Resumen de ecuaciones útiles

Estructura de la Materia 1 - 1er. cuatrimestre de 2021

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Notación y definiciones generales

En este resumen, los vectores se notarán en negritas; por ejemplo el campo de velocidades se simboliza mediante $\mathbf{u}$. Los versores se denotan mediante un sombrero; de acuerdo a esto el versor que apunta en la dirección del eje x en coordenadas cartesiadas viene dada por $\hat{x}$. Cada una de las componentes de un vector, según su dirección, se denotan mediante el subíndice correspondiente; e.g., la componente radial del campo de velocidades viene dada por u_r.

Los operadores diferenciales rotor y divergencia de un campo $\mathbf{A}$ se simbolizan mediante $\nabla \times \mathbf{A}$ y $\nabla \cdot \mathbf{A}$, respectivamente. El gradiente y el laplaciano de ese mismo campo se denotan mediante $\nabla \mathbf{A}$ y $\nabla^2 \mathbf{A}$ (notar que estas operaciones están definidas tanto para un campo escalar como para uno vectorial).

La componente ij del tensor de deformación del campo de velocidades se simboliza mediante e_{ij}. Recordemos que la relación entre dicho tensor y el de esfuerzos viscosos, que aquí denotamos por σ_{ij}, es

$$\sigma_{ij} = 2\mu e_{ij},$$

siendo μ el coeficiente de viscosidad dinámico, vinculado con la viscosidad cinemática ν y la densidad ρ mediante la relación $\mu = \rho\, \nu$.

1 Coordenadas polares (en el plano)

En este caso se trata del sistema de coordenadas polares planas usuales, donde r representa la distancia al origen y θ es el ángulo entre el eje $\hat{x}$ y el versor radial.

1.1 Gradiente de un escalar

$$\nabla \psi = \frac{\partial \psi}{\partial r}\, \hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\, \hat{\theta}$$

1.2 Laplaciano de un escalar

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$$

1.3 Divergencia de un vector

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

1.4 Rotor de un vector

$$\nabla \times \mathbf{u} = \left[\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right]\, \hat{z}$$

1.5 Laplaciano de un vector

$$\nabla^2 \mathbf{u} = \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} \right) \hat{r} + \left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) \hat{\theta}$$

1.6 Tasa de deformación y esfuerzo viscoso

$$e_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr}$$
$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta}$$
$$e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta} = \frac{1}{2\mu}\sigma_{r\theta}$$

1.7 Vorticidad

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

1.8 Ecuación de continuidad

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) = 0$$

1.9 Ecuación de Euler sin fuerzas externas

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

donde

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

1.10 Ecuación de Navier-Stokes con ν y ρ constantes; sin fuerzas externas

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$
$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

donde

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

2 Coordenadas cilíndricas

En este sistema de coordenadas tenemos que todo punto del espacio tridimensional puede identificarse biunívocamente por medio de sus coordenadas (r, θ, z). Las dos primeras denotan, respectivamente, la distancia al origen y el ángulo entre el eje de abcisas $\hat{x}$ y el versor radial. La tercera denota la distancia perpendicular de dicho punto al plano r-θ.

2.1 Gradiente de un escalar

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\hat{\theta} + \frac{\partial\psi}{\partial z}\,\hat{z}$$

2.2 Laplaciano de un escalar

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

2.3 Divergencia de un vector

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$$

2.4 Rotor de un vector

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right)\,\hat{r} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\,\hat{\theta} + \left[\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]\,\hat{z}$$

2.5 Laplaciano de un vector

$$\nabla^2\mathbf{u} = \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)\,\hat{r} + \left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right)\,\hat{\theta} + \nabla^2 u_z\,\hat{z}$$

2.6 Tasa de deformación y esfuerzo viscoso

$$\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr}\\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta}\\
e_{zz} &= \frac{\partial u_z}{\partial z} = \frac{1}{2\mu}\sigma_{zz}\\
e_{r\theta} &= \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta} = \frac{1}{2\mu}\sigma_{r\theta}\\
e_{\theta z} &= \frac{1}{2r}\frac{\partial u_z}{\partial\theta} + \frac{1}{2}\frac{\partial u_\theta}{\partial z} = \frac{1}{2\mu}\sigma_{\theta z}\\
e_{zr} &= \frac{1}{2}\frac{\partial u_r}{\partial z} + \frac{1}{2}\frac{\partial u_z}{\partial r} = \frac{1}{2\mu}\sigma_{zr}
\end{aligned}$$

2.7 Vorticidad

$$\begin{aligned}
\omega_r &= \frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\\
\omega_\theta &= \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\\
\omega_z &= \frac{1}{r}\frac{\partial}{\partial r}\left(ru_\theta\right) - \frac{1}{r}\frac{\partial u_r}{\partial\theta}
\end{aligned}$$

2.8 Ecuación de continuidad

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

2.9 Ecuación de Euler sin fuerzas externas

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)\,u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)\,u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
\frac{\partial u_z}{\partial t} + (\mathbf{u}\cdot\nabla)\,u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}$$

donde

$$\begin{aligned}
\mathbf{u}\cdot\nabla &= u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z} \\
\nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}$$

2.10 Ecuación de Navier-Stokes con ν y ρ constantes; sin fuerzas externas

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)\,u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) \\
\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)\,u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) \\
\frac{\partial u_z}{\partial t} + (\mathbf{u}\cdot\nabla)\,u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z
\end{aligned}$$

donde

$$\begin{aligned}
\mathbf{u}\cdot\nabla &= u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z} \\
\nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}$$

3 Coordenadas esféricas

En este sistema de coordenadas, un punto del espacio 3D puede identificarse biunívocamente por medio de tres coordenadas que denominamos (r, φ, θ). La primera de ellas representa la distancia del origen al punto de interés. La segunda, φ, indica el ángulo entre el eje de abcisas $\hat{x}$ y la proyección del versor radial sobre el plano x-y. La tercera coordenada, θ señala el ángulo comprendido entre el versor $\hat{z}$ y el versor radial.

3.1 Gradiente de un escalar

$$\nabla\psi = \frac{\partial \psi}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \varphi}\,\hat{\varphi}$$

3.2 Laplaciano de un escalar

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}$$

3.3 Divergencia de un vector

$$\nabla \cdot u = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (u_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi}$$

3.4 Rotor de un vector

$$\nabla \times \mathbf{u} = \frac{1}{r\sin\theta}\left[\frac{\partial (u_\varphi \sin\theta)}{\partial \theta} - \frac{\partial u_\theta}{\partial \varphi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \varphi} - \frac{\partial (r u_\varphi)}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right]\hat{\varphi}$$

3.5 Laplaciano de un vector

$$\begin{aligned}
(\nabla^2 \mathbf{u})\cdot\hat{r} &= \nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial (u_\theta \sin\theta)}{\partial \theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}\\
(\nabla^2 \mathbf{u})\cdot\hat{\theta} &= \nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial \varphi}\\
(\nabla^2 u)\cdot\hat{\varphi} &= \nabla^2 u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2\sin^2\theta}
\end{aligned}$$

3.6 Tasa de deformación y tensor de esfuerzos viscosos

$$\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr}\\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta}\\
e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r} = \frac{1}{2\mu}\sigma_{\varphi\varphi}\\
e_{\theta\varphi} &= \frac{\sin\theta}{2r}\frac{\partial}{\partial \theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial \varphi} = \frac{1}{2\mu}\sigma_{\theta\varphi}\\
e_{\varphi r} &= \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right) = \frac{1}{2\mu}\sigma_{\varphi r}\\
e_{r\theta} &= \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta} = \frac{1}{2\mu}\sigma_{r\theta}
\end{aligned}$$

3.7 Vorticidad

$$\begin{aligned}
\omega_r &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(u_\varphi \sin\theta) - \frac{\partial u_\theta}{\partial \varphi}\right]\\
\omega_\theta &= \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \varphi} - \frac{\partial (r u_\varphi)}{\partial r}\right]\\
\omega_\varphi &= \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right]
\end{aligned}$$

3.8 Ecuación de continuidad

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\rho u_\varphi) = 0$$

3.9 Ecuación de Euler sin fuerzas externas

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi \cot\theta}{r} &= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \varphi}
\end{aligned}$$

donde

$$\begin{aligned}
\mathbf{u}\cdot\nabla &= u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \\
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}
\end{aligned}$$

3.10 Ecuación de Navier-Stokes con ν y ρ constantes; sin fuerzas externas

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right] \\
\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right] \\
\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi \cot\theta}{r} &= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \varphi} + \nu\left[\nabla^2 u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right]
\end{aligned}$$

donde

$$\begin{aligned}
\mathbf{u}\cdot\nabla &= u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \\
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}
\end{aligned}$$