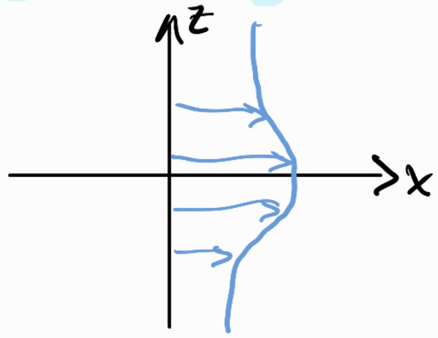
Inestabilidades HD: $\underline{u} = U(z)\hat{x}$

perturbarlo y analizar la estabilidad



$$\begin{cases} u_x = U(z) + \delta u_x(x,z,t) \\ u_z = \delta u_z(x,z,t) \\ p = p_0 + \delta_p(x,z,t) \end{cases}$$

Flujo plano incompresible: $\delta\underline{u} = \underline{\nabla}\psi \times (-\hat{y}) = \delta u_x \hat{x} + \delta u_z \hat{z}$

ψ contiene a la perturbación

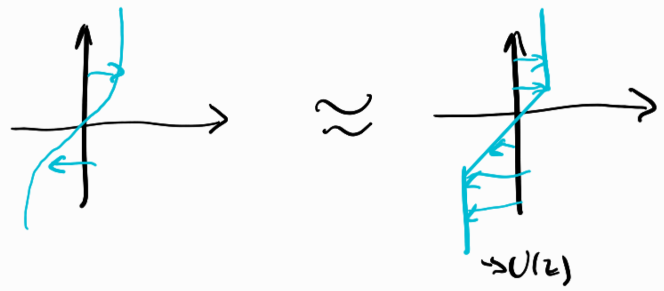
Desarrollo en modos normales $\Rightarrow$ $\psi(x,z,t) = \sum_{k,\omega} f(z)\, e^{i(kx-\omega t)} = f(z) e^{ik(x-ct)}$

$c = \frac{\omega}{k}$

Ecuación de Rayleigh:

$$\boxed{[U(z) - c]\,(f'' - k^2 f) - U''(z) f = 0}$$

Aproximar



Lineal a trozos

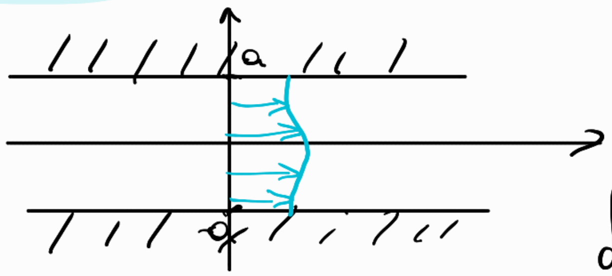
$\Rightarrow U''(z) = 0 \quad \forall z$

$\Rightarrow$ Ec. de Rayleigh: $f''(z) - k^2 f(z) = 0$

$$\Leftrightarrow \begin{cases} f(z) = A e^{kz} + B e^{-kz} \\ f(z) = \alpha \sinh(kz + \varphi) \\ f(z) = \beta \cosh(kz + \theta) \end{cases}$$

Condiciones de Contorno:

Confinado



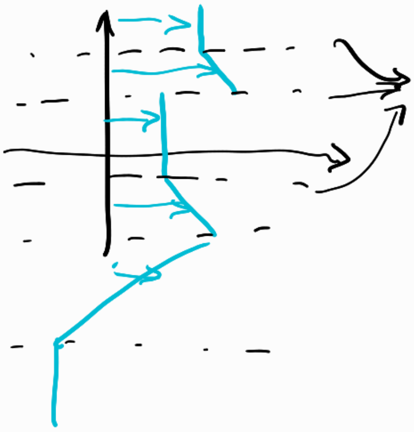
$\underline{u} = (U + \delta u_x)\hat{x} + \delta u_z \hat{z}$

$\delta u_z(z=-a) = \delta u_z(z=a) = 0$

$\delta u_z = -\frac{\partial \psi}{\partial x} \Rightarrow \left.\frac{\partial \psi}{\partial x}\right|_{cont} = 0$

$\psi = f(z) e^{i(kx-\omega t)} \Rightarrow \frac{\partial \psi}{\partial x} = ik\, f(z)\, e^{i(kx-\omega t)}$

$\Rightarrow \left. f(z)\right|_{cont} = 0$



Interfaces: conti. de presión y la comp. z de la vel.

Presión: $\Delta\left[\rho\left(U'f-(U-c)f'\right)\right]\Big|_{int}=0$

u_z: $\Delta\left(\dfrac{f(z)}{U(z)-c}\right)\Big|_{int}=0$

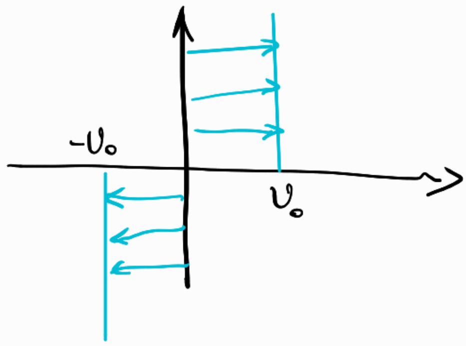
Estudio de la inestabilidad: $\Psi=f(z)\,e^{i(kx-\omega t)}$ $\qquad \omega(k)=\omega_{Re}+i\omega_{Im}$

$\Psi \propto e^{-i(i\omega_{Im}t)}=e^{\omega_{Im}t}\in\mathbb{R}e$ $\quad$ si $\omega_{Im}>0$ crece en el tiempo

Criterio de Rayleigh: Para que exista inestabilidad deben existir puntos de inflexión en el flujo base:

$\exists\, z_* \;/\; U''(z_*)=0$ (Condición necesaria pero no suficiente)

Problema 1: $U(z)=U_0\begin{cases}1 & \text{si } z>0\\ -1 & \text{si } z<0\end{cases}$



$U'=0=U''\quad\forall z$

$\Rightarrow$ Ec. de Rayleigh: $f''-k^2f=0 \;\rightarrow\; f(z)=\begin{cases}A_1e^{-kz}+B_1e^{kz} & z>0\\ A_2e^{-kz}+B_2e^{kz} & z<0\end{cases}$

$z\rightarrow\pm\infty \Rightarrow$ (Para que no diverja) $f(z)=\begin{cases}Ae^{-kz} & z>0\\ Be^{kz} & z<0\end{cases} \Rightarrow f'(z)=\begin{cases}-kAe^{-kz} & z>0\\ kBe^{kz} & z<0\end{cases}$

Interfaz: $\Delta\left(\rho\left(\cancel{U'f}-(U-c)f'\right)\right)\Big|_{int}=0$

Abajo ($z\to0^-$) $\qquad$ Arriba ($z\to0^+$)

$+\cancel{\rho}(-U_0-c)f'(z)\Big|_{z=0^-}=+\cancel{\rho}(U_0-c)f'(z)\Big|_{z=0^+} \Longleftrightarrow -(U_0+c)kBe^{k\cdot0}=(U_0-c)(-kAe^{-k\cdot0})$

$$A(U_0-c) - B(U_0+c) = 0 \qquad M\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

- $$\Delta\left(\frac{f(z)}{U-c}\right)\Bigg|_{int} = 0 \Rightarrow \frac{f(z=0^-)}{-U_0-c} = \frac{f(z=0^+)}{U_0-c} \Leftrightarrow \frac{B}{-U_0-c} = \frac{A}{U_0-c}$$

$$A(U_0+c) + B(U_0-c) = 0$$

Quiero $\omega(k)$:

$$\det\overbrace{\begin{pmatrix} U_0-c & -(U_0+c) \\ U_0+c & U_0-c \end{pmatrix}}^{M} = 0 \Rightarrow (U_0-c)^2 + (U_0+c)^2 = 0$$

$$U_0^2 \cancel{-2U_0c} + c^2 + U_0^2 \cancel{+2U_0c} + c^2 = 0$$

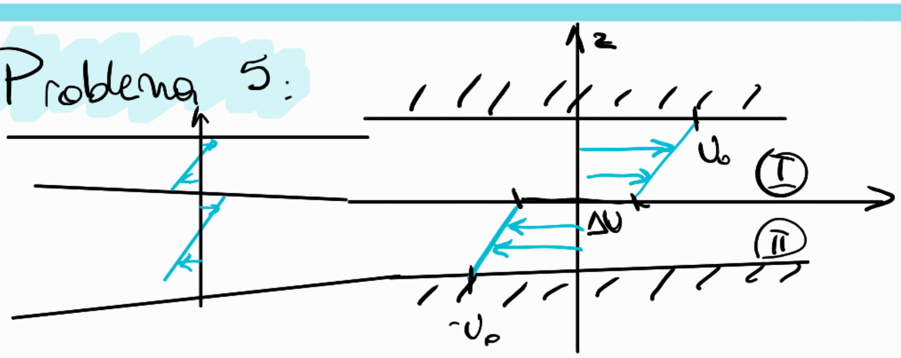
$$2(U_0^2+c^2) = 0 \Rightarrow c^2 = -U_0^2 \Rightarrow c = \pm iU_0 = \frac{\omega}{k}$$

$$\omega(k) = \pm iU_0k$$

Siempre inestable $\forall k$

$$\psi = \sum_j f(z)\, e^{i(kx-\omega_j t)} = f(z)\, e^{ikx}\Big[\underbrace{e^{\overbrace{-i(iU_0k)t}^{U_0kt}}}_{\xrightarrow[t\to\infty]{}\infty} + \underbrace{e^{\overbrace{-i(-iU_0k)t}^{-U_0kt}}}_{\xrightarrow[t\to\infty]{}0}\Big]$$

Problema 5:

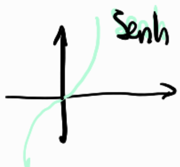


$$U(z) = \begin{cases} U_0\frac{z}{L} + \frac{\Delta U}{2} & \text{I} \quad 0 \le z \le L \\ U_0\frac{z}{L} - \frac{\Delta U}{2} & \text{II} \quad -L \le z \le 0 \end{cases}$$

$$\Delta U \geq 0$$

$U' = \frac{U_0}{L} \ \forall z$ y $U'' = 0 \ \forall z \Rightarrow$ Ec. de R.: $f'' - k^2 f = 0$

$$f_i(z) = A_i e^{kz} + B_i e^{-kz} = \alpha \,\mathrm{senh}(kz+\varphi)$$



Contornos: $\left.\frac{\partial\psi}{\partial x}\right|_{z=\pm L} = 0 \Rightarrow \left. f(z)\right|_{z=\pm L} = 0$

evaluar en L y encontrar φ/

Me salté pasos

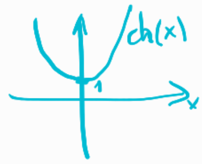
$$\begin{cases} f_{I}(z) = A\,\mathrm{senh}(k(z-L)) \\ f_{II}(z) = B\,\mathrm{senh}(k(z+L)) \end{cases}$$

$$f'(z) = \begin{cases} kA\cosh\big(k(z-L)\big) & z>0 \\ kB\cosh\big(k(z+L)\big) & z<0 \end{cases}$$

Empalmes: $\bullet\ \Delta\left(\frac{f(z)}{U-c}\right)\Big|_{\substack{int \\ \downarrow \\ z=0}} = 0 \Rightarrow \frac{f_{I}(z=0)}{\underbrace{U(z=0^+)}_{\frac{\Delta U}{2}}-c} = \frac{f_{II}(z=0)}{\underbrace{U(z=0^-)}_{-\frac{\Delta U}{2}}-c}$

$$\Rightarrow \frac{A\overbrace{\operatorname{senh}(k(0-L))}^{\operatorname{senh}(-kL) = -\operatorname{senh}(kL)}}{\frac{\Delta U}{2}-c} = \frac{B\operatorname{senh}(k(0+L))}{-\frac{\Delta U}{2}-c} \Leftrightarrow \frac{+A}{\frac{\Delta U}{2}-c} = \frac{+B}{\frac{\Delta U}{2}+c}$$

$$A\left(\frac{\Delta U}{2}+c\right) - B\left(\frac{\Delta U}{2}-c\right) = 0$$



$\bullet\ \Delta\big(\rho(U'f-(U-c)f')\big)\Big|_{int} = 0 \Rightarrow \frac{U_0}{L}\underbrace{f_I(z=0)}_{\substack{A\,sh(-kL) \\ \| \\ -A\,sh(kL)}} - \left(\frac{\Delta U}{2}-c\right)\underbrace{f_I'(z=0)}_{\substack{KA\,ch(-kL) \\ \| \\ KA\,ch(kL)}} = \frac{U_0}{L}\underbrace{f_{II}(z=0)}_{B\,sh(kL)} - \left(-\frac{\Delta U}{2}-c\right)\underbrace{f_{II}'(z=0)}_{KB\,ch(kL)}$

$$\left[+\frac{U_0}{L}\frac{sh(kL)}{ch(kL)} + k\left(\frac{\Delta U}{2}-c\right)\frac{ch(kL)}{ch(kL)}\right]A + B\left[+\frac{U_0}{L}\frac{sh(kL)}{ch(kL)} + \left(\frac{\Delta U}{2}+c\right)\frac{ch(kL)}{ch(kL)}\right] = \frac{0}{ch(kL)}$$

$$A\left[\frac{U_0}{L}tgh(kL) + \frac{k\Delta U}{2} - \overset{kc}{\overset{\|}{\omega}}\right] + B\left[\frac{U_0}{L}tgh(kL) + \frac{k\Delta U}{2} + \omega\right] = 0$$

$$\det\begin{pmatrix} k\frac{\Delta U}{2}+\omega & -\left(\frac{k\Delta U}{2}-\omega\right) \\ \frac{U_0}{L}tgh(kL)+\frac{k\Delta U}{2}-\omega & \frac{U_0}{L}tgh(kL)+\frac{k\Delta U}{2}+\omega \end{pmatrix} = 0$$

$$\left(k\frac{\Delta U}{2}+\omega\right)\left(\frac{U_0}{L}tgh(kL)+\frac{k\Delta U}{2}+\omega\right) + \left(\frac{k\Delta U}{2}-\omega\right)\left(\frac{U_0}{L}tgh(kL)+\frac{k\Delta U}{2}-\omega\right) = 0$$

Hacer cuentas ....

distribuyendo:

$$\omega^2 = -k\frac{\Delta U}{2}\left[\frac{k\Delta U}{2} + \frac{U_0}{L}\,tgh(kL)\right]$$

Si $\Delta U > 0 \Rightarrow \omega$ es imag. $\Rightarrow$ Inestable

Si $\Delta U < 0 \Rightarrow \frac{k\Delta U}{2} + \frac{U_0}{L}\,tgh(kL) < 0$

$$\frac{tgh(kL)}{kL} < -\frac{\Delta U}{2U_0}$$ Condición para que sea inestable