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The quadratically damped oscillator: A case study of a non-linear equation of motion

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The equation of motion for a quadratically damped oscillator, where the damping is proportional to the square of the velocity, is a non-linear second-order differential equation. Non-linear equations of motion such as this are seldom addressed in intermediate instruction in classical dynamics; this one is problematic because it cannot be solved in terms of elementary functions. Like all second-order ordinary differential equations, it has a corresponding first-order partial differential equation, whose independent solutions constitute the constants of the motion. These constants readily provide an approximate solution correct to first order in the damping constant. They also reveal that the quadratically damped oscillator is never critically damped or overdamped, and that to first order in the damping constant the oscillation frequency is identical to the natural frequency. The technique described has close ties to standard tools such as integral curves in phase space and phase portraits.
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I. INTRODUCTION

Newton's second law reduces the science of mechanics to the solution of second-order ordinary differential equations. Constants of the motion, where they can be found, provide important insight into the solution of the equations of motion of a physical system. They also elevate the science of mechanics above a purely mathematical exercise by facilitating (if not actually constituting) our physical understanding of the system. Kepler's law of areas was perhaps the first constant of the motion to be discovered, later to be recognized as an example of the conservation of angular momentum. Huygens implicitly used conservation of linear momentum in his debate with Leibniz over the true measure of the "quantity of motion." The work-energy theorem applied to conservative forces evolved into the conservation of energy, a law that reaches far beyond its purely mechanical origins. Techniques for finding constants of the motion for a physical system are valuable tools in theoretical physics: think of the importance of cyclic or ignorable coordinates in Lagrangian and Hamiltonian fomulations, of Noether's theorem linking conserved quantities to symmetry and invariance properties, and of the link between conserved quantities and eigenvalues in quantum mechanics.

These seminal constants and conservation principles are core to the discipline of theoretical physics. A student just acquiring an appreciation for their importance and utility may unconsciously conclude that they are rare and exceptional. In actuality, any system of ordinary differential equations of any order can be expanded into a larger system of first-order differential equations by identifying each derivative as a new variable; for example,

$$\frac{dx^i}{dt} = y^i, \tag{1a}$$

$$\frac{d^2x^i}{dt^2} = \frac{dy^i}{dt}. \tag{1b}$$

The expanded system has an associated first-order partial differential equation[1]

$$0 = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q^i}\frac{dq^i}{dt}, \quad 1 \le i \le n, \tag{2}$$

where the $q^i(t)$ include the original and all of the newly identified variables, and n is the total number of first-order ordinary differential equations in the expanded system. There are an infinite number of solutions, but only n independent solutions. Once n independent solutions $S_i(q^j;t)$ have been found, they may be inverted to obtain $q^i(S_j;t)$. Among the $q^i(S_j;t)$ are the sought-after solutions to the original differential equations, complete with integration constants.

Newton's second law for a single particle provides the equations of motion

$$m\ddot{x}^i = f^i(x^j, \dot{x}^j; t), \quad i,j \le 3. \tag{3}$$

The expanded system of first-order ordinary differential equations is

$$v^i = \dot{x}^i, \tag{4a}$$

$$m\dot{v}^i = ma^i = f^i(x^j, v^j; t). \tag{4b}$$

The velocities are now considered independent variables on an equal par with the coordinates. Equation (2) becomes[2]

$$0 = \frac{\partial S}{\partial t} + \sum_{i=1}^{3} \frac{\partial S}{\partial x^i}v^i + \sum_{i=1}^{3} \frac{\partial S}{\partial v^i}a^i. \tag{5}$$

Solutions of this first-order partial differential equation are constants of the motion.

Reference 2 gives four examples, all important classroom problems, where the solution of Eq. (5) can be obtained by elementary means. In many other cases, finding a complete set of independent solutions to Eq. (5) is harder than solving the equations of motion themselves. However, even one solution provides significant insight into the problem. Consider, for example, that if all the forces are conservative, so that

$$a^i = -\frac{1}{m}\frac{\partial \phi}{\partial x^i} \tag{6}$$

for some $\phi(x^i)$, then Eq. (5) yields

$$0 = \frac{\partial S}{\partial t} + \sum_{i=1}^{3} \frac{\partial S}{\partial x^i} v^i - \frac{1}{m} \sum_{i=1}^{3} \frac{\partial S}{\partial v^i} \frac{\partial \phi}{\partial x^i}, \tag{7}$$

which has the immediate time-independent solution

$$S = \frac{1}{2} m \sum_{i=1}^{3} v^i v^i + \phi. \tag{8}$$

This is the well-known result that the motion proceeds in such a way that the total mechanical energy is constant. In this example, S is a time-independent scalar.

The nature of S varies. It may be a scalar, as in the example of constant mechanical energy, or it may be a vector as in the case of total angular momentum L^i for central force motion. If a scalar, it need not be energy or even something resembling energy. Indeed, it may be something for which a meaningful physical interpretation is hard to find. For the strictly utilitarian purposes of solving the equations of motion, the simplest possible expressions for S, of whatever type, are preferred.

One-dimensional systems possess only two independent constants of the motion. Restricting the discussion to forces that are not explicitly a function of time, the equation of motion

$$m\dot{v} = f(x, v) \tag{9}$$

can be multiplied by the identity

$$v = \dot{x} \tag{10}$$

to yield the Pfaffian form

$$mv\, dv = f(x, v)\, dx. \tag{11}$$

All two-variable Pfaffian forms can be transformed into total integrals by the use of an integrating factor[3] μ such that

$$0 = dS = m\mu v\, dv - \mu f(x, v)\, dx. \tag{12}$$

The integrating factor can be found from the integrability condition

$$m \frac{\partial(\mu v)}{\partial x} = -\frac{\partial(\mu f)}{\partial v}. \tag{13}$$

Once the integrating factor is known, Eq. (12) can be integrated to yield the first constant of the motion,

$$S(x, v) = S(x_0, v_0). \tag{14}$$

The second independent constant of the motion must be explicitly a function of time

$$T(x, v; t) = T(x_0, v_0; t_0), \tag{15}$$

in order for the equation of motion to be found. Equations (14) and (15) constitute two equations in the three independent variables x, v, and t. One equation may be used to eliminate v in the other, yielding the equation of motion

$$x = x\Big(S(x_0, v_0), T(x_0, v_0; t_0); t\Big). \tag{16}$$

Finding T is, in principle, straightforward for one-dimensional systems. Equation (5) is separable in t with a solution of the form

$$T(x, v; t) = \omega t + \hat{T}(x, v), \tag{17}$$

where

$$\frac{\partial T}{\partial t} = \omega, \tag{18a}$$

$$v \frac{\partial \hat{T}}{\partial x} + \frac{f}{m} \frac{\partial \hat{T}}{\partial v} = -\omega. \tag{18b}$$

Equation (10) can be solved to yield

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{v}. \tag{19}$$

While the constants of the motion treat x and v as independent variables, for one-dimensional motion they are linked through Eq. (14). With v considered a function of x, Eq. (19) can be integrated by parts to yield

$$t - t_0 = \frac{x}{v(x)}\bigg|_{x_0}^{x} + \int_{x_0}^{x} \frac{x}{v^2(x)} \frac{dv}{dx}\, dx, \tag{20a}$$

$$= \frac{x}{v(x)}\bigg|_{x_0}^{x} + \int_{x_0}^{x} \frac{x}{v^2(x)} \frac{dv/dt}{dx/dt}\, dx, \tag{20b}$$

$$= \frac{x}{v(x)}\bigg|_{x_0}^{x} + \int_{x_0}^{x} \frac{x}{v^2(x)} \frac{a}{v}\, dx, \tag{20c}$$

$$= \frac{x}{v(x)}\bigg|_{x_0}^{x} + \int_{x_0}^{x} \frac{xf(x, v)}{mv^3(x)}\, dx. \tag{20d}$$

Repeatedly integrating by parts yields an infinite series on the right-hand side,

$$t - t_0 = \sum_{n=0}^{\infty} F_n(x, v) \left(\frac{1}{v}\right)^n \bigg|_{x_0}^{x}, \tag{21}$$

which, if questions of convergence can be adequately dealt with, represents a function $\hat{T}(x, v) - \hat{T}(x_0, v_0)$ that brings Eq. (19) into the form

$$T(x, v; t) = \omega t - \hat{T}(x, v) = \omega t_0 - \hat{T}(x_0, v_0). \tag{22}$$

Alternatively, x can be regarded as a function of v, yielding a second expression for T that can be found by writing Eq. (19) as

$$t - t_0 = \int_{v_0}^{v} \frac{dx}{dv} \frac{dv}{v} = \int_{v_0}^{v} \frac{m\, dv}{f(x, v)}. \tag{23}$$

Repeated integration by parts again yields an infinite series,

$$t - t_0 = \frac{mv}{f(x, v)}\bigg|_{v_0}^{v} + \int_{v_0}^{v} \frac{mv}{f^2(x, v)} \frac{df}{dv}\, dv, \tag{24a}$$

$$= \frac{mv}{f(x, v)}\bigg|_{v_0}^{v} + \int_{v_0}^{v} \frac{mv}{f^2(x, v)} \left(\frac{\partial f}{\partial x}\frac{dx}{dv} + \frac{\partial f}{\partial v}\right) dv, \tag{24b}$$

$$= \frac{mv}{f(x,v)}\Big|_{v_0}^{v} + \frac{mv^2}{2}\frac{1}{f^2(x,v)}\left(\frac{\partial f}{\partial x}\frac{mv}{f} + \frac{\partial f}{\partial v}\right)\Big|_{v_0}^{v} \quad (24c)$$

$$- \int_{v_0}^{v} \frac{mv^2}{2}\frac{d}{dv}\left[\frac{1}{f^2(x,v)}\left(\frac{\partial f}{\partial x}\frac{mv}{f} + \frac{\partial f}{\partial v}\right)\right] dv. \quad (24d)$$

If this infinite series convergences properly, then a second representation for $T(x,v;t)$ exists.

II. THE HARMONIC OSCILLATOR

The simple harmonic oscillator will be used to illustrate the process. As the restoring force for the simple harmonic oscillator,

$$m\dot{v} = -m\omega^2 x, \quad (25)$$

is conservative, Eq. (12) immediately yields the first constant,

$$S = \frac{1}{2}m(v^2 + \omega^2 x^2) = \frac{1}{2}m(v_0^2 + \omega^2 x_0^2). \quad (26)$$

Equation (19) becomes

$$t - t_0 = \frac{x}{v}\Big|_{x_0}^{x} + \int_{x_0}^{x} \frac{x}{v^2(x)}\frac{-\omega^2 x}{v}\, dx, \quad (27a)$$

$$= \frac{x}{v}\Big|_{x_0}^{x} - \frac{1}{3}\frac{\omega^2 x^3}{v^3}\Big|_{x_0}^{x} + \int_{x_0}^{x} \frac{\omega^4 x^4}{v^5} dx, \quad (27b)$$

$$\omega(t - t_0) = \frac{\omega x}{v}\Big|_{x_0}^{x} - \frac{1}{3}\frac{\omega^3 x^3}{v^3}\Big|_{x_0}^{x} + \frac{1}{5}\frac{\omega^5 x^5}{v^5}\Big|_{x_0}^{x} - \dots. \quad (27c)$$

The sequence converges for $(\omega x/v)^2 < 1$ to yield

$$T(x,v;t) = \omega t - \tan^{-1}\left(\frac{\omega x}{v}\right) = \omega t_0 - \tan^{-1}\left(\frac{\omega x_0}{v_0}\right). \quad (28)$$

This expression for $T(x,v;t)$ can be used to immediately calculate the period τ, and therefore the frequency, of the oscillator. At the two turning points of the motion, $v = 0$, and the interval between them is

$$\omega\left(\frac{\tau}{2}\right) = \omega(t_+ - t_-) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi, \quad (29)$$

so that the period of the motion is $2\pi/\omega$, as expected for the simple harmonic oscillator. In exactly the same way, Eq. (24a) yields

$$T(x,v;t) = \omega t + \tan^{-1}\left(\frac{v}{\omega x}\right) = \omega t_0 + \tan^{-1}\left(\frac{v_0}{\omega x_0}\right). \quad (30)$$

It bears repeating that the constants of the motion are not unique; the two expressions here for $T(x,v;t)$ are related through the trigonometric identity

$$\tan^{-1} z = -\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{z}\right). \quad (31)$$

It is left as an exercise for the reader to solve Eq. (26) for $v(x)$ and substitute this into either form of $T(x,v;t)$ to obtain the standard equation of motion for the simple harmonic oscillator.

A similar analysis can be done for the linearly (viscously) damped harmonic oscillator. The time-independent constant[4,5] is

$$S = \frac{1}{2}m[(v + \gamma x)^2 + \omega^2 x^2]\exp\left[\frac{2\gamma}{\omega}\tan^{-1}\left(\frac{\omega x}{v + \gamma x}\right)\right]. \quad (32)$$

The time-dependent constant is

$$T(x,v;t) = \omega t - \tan^{-1}\left(\frac{\omega x}{v + \gamma x}\right), \quad (33)$$

where

$$\omega = \sqrt{\omega_0^2 + \gamma^2} \quad (34)$$

is the damped frequency and ω_0 is the undamped (natural) frequency of the oscillator.

III. THE QUADRATICALLY DAMPED HARMONIC OSCILLATOR

Both the simple and the viscously damped harmonic oscillators can be adequately solved with conventional approaches. The value to the student of the approach offered here for those simple problems is more pedagogical than utilitarian. The balance begins to shift toward the utilitarian for the quadratically damped harmonic oscillator. For damping proportional to v^2, Newton's second law takes the form

$$m\ddot{x} = \begin{cases} -m\omega^2 x - m\gamma\dot{x}^2, & \dot{x} > 0, \\ -m\omega^2 x + m\gamma\dot{x}^2, & \dot{x} < 0, \end{cases} \quad (35)$$

and the student is faced with two different non-linear differential equations. The damping coefficient must change sign whenever $v = 0$ since the damping force always opposes the velocity; for oscillatory motion, this happens every half-cycle. As a consequence, each equation must be solved separately and the solutions mated at the turning points to yield a continuous solution valid for all t. Assuming that x has its maximum displacement x_0 at $t = 0$, the first turning point $x(T_1)$ must be calculated from the $v < 0$ solution to Eq. (35) and used as a boundary condition for the $v > 0$ solution. Similarly, $x(T_2)$ must be calculated from the $v > 0$ solution and used as a boundary condition for the next $v < 0$ solution, and so on. The two differential equations are related through a parity transformation: $\ddot{x}_{v<0} = -\ddot{x}_{v>0}$. Once the solution is known for $v < 0$, the solution for $v > 0$ is obtained by simply changing sign.

The brute-force approach to Eq. (35) is a power series expansion $x(t) = \sum_{n=0}^{\infty} a_n t^n$. The expressions for a_n get very complex very fast. While formally correct, this approach fails to provide any significant physical insight into the behavior of the quadratically damped oscillator. It does not easily provide an expression for the damped frequency of the oscillator in terms of the natural frequency, for example, nor does it provide any convenient method of calculating the displacements at the turning points.

Instead, we approach the solution through the constants of the motion. First, write Eq. (35) as two-variable Pfaffian forms

$$0 = dA = \begin{cases} v\,dv + (\omega^2 x - \gamma v^2)\,dx, & v < 0, \\ v\,dv + (\omega^2 x + \gamma v^2)\,dx, & v > 0. \end{cases} \tag{36}$$

These are not quadratures, as they do not obey the integrability condition. The required integrating factors are found by solving

$$\frac{\partial \mu}{\partial x} v = \frac{\partial \mu}{\partial v}(\omega^2 x \mp \gamma v^2) \mp 2\gamma v \mu. \tag{37}$$

The most general solutions are not necessary; the simplest solutions[5–7] $\mu = e^{\mp 2\gamma x}$ suffice. Equation (36) becomes

$$\mu\,dA = dS^{\mp} = (e^{\mp 2\gamma x} v\,dv \mp v^2 \gamma e^{\mp 2\gamma x}\,dx) + e^{\mp 2\gamma x}\omega^2 x\,dx, \tag{38a}$$

$$S^{\mp} = e^{\mp 2\gamma x}\left[\frac{v^2}{2} \mp \frac{\omega^2}{2\gamma}\left(x \pm \frac{1}{2\gamma}\right)\right]. \tag{38b}$$

Note that S^- and S^+ are related by a parity transformation: $S^-(-x,-v) = S^+(x,v)$.

The quantities $S^{\pm}$, the first constants of the motion for this problem, are each graphically illustrated over one-half cycle in Fig. 1. The phase space axes are expressed in terms of the unitless parameters

$$w = \frac{\gamma v}{\omega}, \tag{39a}$$

$$u = \gamma x. \tag{39b}$$

The quantities $S^{\mp}$ normalize to

$$\hat{S}^{\mp} = \frac{\gamma^2}{\omega^2} S^{\mp} = e^{\mp 2u}\left(\frac{w^2}{2} \mp \frac{u}{2} - \frac{1}{4}\right). \tag{40}$$

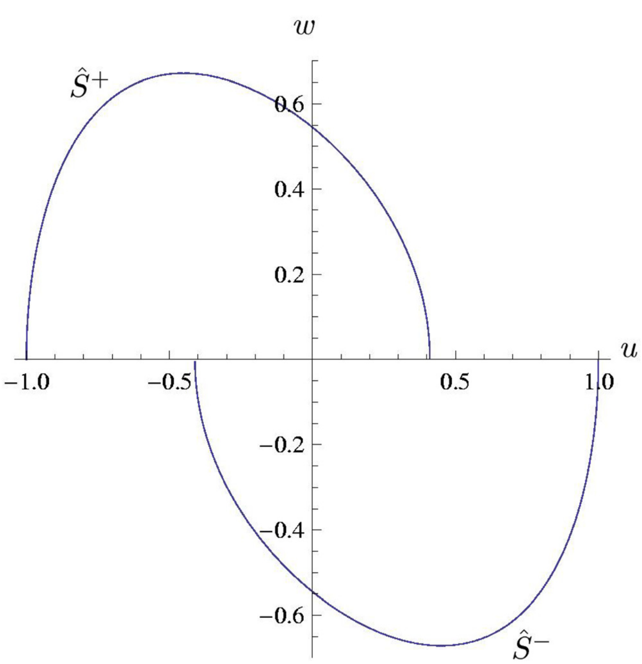

Fig. 1. The functions $\hat{S}^-$ and $\hat{S}^+$ when $\hat{S}^- = \hat{S}^+$. The axes have been normalized to $u = \gamma x$ and $w = \gamma v/\omega$.

The plot of constant $\hat{S}$ lays out the trajectory of the damped oscillator in phase space. Since $\hat{S}^-$ is a constant, its value can be conveniently found by setting $v = 0$ in Eq. (40). The condition $\hat{S}^- < 0$ holds for all positive values of u_0 and $\hat{S}^- \to 0$ from below as $u_0 \to \infty$, limiting $\hat{S}^-$ to the range

$$0 > \hat{S}^- \geq -\frac{1}{4}. \tag{41}$$

Starting from maximum displacement u_0 where $w = 0$, the trajectory will follow a path set by

$$\hat{S}^- = -e^{-2u_0}\left(\frac{u_0}{2} + \frac{1}{4}\right). \tag{42}$$

The turning point at the end of this half-cycle is found by iteratively solving

$$e^{-2u_0}(2u_0 + 1) = e^{-2u_1}(2u_1 + 1). \tag{43}$$

A turning point implies that the velocity is zero, so that a turning point at the origin $u_1 = 0$ means that the oscillator is critically damped or overdamped. Equation (43) then becomes

$$e^{-2u_0}(2u_0 + 1) = 1, \tag{44}$$

which has no solution for $u_0 > 0$; therefore, the quadratically damped oscillator is never critically damped or overdamped.

The velocity at the first passage through $u = 0$ is

$$w = -\sqrt{\frac{1 - e^{-2u_0}(2u_0 + 1)}{2}}. \tag{45}$$

The point of maximum velocity x_f, i.e., the point where the acceleration is zero, is found from Eq. (38b)

$$2\gamma S^- e^{2\gamma x} = \gamma v^2 - \omega^2 x - \frac{\omega^2}{2\gamma}, \tag{46a}$$

$$= \ddot{x} - \frac{\omega^2}{2\gamma}, \tag{46b}$$

$$u_f = u_0 - \frac{\ln(2u_0 + 1)}{2}. \tag{46c}$$

At the first turning point, the expression for $\hat{S}$ changes from $\hat{S}^-$ to $\hat{S}^+$. It is clear from Fig. 1 that if $\hat{S}^- = \hat{S}^+$, $\hat{S}$ will not be continuous across the u axis. This happens because the solutions given in Eq. (38b) are only determined up to an integration constant $\hat{S}_n$. For the two expressions to match smoothly and form a single function,

$$\hat{S}^- = -e^{-2u_0}\left(\frac{u_0}{2} + \frac{1}{4}\right) = \hat{S}^+ + \hat{S}_1, \tag{47}$$

where the value of the constant $\hat{S}^+$ must be

$$\hat{S}^+ = -e^{2u_1}\left(\frac{u_1}{2} - \frac{1}{4}\right). \tag{48}$$

Therefore, $\hat{S}_1$ is

$$\hat{S}_1 = -e^{2u_1}\left(\frac{u_1}{2} - \frac{1}{4}\right) - e^{-2u_0}\left(\frac{u_0}{2} - \frac{1}{4}\right). \tag{49}$$

The result is shown in Fig. 2. Measuring the angle θ clockwise from the positive u axis, the function $\hat{S}$ for the first full cycle is

$$\hat{S} = -e^{-2u_0}\left(\frac{u_0}{2} + \frac{1}{4}\right) = \begin{cases} e^{-2u}\left(\frac{w^2}{2} - \frac{u}{2} - \frac{1}{4}\right), & 0 \le \theta < \pi, \\ e^{2u}\left(\frac{w^2}{2} + \frac{u}{2} - \frac{1}{4}\right) + \hat{S}_1, & \pi \le \theta < 2\pi. \end{cases} \tag{50}$$

The next turning point is found by interatively solving

$$\hat{S} - \hat{S}_1 = e^{2u_2}[2u_2 - 1]. \tag{51}$$

The new turning point u_2 is fed back into Eq. (42) to determine $\hat{S}^-$ for the second cycle, and the process is repeated. This process generates a sequence of integration constants labeled $\hat{S}_n$, where $\hat{S}_0 = 0$, and n numbers the half-cycles.

This kind of functional patch needs to be made at every turning point. The general equations are

$$\hat{S} - \hat{S}_{2n} = e^{-2u_{2n+1}}(2u_{2n+1} + 1) = e^{-2n_{2n}}(2n_{2n} + 1), \tag{52a}$$

$$\hat{S} - \hat{S}_{2n+1} = e^{2u_{2n+2}}(2u_{2n+2} - 1) = e^{2n_{2n+1}}(2n_{2n+1} - 1), \tag{52b}$$

$$\hat{S} = -e^{-2u_0}\left(\frac{u_0}{2} + \frac{1}{4}\right) = \begin{cases} e^{-2u}\left(\frac{w^2}{2} - \frac{u}{2} - \frac{1}{4}\right) + \hat{S}_{2n}, & 2n\pi \le \theta < (2n+1)\pi, \\ e^{2u}\left(\frac{w^2}{2} + \frac{u}{2} - \frac{1}{4}\right) + \hat{S}_{2n+1}, & (2n+1)\pi \le \theta < (2n+2)\pi. \end{cases} \tag{52c}$$

The results for three full cycles are displayed in Table I. Figure 3 shows three full cycles of the quadratically damped oscillator in phase space. See Ref. 7 for a parallel, more mathematically sophisticated, development of the material in this section.

IV. THE SOLUTION OF THE EQUATION OF MOTION

Equation (52c) leads directly to the equation of motion for a given half-cycle, once the value of the corresponding constant has been determined. Returning to the configuration variables v and x, using

$$E_n = \frac{\omega^2}{\gamma^2}(\hat{S} - \hat{S}_n), \tag{53}$$

then solving for $v(x)$ and separating the variables, yields the quadrature

$$\int dt = \int \frac{dx}{\sqrt{2e^{2\gamma x}E_n + \frac{\omega^2}{2\gamma}\left(x + \frac{1}{2\gamma}\right)}}. \tag{54}$$

The duration of the half-cycle is the transit time from one turning point to the next,

$$\frac{\tau}{2} = \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2e^{2\gamma x}E_n + \frac{\omega^2}{2\gamma}\left(x + \frac{1}{2\gamma}\right)}}. \tag{55}$$

For a single half-cycle, $x(t)$ is one-half of the periodic function $y(t)$ obtained by performing the integration of Eq. (54). The phase space picture of $y(t)$ is shown in Fig. 4.

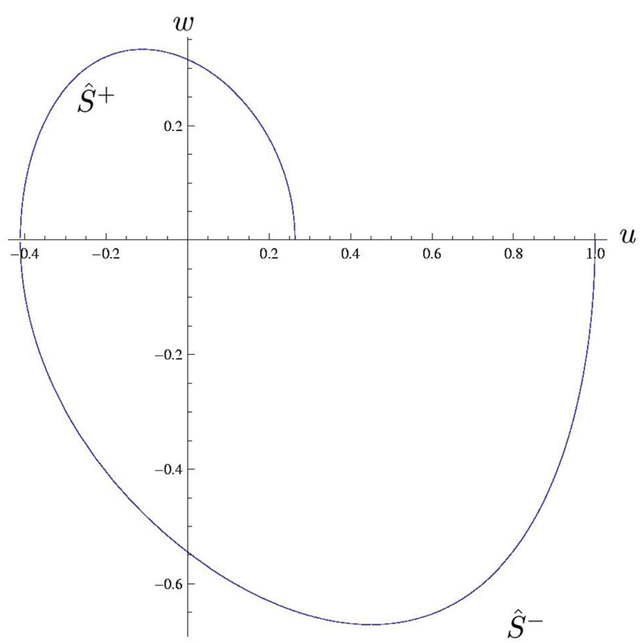

Fig. 2. One full cycle of the quadratically damped oscillator.

Table I. $\hat{S} = -0.101501$.

nth Half-cycle	$\hat{S}_{n-1}$	u_{n-1}
First	0	1
Second	0.098765	−0.41072
Third	0.123863	0.263568
Fourth	0.133834	−0.194566
Fifth	0.138780	0.154318
Sixth	0.141589	−0.127908

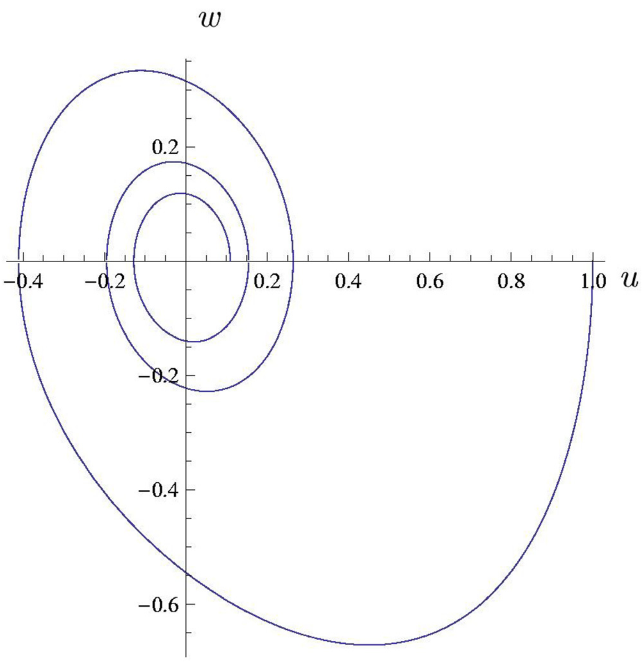

Fig. 3. Three full cycles of the quadratically damped oscillator.

The equation of the trajectory $x(t)$ is not easily found; Eq. (54) is not found in standard tables of integrals. Using Eq. (24a),

$$\omega(t-t_0)=\int_{v_0}^{v}\frac{\omega\,dv}{\gamma v^2-\omega^2 x}, \tag{56a}$$

$$=\frac{\omega v}{\gamma v^2-\omega^2 x}\bigg|_{v_0}^{v}+\int_{v_0}^{v}\frac{\omega v[2\gamma v\,dv-\omega^2\,dx]}{(\gamma v^2-\omega^2 x)^2}, \tag{56b}$$

$$=\frac{\omega v}{\gamma v^2-\omega^2 x}\bigg|_{v_0}^{v}+\int_{v_0}^{v}\frac{2\gamma\omega v^2\,dv}{(\gamma v^2-\omega^2 x)^2} -\int_{v_0}^{v}\frac{\omega^3 v^2\,dv}{(\gamma v^2-\omega^2 x)^3}, \tag{56c}$$

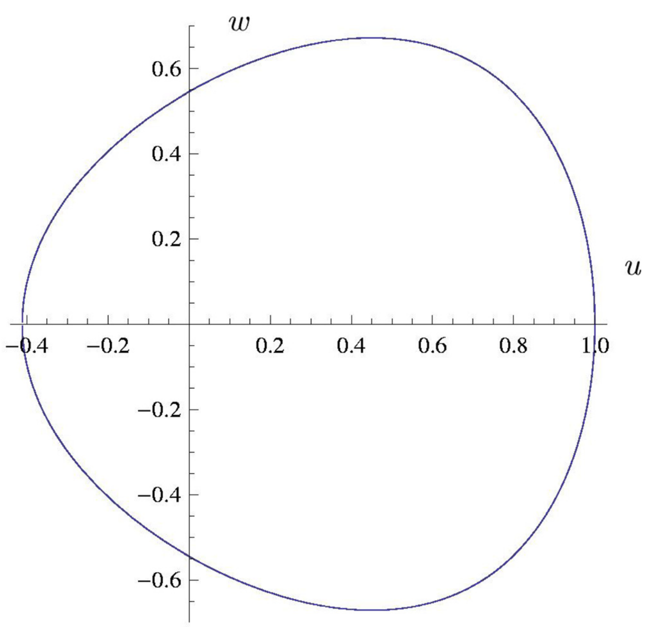

Fig. 4. The phase space plot of $y(t)$.

$$=\frac{\omega v}{\gamma v^2-\omega^2 x}\bigg|_{v_0}^{v}+\frac{2}{3}\left(\frac{\gamma v}{\omega}\right)\frac{\omega^2 v^2}{(\gamma v^2-\omega^2 x)^2}\bigg|_{v_0}^{v} -\frac{1}{3}\frac{\omega^3 v^3}{(\gamma v^2-\omega^2 x)^3}\bigg|_{v_0}^{v} +\int_{v_0}^{v}\frac{4}{3}\left(\frac{\gamma v}{\omega}\right)^3\left(\frac{\omega^4}{\gamma^2}\right)\frac{2\gamma v\,dv-\omega^2\,dx}{(\gamma v^2-\omega^2 x)^3} -\int_{v_0}^{v}\frac{\omega^3 v^3(2\gamma v\,dv-\omega^2\,dx)}{(\gamma v^2-\omega^2 x)^4}. \tag{56d}$$

This suggests that $T(x,v;t)$ may be written as

$$T(x,v;t)=\omega t-\sum_{n=1}^{\infty}f_n\left(\frac{\gamma v}{\omega}\right)\left(\frac{\omega v}{\gamma v^2-\omega^2 x}\right)^n. \tag{57}$$

Substituting this into Eq. (5) yields

$$0=\omega-\omega\sum_{n=2}^{\infty}(n-1)f_{n-1}\left(\frac{\omega v}{\gamma v^2-\omega^2 x}\right)^n -\omega\sum_{n=0}^{\infty}\left(\frac{\gamma v}{\omega}\right)f'_{n+1}\left(\frac{\omega v}{\gamma v^2-\omega^2 x}\right)^n -\omega\sum_{n=0}^{\infty}(n+1)f_{n+1}\left(\frac{\omega v}{\gamma v^2-\omega^2 x}\right)^n +2\omega\left(\frac{\gamma v}{\omega}\right)\sum_{n=1}^{\infty}nf_n\left(\frac{\omega v}{\gamma v^2-\omega^2 x}\right)^n. \tag{58}$$

Setting the coefficients of each order equal to zero yields this set of equations

$$w=\frac{\gamma v}{\omega}, \tag{59a}$$

$$0=1-f_1-wf'_1, \tag{59b}$$

$$0=wf'_2+2f_2-2wf_1, \tag{59c}$$

$$0=(n-2)f_{n-2}+wf'_n+nf_n-2w(n-1)f_{n-1}, \quad 0=(n-2)w^{n-1}f_{n-2}+(w^n f_n)'-2(n-1)w^n f_{n-1}. \tag{59d}$$

These equations generate the following solutions:

$$f_1=1, \tag{60a}$$

$$f_2=\frac{2}{3}\left(\frac{\gamma v}{\omega}\right), \tag{60b}$$

$$f_3=\frac{8}{15}\left(\frac{\gamma v}{\omega}\right)^2-\frac{1}{3}, \tag{60c}$$

$$f_4=\frac{16}{35}\left(\frac{\gamma v}{\omega}\right)^3-\frac{2}{3}\left(\frac{\gamma v}{\omega}\right), \tag{60d}$$

$$f_5=\frac{128}{315}\left(\frac{\gamma v}{\omega}\right)^4-\frac{104}{105}\left(\frac{\gamma v}{\omega}\right)^2+\frac{1}{5}, \tag{60e}$$

$$f_6=\frac{256}{693}\left(\frac{\gamma v}{\omega}\right)^5-\frac{176}{135}\left(\frac{\gamma v}{\omega}\right)^3+\frac{2}{3}\left(\frac{\gamma v}{\omega}\right), \tag{60f}$$

$$f_7 = \frac{1024}{3003}\left(\frac{\gamma v}{\omega}\right)^6 - \frac{1856}{1155}\left(\frac{\gamma v}{\omega}\right)^4 + \frac{272}{189}\left(\frac{\gamma v}{\omega}\right)^2 - \frac{1}{7}, \tag{60g}$$

$$f_n = \sum_{i=0}^{n-1} C_i^n \left(\frac{\gamma v}{\omega}\right)^i. \tag{60h}$$

Equation (57) can be now be re-written as

$$T(x, v; t) = \omega t - \sum_{i=0}^{\infty}\left(\frac{\gamma v}{\omega}\right)^i \sum_{n=i+1}^{\infty} C_i^n \left(\frac{\omega v}{\gamma v^2 - \omega^2 x}\right)^n. \tag{61}$$

From Eq. (59d),

$$C_i^n = \frac{2(n-1)}{n+i} C_{i-1}^{n-1} - \frac{n-2}{n+i} C_i^{n-2}, \tag{62a}$$

$$= \sum_{k=1}^{\frac{n-i+1}{2}} \frac{2(-1)^k (n+i-2k)!!(n-2)!!(n+1-2k)}{(n+i)!!(n-2k)!!} C_{i-1}^{n+1-2k}, \tag{62b}$$

where n and i have opposite parity. It follows that for $i = 0$,

$$C_0^{2m+1} = \begin{cases} -\dfrac{n-2}{n} C_0^{n-2} = \dfrac{n-4}{n} C_0^{n-4} = \dfrac{(-1)^{(n-1)/2}}{n}, & n = \text{odd}, \\ 0, & n = \text{even}. \end{cases} \tag{63}$$

The corresponding summation in Eq. (61) is

$$g_0 = \tan^{-1}\left(\frac{\omega v}{\gamma v^2 - \omega^2 x}\right). \tag{64}$$

For $i = 1$,

$$C_1^n = \begin{cases} \dfrac{2(n-1)}{n+1} C_0^{n-1} - \dfrac{n-2}{n+1} C_1^{n-2} = 2\dfrac{(-1)^{(n-2)/2}}{n+1} - \dfrac{n-2}{n+1} C_1^{n-2}, & n = \text{even}, \\ 0, & n = \text{odd}. \end{cases} \tag{65}$$

It is easy to see that if

$$C_1^{n-2} = (-1)^{(n-4)/2} \cdot \frac{2}{3}, \tag{66}$$

for n even, then[8]

$$C_1^n = (-1)^{(n-2)/2} \cdot \frac{2}{3}. \tag{67}$$

Since it is true for $n = 2$, then by mathematical induction it is true for all even n. The corresponding summation in Eq. (61) is

$$g_1 = \frac{2}{3} \frac{\gamma \omega v^3}{\left[\omega^2 v^2 + (\gamma v^2 - \omega^2 x)^2\right]}. \tag{68}$$

From $i = 2$ on, the summations do not reduce to readily identifiable functions.

In the limit $\gamma \to 0$, the equations should reduce to those for the simple harmonic oscillator. To first order in $\gamma v/\omega$, Eq. (61) becomes

$$0 = \omega t - \tan^{-1}\left(\frac{\omega v}{\gamma v^2 - \omega^2 x}\right) - \frac{2}{3} \frac{\omega \gamma v^3}{\left[\omega^2 v^2 + (\gamma v^2 - \omega^2 x)^2\right]}, \tag{69}$$

which agrees with the harmonic oscillator result when the damping vanishes. The transit time from one turning point to the other is $\omega(t_+ - t_-)$. The last term on the right in Eq. (69) vanishes since $v = 0$ at the turning points; the argument of the $\tan^{-1}$ also vanishes at each turning point, but since the argument changes sign and diverges at the intermediate point where the denominator is zero, π must be added to keep ωt continuous. The transit time is also the half-period so that

$$\tau = \frac{2\pi}{\omega}. \tag{70}$$

To first order in γ, the frequency of the quadratically damped oscillator is identical to the natural frequency.

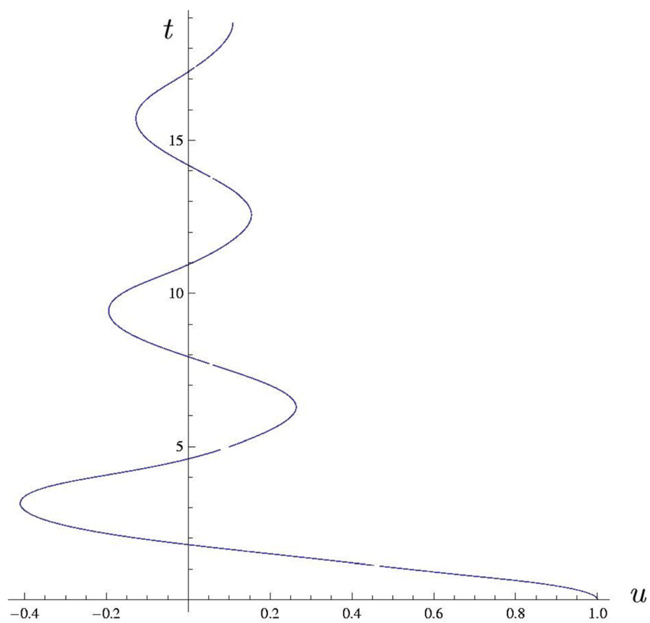

Fig. 5. Three full cycles of the quadratically damped oscillator to first-order in $\gamma v/\omega$.

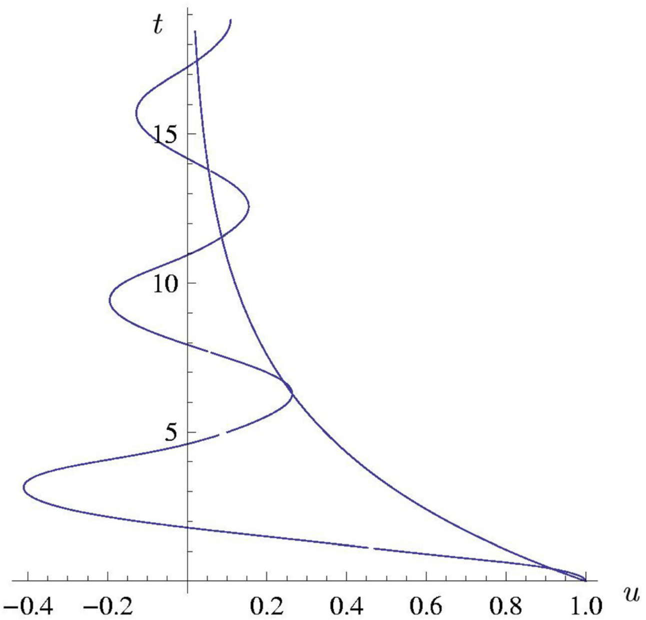

Fig. 6. The exponential envelope for Fig. 5 has been set to intersect $u(t)$ at the beginning and at the end of the first period.

A graph of $x(t)$ for any given half-cycle is constructed by inserting $v_n(x)$ from Eq. (54) into Eq. (69) and calculating $t_n(x)$, then plotting the results as $x_n(t)$. At the end of each half-cycle, the calculation starts anew with $v_{n+1}(x)$. The graphs are mated at the end of the half-cycle:

$$x_n(0) = x_{n-1}(\pi/\omega) = x([n-1]\pi/\omega). \tag{71}$$

The result for three full cycles is shown in Fig. 5. The decaying oscillation in Fig. 5 may be compared with the exponential amplitude decay seen in the linearly damped oscillator. The differences can be brought out by plotting a similar exponential fit as shown in Fig. 6. The exponential decay has been arbitrarily forced to fit the turning points at the beginning and end of the first cycle. Clearly, the decay of the quadratically damped oscillator is much slower than exponential. The equivalent comparison in phase space, as shown in Fig. 7, is equally illuminating.

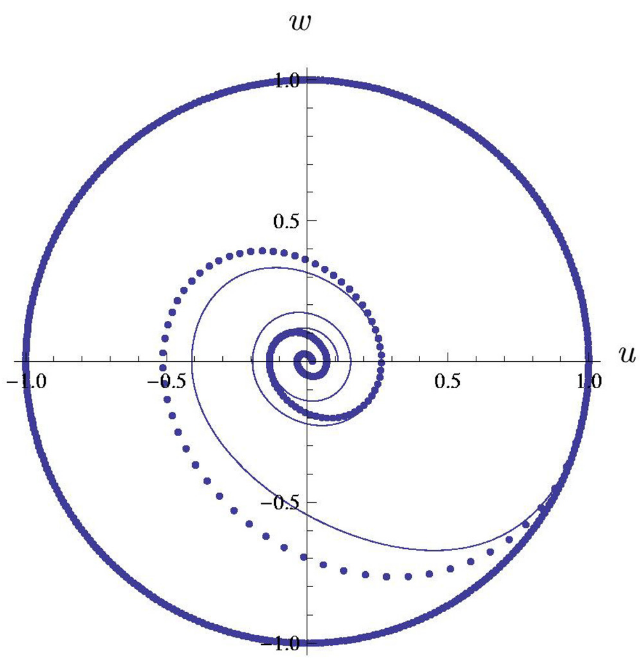

Fig. 7. Three full cycles in phase space of the quadratically damped oscillator (thin solid line) superimposed on the viscously (linearly) damped oscillator (dotted line). As in Fig. 6, the two have been set to intersect at the beginning and end of the first cycle. The outer circle represents the undamped oscillator.

The analytical expression corresponding to the curve in Fig. 5 is

$$\begin{aligned} 0 = \omega t_n - \tan^{-1}&\left(\frac{\omega\sqrt{2e^{2\gamma x}E_n + \frac{\omega^2 x}{\gamma} + \frac{\omega^2}{2\gamma^2}}}{2\gamma e^{2\gamma x}E_n + \frac{\omega^2}{2\gamma}}\right) \\ &- \frac{2}{3}\left(\frac{\gamma\omega\left[2e^{2\gamma x}E_n + \frac{\omega^2 x}{\gamma} + \frac{\omega^2}{2\gamma^2}\right]^{3/2}}{4\gamma^2 E_n^2 e^{4\gamma x} + 3\omega^2 E_n e^{2\gamma x} + \frac{\omega^4 x}{\gamma} + \frac{3\omega^4}{4\gamma^2}}\right), \\ \frac{n\tau}{2} < t_n \le \frac{(n+1)\tau}{2}&. \end{aligned} \tag{72}$$

V. CONCLUSION

Not surprisingly, the method used here lies in extremely close analogy to other, more standard, methods of solving this problem. It is, after all, a one-dimensional problem, for which every approach is at heart a variation on a single theme. But choice of approach is suggested by point of view. The "second-order differential equation" point of view suggests the straightforward "brute force" approach of the power series expansion. By contrast, the use of an integrating factor to find the first integral of Eq. (14) suggests the standard approach of analyzing the integral curves and the phase space portraiture[9] for insight into the dynamics; inverting the first integral to get $v(x;S)$, and separating to obtain the quadrature of Eq. (19), follows. Focusing on $v(x)$ and proceeding with the integration over x is very natural at this point.

This paper presents a third approach that starts from the point of view that Eq. (15) provides a second constant of the motion, defined in such a way that x and v have equal standing as independent variables. This approach suggests the alternate quadrature of Eq. (23) on an equal footing with Eq. (19).

As demonstrated here, this third point of view can be rewarding.

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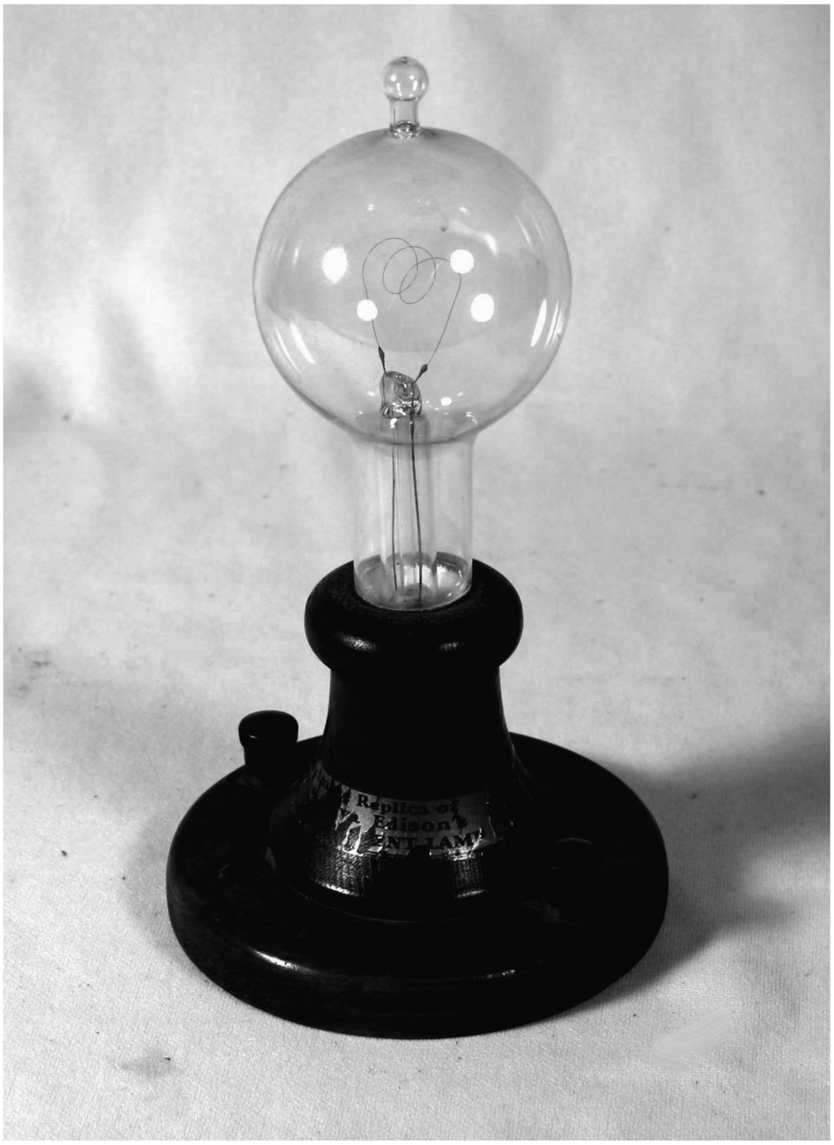

Replica Edison Light Bulb

In late 1879, Thomas Edison succeeded in making a carbon-filament light bulb that burned for thirteen-and-a-half hours, and early in the next year the light bulb was commercially available. Fifty years later replicas commemorating the event were put on sale. These bulbs had an evacuated interior to keep the filament from burning up in atmospheric oxygen. The carbon eventually evaporated onto the interior surface of the bulb, dimming its output. In an attempt to prevent this, Edison inserted a wire electrode into the evacuated space, and his observations of what is now known as the Edison effect form the basis for the vacuum diode. This replica is in the Greenslade Collection. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)