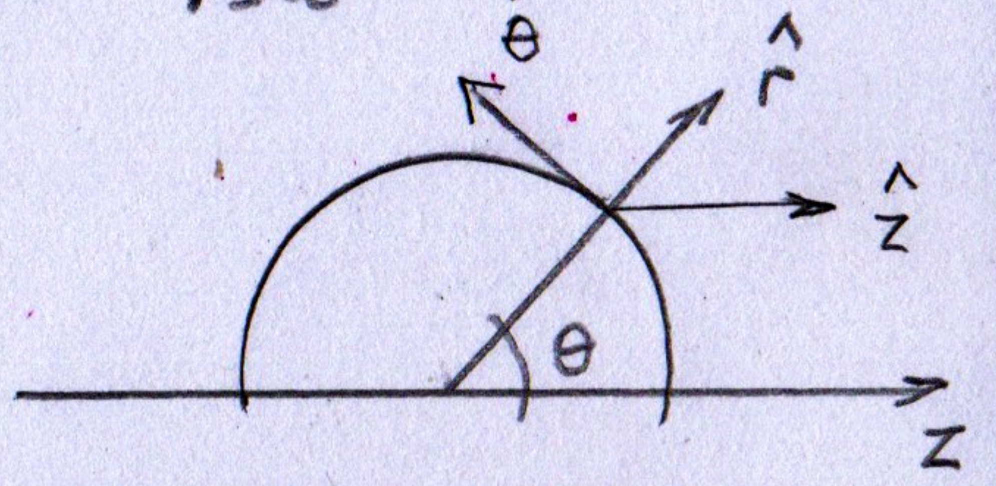
$$\left.\begin{array}{l} A = \dfrac{u_0}{4}a^3 \\[2ex] B = -\dfrac{3}{4}u_0 a \end{array}\right\} \longrightarrow \psi(r,\theta) = \frac{u_0}{4}\sin^2\theta\left(2r^2 + \frac{a^3}{r} - 3ar\right)$$

De cualquiera de las componentes de N-S integramos para obtener la presión:

$$p(r,\theta) = p_0 - \frac{3}{2}\frac{\mu u_0 a}{r^2}\cos\theta$$

Finalmente, para calcular la fuerza sobre la esfera:

$$\underline{F} = \oiint_{r=a} \underline{\underline{\sigma}}\cdot\hat{r}\,dS \xrightarrow[\partial_\varphi = 0]{} F_z = \oiint_{r=a} \hat{z}\cdot\underline{\underline{\sigma}}\cdot\hat{r}\,dS$$



Como:

$$\sigma_{rr} = -p + 2\mu\,\partial_r u_r$$

$$\sigma_{r\theta} = \mu r\,\partial_r\left(\frac{u_\theta}{r}\right) + \frac{\mu}{r}\partial_\theta u_r$$

$$\hat{z}\cdot\underline{\underline{\sigma}}\cdot\hat{r} = \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta$$

Ley de Stokes

Necesitamos calcular

$$u_r = \frac{1}{r^2 \sin\theta} \partial_\theta \psi = \frac{u_0 \cos\theta}{2r^2}\left(2r^2 + \frac{a^3}{r} - 3ar\right)$$

$$u_\theta = -\frac{1}{r \sin\theta} \partial_r \psi = -\frac{u_0 \sin\theta}{4r}\left(4r - \frac{a^3}{r^2} - 3a\right)$$

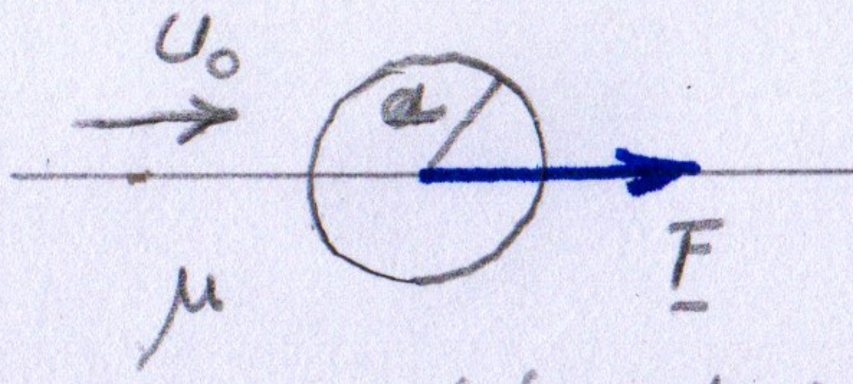
y luego:

$$\partial_r u_r\Big|_{r=a} = 0 \qquad \partial_\theta u_r\Big|_{r=a} = 0$$

$$\partial_r\left(\frac{u_\theta}{r}\right)\Big|_{r=a} = -\frac{3}{2}\frac{u_0 \sin\theta}{a^2}$$

Finalmente

$$F_z = 6\pi\mu a u_0$$



- Las líneas contínuas son líneas de corriente ($\psi = cte$)
- Las punteadas son de $p = cte$.
- La paleta de colores muestra vorticidad ω_φ

$$\omega_\varphi = -\frac{3}{2}\frac{u_0 a \sin\theta}{r^2}$$