The equation of evolution of the angular momentum and the instantaneous axis of rotation

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We present a discussion about the "instantaneous axis of rotation" as an adequate origin of momenta in the equation of evolution of the angular momentum, a subject that usually is ill interpreted at the undergraduate level. Three simple examples are presented that can contribute to a better understanding of the topic.

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I. INTRODUCTION

In the discussion about the angular momentum $\vec{L}_O$ of a system of particles in Elementary Mechanics, a subject that always deserves special attention is the choice of an adequate origin of momenta[1]. In particular, the equation of evolution of $\vec{L}_O$ has different forms, according to this choice. We have observed that in many textbooks[2,3,4,5,6], this subject is not dealt with in a complete way and a full discussion is avoided, perhaps in

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order to present it at a more elementary level. However, we observe that this “simplified” treatment of the topic may lead, in general, to some misinterpretations, by most students in undergraduate courses. In particular, the correct equation of motion of $\vec{L}_O$ in rigid body dynamics taking the “instantaneous axis of rotation” (IAR) as origin is a subject that must be considered with care, as a superficial analysis can lead to erroneous conclusions. We present here a discussion, based on two simple examples, that, we believe sheds some clarity about this particular subject.

Equation of evolution of the angular momentum

The angular momentum of a system of N particles about an origin O is expressed as:

$$\vec{L}_O = \sum_{i=1}^{N} (\vec{r}_i - \vec{r}_O) \times \vec{p}_i \qquad (1)$$

where $\vec{r}_i$ is the position of mass m_i, $\vec{p}_i$ is its linear momentum and $\vec{r}_O$ is the position of point O, all of them relative to the reference frame. Assuming that point O is moving with velocity $\vec{v}_O(t)$, we can derive an equation of evolution for the angular momentum $\vec{L}_O$:

$$\frac{d\vec{L}_O}{dt} = \sum_{i=1}^{N} (\vec{r}_i - \vec{r}_O) \times \frac{d\vec{p}_i}{dt} + (\vec{v}_i - \vec{v}_O) \times \vec{p}_i \qquad (2)$$

In this last expression, $\vec{v}_i \times \vec{p}_i = \vec{0}$ and $\frac{d\vec{p}_i}{dt} = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij}$, where $\vec{F}_i^{ext}$ is the resultant of external forces acting on m_i and $\sum_{j \neq i} \vec{F}_{ij}$ is the sum of internal forces on particle *i*, due to the rest of the particles. It is easy to see that $\sum_{i=1}^{N} (\vec{r}_i - \vec{r}_O) \times \sum_{j \neq i} \vec{F}_{ij} = \vec{0}$, if internal forces are collinear with the relative position of each pair of particles. Therefore:

$$\frac{d\vec{L}_O}{dt} = \sum_{i=1}^{N} (\vec{r}_i - \vec{r}_O) x \vec{F}_i^{ext} - \vec{v}_O x \sum_{i=1}^{N} \vec{p}_i \qquad (3)$$

and thus, the time rate of change of the angular momentum about a certain moving origin O of a system of particles is:

$$\frac{d\vec{L}_O}{dt} = \vec{N}_O - \vec{v}_O(t)\times\vec{P}_{CM} \qquad (4)$$

where $\vec{N}_O$ stands for the sum of all external torques acting on the system about the same point as $\vec{L}_O$, $\vec{v}_O(t)$ is the linear velocity of the origin O and $\vec{P}_{CM}$ is the linear momentum of the system.

Eq. 4 is reduced to the well-known form:

$$\frac{d}{dt}\vec{L}_O = \vec{N}_O \qquad (5)$$

in particular cases, i.e., if and only if the origin O is chosen as a) a fixed point in space, b) the center of mass (CM) of the system, or c) a point in space that moves relative to the reference frame, with its velocity $\vec{v}_O(t)$ parallel to the linear momentum of the system, $\vec{P}_{CM}$. The particular case of a point lying down on the IAR of a rigid body chosen as origin of momenta is considered now in detail.

II. THREE EXAMPLES

1) A particular case, which is explained in most elementary courses of Mechanics is that of a rigid cylinder which rolls down an inclined plane without sliding. In this case, it is usual to refer the equation of evolution of the angular momentum to a) the CM of the body or b) a point lying down on the instantaneous axis of rotation (IAR). In both cases, the dynamic equation considered is Eq. 5, with the torque of the unknown static frictional force in the former, and the torque of the (known) weight force in the latter. However, the reason of the validity of this equation for case b) is in general not given explicitly. This

fact could lead to the conclusion (and most textbooks overlook this discussion) that Eq. 5 is also valid if the origin O is a point on the IAR, in any other situation. Let's analyze two possible interpretations for a general situation:

(i) The equation of evolution Eq. 5 holds choosing as origin a point O lying on the IAR, because $\vec{v}_O = \vec{0}$: the origin O is instantly at rest.

(ii) The equation of evolution, Eq. 5, does not hold in general if O is on the IAR, as $\vec{v}_O(t)$ in Eq. 4 is the velocity of the *point of space* (not the material point of the system) taken as origin of momenta, and its velocity is not null, in the general case, although the velocity of the material point on the IAR is zero at every instant.

As it will be seen, the correct answer is interpretation (ii). A second example will illustrate this conclusion.

2) Let's consider the movement of a particle of mass *m* at the end of a light rope, which is winding around a fixed cylinder. Consider that all the movement is contained in a frictionless horizontal plane (Figure 1). Point Q is the point of contact between the string and the cylinder. As the cylinder is at rest and the string does not slip, point Q on the string has the same velocity as the contact point on the cylinder, that is, a zero velocity. Then, the movement of mass *m* can be considered as an instantaneous circular motion with center in point Q (which is changing with time), and the (changing) string length $l = |\vec{r}'|$ is the curvature radius of the motion. Therefore, the IAR passes through point Q. Hence, the tension $\vec{T}$ in the rope is always perpendicular to the velocity $\vec{v}$ of the mass and the (kinetic) energy of the mass is conserved. Therefore, it is concluded that the speed $|\vec{v}|$ is constant. Let's consider now the problem analyzing the angular momentum of the

mass. Taking as origin the fixed point O in Fig. 1, the equation of evolution of the angular momentum is Eq. 5, which results:

$$\frac{d\vec{L}_O}{dt} = \vec{N}_O = -RT\hat{z} \qquad (6)$$

where the torque is clearly not null. If now the origin of momentum is taken as point Q, the torque about this point on the IAR is null. Then, if Eq. 5 was correct, it could be concluded that $|\vec{v}|$ is not constant! In fact:

$$\frac{d\vec{L}_Q}{dt} = \vec{N}_Q = \vec{0} \Rightarrow \vec{L}_Q = mr'|\vec{v}|\hat{z} = L\hat{z} \qquad (7)$$

where L is a constant. As r' changes with time, then $|\vec{v}| = L/mr'$ would change with time too. The mistake is due to interpretation (i). As it was already mentioned, in the general equation of evolution of the angular momentum, Eq. 4, $\vec{v}_O(t)$ is the velocity of the point in space chosen as origin (following interpretation (ii)), and it is not the velocity of the material points which lie, in each instant, on the IAR (interpretation (i)). Then, the correct equation of evolution of the angular momentum with origin in Q is:

$$\frac{d\vec{L}_Q}{dt} = -\vec{v}_Q(t) \times \vec{P}_{CM} \qquad (8)$$

which leads to the same result as the energy conservation equation. This is easily seen taking into account that the rhs of Eq. 8 is

$$-\vec{v}_Q(t) \times \vec{P}_{CM} = mv_Q v\hat{z} \qquad (9)$$

and its lhs is:

$$\frac{d\vec{L}_Q}{dt} = \frac{d(mr'v)}{dt}\hat{z} = m\left[v\frac{dr'}{dt} + r'\frac{dv}{dt}\right]\hat{z} = m\left[vv_Q + r'\frac{dv}{dt}\right]\hat{z} \tag{10}$$

Therefore, Eq. 8 also leads to $\frac{dv}{dt} = 0$

In the case of the rolling body, the equation of evolution to be considered by choosing an origin lying down on the IAR is Eq. 5, because the chosen point in space moves following a trajectory that is congruent to the trajectory of the CM of the system. That is, the additional term in Eq. 4 is zero in this case, as $\vec{v}_O(t)$ is parallel to $\vec{P}_{CM}$ at every instant. Example 2, on the other hand, is a very simple situation in which the IAR and the CM follow different trajectories, thus allowing a clarifying discussion of the general case. This particular example was chosen because of its simplicity and, in fact, it involves the movement of a single point particle. Once the correct way of handling this kind of situation is established, it is not difficult to find other examples.

3) Consider, for instance, an inhomogeneous cylinder that rolls without sliding on a horizontal plane (Figure 2). The center of mass (CM) lies at a point a distance *d* apart from the central axis of the cylinder. As in the first example, the IAR of the cylinder passes through the point of contact O with the surface. The CM rotates around the center axis of the cylinder, describing a cycloid. Therefore, points O and CM describe very different trajectories.

The equation of motion (EOM) of the system can be readily obtained from the conservation of energy, taking into account that the movement of the cylinder can be considered, instantaneously, as a pure rotation around the IAR passing through point O:

$$E = \frac{1}{2}I_O(\theta)(\frac{d\theta}{dt})^2 + mgdsin\theta \tag{11}$$

where $I_O(\theta)$ is the moment of inertia of the cylinder with respect to the axis passing through O. It is a function of θ because it takes different values depending on the position of the CM. As $L_O = I_O(\theta)(\frac{d\theta}{dt})$, the conserved energy E can be written as:

$$E = \frac{L_O^2}{2I_O(\theta)} + mgdsin\theta \quad (12)$$

The EOM can be obtained in terms of $L_O = I_O(\theta)(\frac{d\theta}{dt})$, by taking the time derivative in Eq. 12. It is obtained:

$$0 = \frac{L_O}{I_O(\theta)}\frac{dL_O}{dt} - \frac{L_O^2}{2I_O^2(\theta)}\frac{dI_O(\theta)}{dt} + mgd\cos\theta\frac{d\theta}{dt} \quad (13)$$

Rearrangement of terms yields:

$$\frac{dL_O}{dt} = -mgd\cos\theta + \frac{1}{2}\frac{dI_O(\theta)}{dt}\frac{d\theta}{dt} \quad (14)$$

It is clearly seen that the time rate of change of L_O does not depend only on the torque of the applied forces (first term in the rhs of Eq. 14), but there is an additional term. It can be proven that this term has the form required by Eq. 4, i.e.:

$$-\vec{v}_O(t)\times\vec{P}_{CM} = -(-R\frac{d\theta}{dt}\hat{x})\times(-mdsin\theta\frac{d\theta}{dt}\hat{x} + md\cos\theta\frac{d\theta}{dt}\hat{y}) = mRd\cos\theta(\frac{d\theta}{dt})^2\hat{z} \quad (15)$$

In fact, it is found:

$$I_O = I_{CM} + m(R^2 + d^2 + 2Rdsin\theta) \quad (16)$$

Taking into account that $\frac{dI_{CM}}{dt} = 0$, Eq. 14 can be re-expressed as:

$$\frac{1}{2}\frac{dI_O(\theta)}{dt}\frac{d\theta}{dt} = mRd\cos\theta(\frac{d\theta}{dt})^2 \quad (17)$$

which is exactly the result in Eq. 15. It is therefore proven that the second term in the EOM of L_O in Eq. 14 accounts for the "inertial" term $-\vec{v}_O(t)\times\vec{P}_{CM}$ of Eq. 4.

III. CONCLUSIONS

The subject of obtaining the correct answers for the correct reasons is certainly a fundamental one in elementary physics courses. In this work we have shown the importance of clarifying explicitly the meaning of $\vec{v}_O(t)$ as the time rate of change of the position of the origin of momenta in Eq. 4. The three examples presented here show the kinds of confusion that can arise if this velocity is incorrectly interpreted as that of the material point of the rigid body occupying such position, when the IAR is taken as origin of momenta. Unexpectedly, this misinterpretation leads to the correct results in many textbook situations. We have shown that this is only the case when the IAR and the center of mass of the system follow parallel trajectories. We have also shown the correct way of handling the problem by considering two simple situations in which this is not the case.

Figure 1: A particle of mass m at the end of a light cord, winding around a fixed cylinder.

All the movement is in a frictionless horizontal plane.

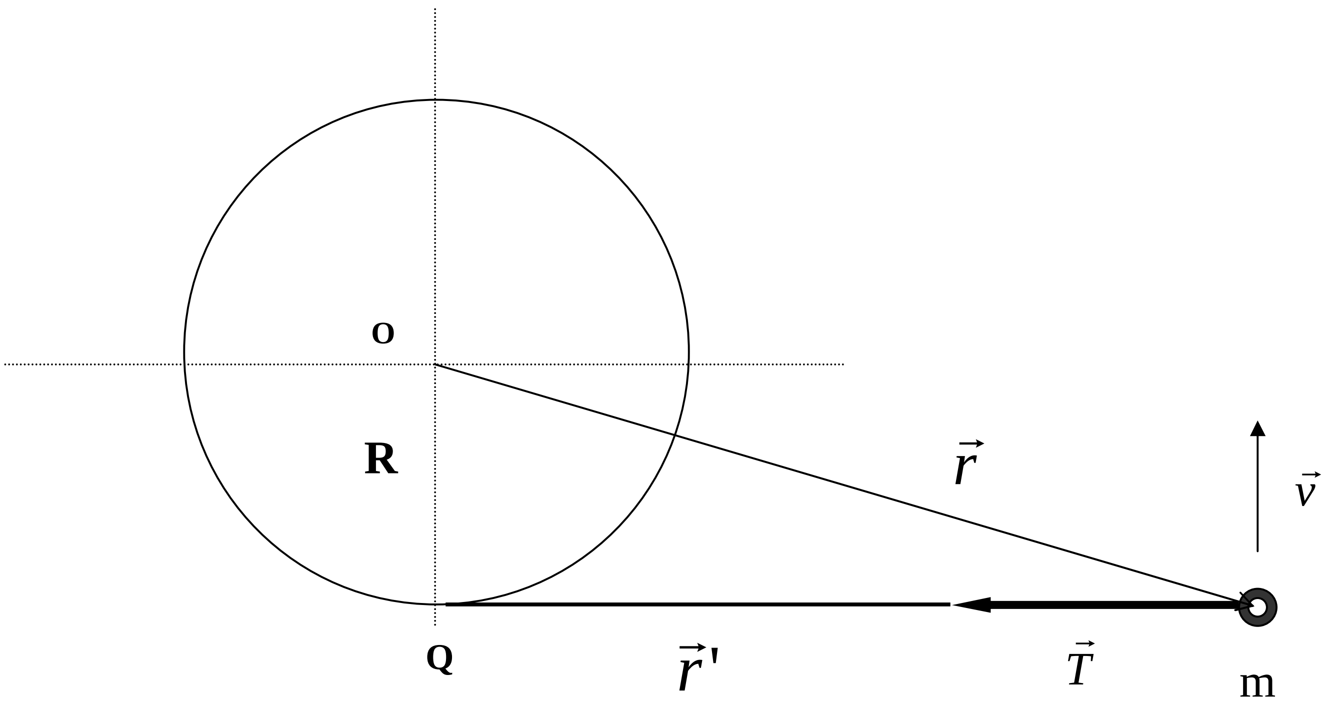
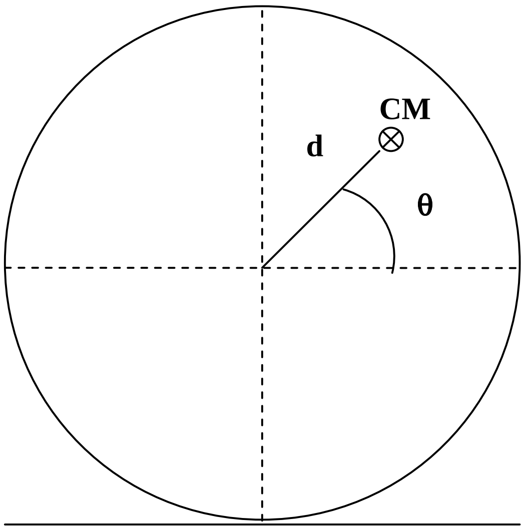


Figure 2: An inhomogeneous cylinder rolling without sliding on a horizontal plane.



[1] M. A. Illarramendi and T. del Rio Gaztellurrutia, Europ. J. Phys. **16**, 249 (1995).

[2] M. Alonso and E. J. Finn, *Fundamental University Physics. Vol. 1: Mechanics and Thermodynamics*, **2nd** edition (Addison-Wesley Publishing Company, Reading, Massachusetts, 1980), Vol. **1**, Chap. 9, p. 234,Chap. 10, p.275.

[3] D. Halliday and R. Resnick, *Physics for Students of Science and Engineering*, **1st** edition (J. Wiley & Sons, New York, 1962), Vol. **1**, Chap. 12, p.231.

[4] U. Ingard and W. L. Kraushaar, *Introduction to Mechanics, Matter and Waves*, **1st** edition (Addison-Wesley Publishing Company, Reading, Massachusetts, 1961), Vol. **1**, Chap. 13.

[5] C. Kittel, W. D. Knight, and M. A. Ruderman, *Mecánica - Berkeley Physics Course*, **2nd** edition (Ed. Reverté, Barcelona - Spain, 1996), Vol. **1**, Chap. 6, p.193.

[6] P. A. Tipler and G. Mosca, *Physics for Scientist and Engineers*, **5**th edition (W. A. Freeman and Co., New York, 2003), Vol. **1**, Chap. 9, p.289.