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Thiago R. de Oliveira and Nivaldo A. Lemos

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Force and torque of a string on a pulley

Thiago R. de Oliveira[a] and Nivaldo A. Lemos[b]
Instituto de Física, Universidade Federal Fluminense, Av. Litorânea, S/N, Boa Viagem, Niterói, CEP.24210-340 Rio de Janeiro, Brazil

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Every university introductory physics course considers the problem of Atwood's machine taking into account the mass of the pulley. In the usual treatment, the tensions at the two ends of the string are offhandedly taken to act on the pulley and be responsible for its rotation. However, such a free-body diagram of the forces on the pulley is not *a priori* justified, inducing students to construct wrong hypotheses such as that the string transfers its tension to the pulley or that some symmetry is in operation. We reexamine this problem by integrating the contact forces between each element of the string and the pulley and show that although the pulley does behave as if the tensions were acting on its ends, this comes only as the final result of a detailed analysis. We also address the question of how much friction is needed to prevent the string from slipping over the pulley. Finally, we deal with the case in which the string is on the verge of sliding and show that this cannot happen unless certain conditions are met by the coefficient of static friction and the masses involved.
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I. INTRODUCTION

A crucial step in solving a problem in mechanics by applying Newton's laws to interacting bodies is to identify the individual forces that act on each object. We dare say that the physics ends there and the rest is only manipulation of equations. Although this may be too strong a statement, it has been recognized in recent years that many students have difficulties with this step and that textbooks and instructors should give more attention to the correct identification of the forces on each body.[1] One type of problem in which most, if not all, textbooks fail to correctly identify the forces on each object are the ones containing strings and massive pulleys. These problems are treated in any university elementary physics course that addresses the rotational dynamics of rigid bodies about a fixed axis.

A staple problem is Atwood's machine,[2] schematically depicted in Fig. 1. In its simplest incarnation, the pulley and the string are idealized as massless and the pulley is assumed to be mounted on a frictionless axle. It is also assumed that the string does not slip on the pulley, which requires enough friction between the string and the pulley. Both the idealized and the realistic case, in which account is taken of the masses of the string and the pulley as well as of the friction in the pulley's bearings, illustrate the principles involved in the application of Newton's laws.[3,4] Atwood's machine is a multipurpose mechanical system, which allows one to investigate from Stokes's law[5] to variable-mass rocket motion.[6] If one of the hanging masses is allowed to swing in a plane, the resulting two-degree-of-freedom system exhibits a very rich variety of motions.[7] The system is integrable if the mass of the swinging body is thrice as small as that of the vertically hanging body,[8] but for most other mass ratios it is nonintegrable.[9]

Here, we focus our attention on the force and torque exerted by the string on the pulley. The standard textbook treatment of Atwood's machine[10] assumes that the tension in the string is somehow transferred to the pulley and is responsible for the net force and net torque on the pulley. But tension is an internal force in the string and it is not made clear how the string can exert a force and a torque on the pulley. This adds to the conceptual difficulties presented by students related to the tension in a massless string.[11,12]

The proper physical analysis consists in taking into account that each element of the string exerts a force on the part of the pulley with which it is in contact.[13] The present study complements that of Krause and Sun by treating in detail not only the total torque but also the net force exerted by the string on the pulley. We also extend their work by addressing an interesting question that, as far as we can tell, is never asked in the textbooks: What is the net friction force that prevents the string from slipping on the pulley? Furthermore, we examine the situation in which the string is on the verge of slipping over the pulley and find that this condition can be reached only if the mass of the pulley and the hanging masses satisfy certain necessary conditions.

II. STANDARD TEXTBOOK TREATMENT OF ATWOOD'S MACHINE

Let us consider Atwood's machine with a pulley whose mass M is not negligible in comparison with m_1 and m_2, depicted in Fig. 1. As usual, we assume that the string is inextensible, its mass is negligible and it does not slide on the pulley, which requires enough static friction between the string and the pulley. On the other hand, we assume that the pulley is mounted on a frictionless axle.

For the sake of definiteness let us assume that $m_2 > m_1$. The standard free-body diagram of forces on the hanging masses and the pulley is shown in Fig. 2. Since $W_1 = m_1 g$ and $W_2 = m_2 g$, Newton's second law applied to the masses gives

$$T_1 - m_1 g = m_1 a, \tag{1a}$$

$$m_2 g - T_2 = m_2 a. \tag{1b}$$

As to the pulley, the standard claim[10] is that T_2 and T_1 are forces exerted by the string *on the pulley* at the points P and Q, respectively. The weight of the pulley and the sustaining force exerted by the axle produce no torque about the rotation axis. Therefore, the net torque on the pulley is $\tau = T_2 R - T_1 R$. The angular acceleration α is given by the so-called

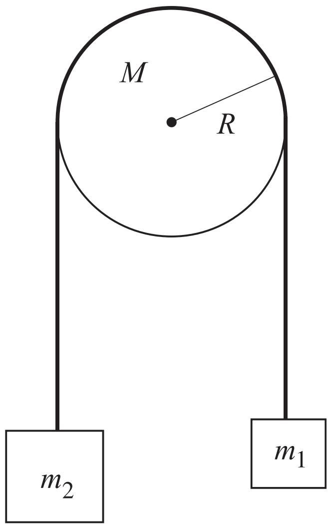

Fig. 1. Atwood's machine.

Newton's second law for rotational motion, $\tau = I\alpha$, so that one has

$$(T_2 - T_1)R = I\alpha. \tag{2}$$

Since the string does not slide on the pulley, the constraint $a = \alpha R$ applies. With the use of this constraint and the assumption that the pulley is a homogeneous disk, whose pertinent moment of inertia is $I = MR^2/2$, Eq. (2) becomes

$$T_2 - T_1 = \frac{M}{2}a. \tag{3}$$

By simply summing Eqs. (1a), (1b), and (3) one gets the acceleration, and with a little more elementary algebra one finds the tensions. The result is

$$a = \frac{m_2 - m_1}{m_1 + m_2 + M/2}g, \quad T_1 = \frac{2m_2 + M/2}{m_1 + m_2 + M/2}m_1 g,$$
$$T_2 = \frac{2m_1 + M/2}{m_1 + m_2 + M/2}m_2 g. \tag{4}$$

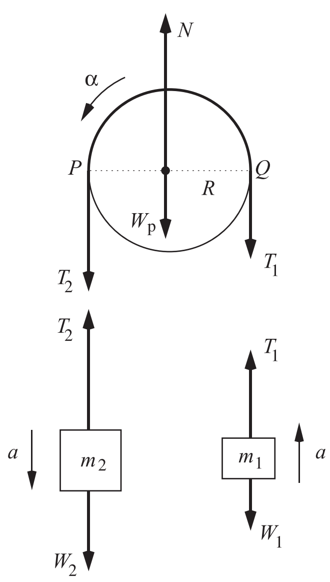

Fig. 2. Standard free-body diagram of forces on the masses and the pulley.

This standard solution to the problem of the motion of Atwood's machine with a massive pulley, as well as the solutions to similar problems that can be found in so many textbooks, is open to a serious physical objection. The forces T_2 and T_1 are *not* forces on the pulley, but forces on the points P and Q *of the string* exerted by the hanging parts of the string at each side of the pulley. This gives rise to the problem of justifying the above results obtained on the basis of a physically unwarranted identification of the forces on the pulley. Note that putting directly the forces T_2 and T_1 on the pulley might reinforce a common misconception among students that all strings do is convey forces from one object to another.[11] Here, the problem is even more subtle since we do consider the string as massless, which usually entails that the tension is constant all along the string. There is also an interesting question that is not usually asked in the textbooks: What is the friction force that prevents the string from slipping on the pulley?

The determination of the net force and the net torque on the pulley requires an integration of the infinitesimal forces and torques exerted on the pulley by each element of the string that touches the pulley. Here, one might be tempted to justify the usual treatment by the seemingly reasonable conjecture that, except for the forces at points P and Q of the pulley, the vector sum of all forces exerted by the string on the other points of the pulley cancel each other owing to an apparent symmetry. This argument turns out to be wrong, and no such symmetry exists.

III. FORCES ON AN ELEMENT OF THE STRING

We follow the approach used in the analysis of the related problem of determining the effect of the friction force on a rope wrapped around a capstan.[14,15]

For the sake of definiteness we make the assumption that $m_2 > m_1$, which implies $T_2 > T_1$. Figure 3 shows the forces on a piece of the string that subtends the small angle $\Delta\theta$. $\mathbf{F}_1$ and $\mathbf{F}_2$ are the tensions at the ends of the string element; $\mathbf{f}$ and $\mathbf{n}$ are, respectively, the tangential (friction) and normal forces exerted by the pulley on the string. Since the string is massless, Newton's second law entails that the vector sum of the forces shown in Fig. 3 is zero

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{f} + \mathbf{n} = 0. \tag{5}$$

Let

$$\hat{\mathbf{r}}(\theta) = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}, \quad \hat{\boldsymbol{\theta}}(\theta) = -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}, \tag{6}$$

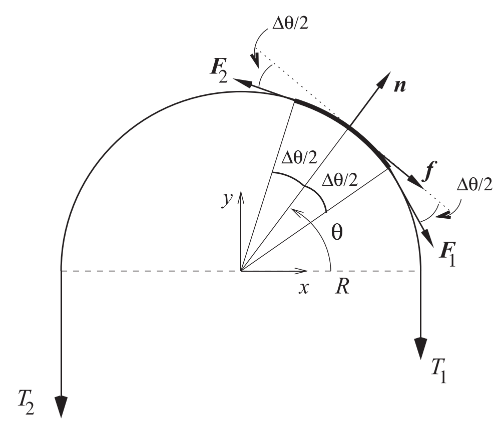

Fig. 3. Forces on a piece of the string that subtends the small angle $\Delta\theta$.

respectively, be the outward normal and tangential (oriented toward increasing θ) unit vectors at the point of the pulley with angular coordinate θ. For the friction force and the normal force of the pulley *on the piece of string,* we write

$$\mathbf{f} = -\tilde{f}(\theta)\Delta\theta\,\hat{\boldsymbol{\theta}}(\theta), \quad \mathbf{n} = \tilde{n}(\theta)\Delta\theta\,\hat{\mathbf{r}}(\theta), \tag{7}$$

where $\tilde{f}$ and $\tilde{n}$ are positive and have dimension of force per unit angle in radians. Since we will eventually let $\Delta\theta \to 0$, Eq. (7) has been written as if $\Delta\theta$ were infinitesimal.[16]

The tangential and normal components of Eq. (5) are

$$F_2 \cos\left(\frac{\Delta\theta}{2}\right) - F_1 \cos\left(\frac{\Delta\theta}{2}\right) - \tilde{f}\Delta\theta = 0, \tag{8a}$$

$$-F_1 \sin\left(\frac{\Delta\theta}{2}\right) - F_2 \sin\left(\frac{\Delta\theta}{2}\right) + \tilde{n}\Delta\theta = 0. \tag{8b}$$

In Eqs. (8), we have

$$F_2 = |\mathbf{F}_2| = T\left(\theta + \frac{\Delta\theta}{2}\right), \quad F_1 = |\mathbf{F}_1| = T\left(\theta - \frac{\Delta\theta}{2}\right), \tag{9}$$

where $T(\theta)$ is the tension at the point of the string with angular coordinate θ with, of course,

$$T(0) = T_1, \quad T(\pi) = T_2. \tag{10}$$

Therefore Eqs. (8) become

$$T\left(\theta + \frac{\Delta\theta}{2}\right)\cos\left(\frac{\Delta\theta}{2}\right) - T\left(\theta - \frac{\Delta\theta}{2}\right)\cos\left(\frac{\Delta\theta}{2}\right) - \tilde{f}(\theta)\Delta\theta = 0, \tag{11a}$$

$$-T\left(\theta - \frac{\Delta\theta}{2}\right)\sin\left(\frac{\Delta\theta}{2}\right) - T\left(\theta + \frac{\Delta\theta}{2}\right)\sin\left(\frac{\Delta\theta}{2}\right) + \tilde{n}(\theta)\Delta\theta = 0. \tag{11b}$$

Now we divide each of the two last equations by $\Delta\theta$ and let $\Delta\theta \to 0$ to obtain

$$\tilde{f}(\theta) = \frac{dT}{d\theta} \tag{12}$$

and also

$$\tilde{n}(\theta) = T(\theta). \tag{13}$$

Equation (12) shows that friction causes the tension in the string to be variable, even though the string is massless.

IV. FORCE ON THE PULLEY

We are now in a position to compute the friction force and the total force exerted by the string on the pulley.

The net friction force exerted by the pulley *on the string* is given by

$$\mathbf{F}_f^{string} = \int_0^\pi \left(-\tilde{f}(\theta)\,\hat{\boldsymbol{\theta}}(\theta)\right)d\theta = \hat{\mathbf{x}}\int_0^\pi \frac{dT}{d\theta}\sin\theta d\theta - \hat{\mathbf{y}}\int_0^\pi \frac{dT}{d\theta}\cos\theta d\theta, \tag{14}$$

where we have used Eqs. (6), (7), and (12). Integrations by parts yield

$$\int_0^\pi \frac{dT}{d\theta}\sin\theta d\theta = T(\theta)\sin\theta\big|_0^\pi - \int_0^\pi T(\theta)\cos\theta d\theta = -\int_0^\pi T(\theta)\cos\theta d\theta \tag{15}$$

and

$$\int_0^\pi \frac{dT}{d\theta}\cos\theta d\theta = T(\theta)\cos\theta\big|_0^\pi + \int_0^\pi T(\theta)\sin\theta d\theta = -(T_1+T_2) + \int_0^\pi T(\theta)\sin\theta d\theta. \tag{16}$$

With these results Eq. (14) becomes

$$\mathbf{F}_f^{string} = (T_1+T_2)\hat{\mathbf{y}} - \int_0^\pi T(\theta)(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}})d\theta. \tag{17}$$

One could think that by symmetry the x-component in Eq. (17) would be zero, but this is not true because $T(\theta) \neq T(\pi-\theta)$. We also note that the friction force on the string cannot be determined unless the tension is known as a function of θ. Thus, in general, the question "What is the friction force that prevents the string from slipping over the pulley?" does not have a definite answer. As will be seen shortly, however, $T(\theta)$ can be found explicitly if the string is on the verge of sliding on the pulley.

The net normal force exerted by the pulley *on the string* is given by

$$\mathbf{F}_n^{string} = \int_0^\pi \tilde{n}(\theta)\,\hat{\mathbf{r}}(\theta)\,d\theta = \int_0^\pi T(\theta)(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}})d\theta, \tag{18}$$

where Eqs. (6), (7), and (13) have been used. From Eqs. (17) and (18), it follows at once that

$$\mathbf{F}_f^{string} + \mathbf{F}_n^{string} - (T_1+T_2)\hat{\mathbf{y}} = 0. \tag{19}$$

Note that $-(T_1+T_2)\hat{\mathbf{y}}$ is the total force on the part of the string in contact with the pulley exerted by the hanging pieces of the string at each side of the pulley. Thus the last result is correct: the total force on the part of the string in contact with the pulley is zero because the string is massless.

By Newton's third law, the total force exerted by the string *on the pulley* is

$$\mathbf{F}_{pulley} = -(\mathbf{F}_f^{string} + \mathbf{F}_n^{string}) = -(T_1+T_2)\hat{\mathbf{y}}. \tag{20}$$

Therefore, although T_2 and T_1 **are not** forces on the pulley, everything happens as if they actually were forces applied by the string at the points P and Q of the pulley shown in Fig. 2. The usual textbook derivation may leave the student with the false impression that the forces exerted by the string on the other points of the pulley cancel each other owing to an apparent symmetry, which in fact does not exist.

V. TORQUE ON THE PULLEY

The torque exerted by the string on an element *of the pulley* that subtends an angle $d\theta$ is

$$d\boldsymbol{\tau}_{pulley} = R\hat{\mathbf{r}} \times (-\mathbf{f}) = R\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}\tilde{f}(\theta)d\theta = \tilde{f}(\theta)R\hat{\mathbf{z}}d\theta, \tag{21}$$

where Eq. (7) has been used. By making use of Eq. (12) we are led to

$$d\boldsymbol{\tau}_{pulley} = \hat{\mathbf{z}}R\frac{dT}{d\theta}d\theta. \tag{22}$$

Therefore,

$$\boldsymbol{\tau}_{pulley} = \hat{\mathbf{z}}R\int_0^\pi \frac{dT}{d\theta}d\theta = R(T_2 - T_1)\hat{\mathbf{z}}. \tag{23}$$

Once again, this is the torque on the pulley obtained by the *a priori* physically unwarranted assumption that T_2 and T_1 are forces on the pulley and that the torques applied by the string on the pulley at points other than the points P and Q shown in Fig. 2 cancel each other owing to an apparent (but nonexistent) symmetry. It should be noted that, differently from what has been done here, one can calculate directly the total torque on the pulley without first finding the net frictional and normal forces.[13]

VI. STRING ON THE VERGE OF SLIDING

Let us suppose that the string is on the verge of sliding on the pulley. If μ is the coefficient of static friction between the string and the pulley we have

$$\tilde{f} = \mu\tilde{n}. \tag{24}$$

This equation deserves some discussion, since it implies that all points of the string that are in contact with the pulley reach the slipping condition simultaneously. Let us first note that, by Eq. (12), the tension $T(\theta)$ is a differentiable function, so it is also continuous. Suppose the string is not on the verge of slipping. As the tension T_2 is gradually increased, by continuity the tension at each point of the string with $0 \le \theta \le \pi$ also gradually increases, no discontinuous jump is possible for $T(\theta)$. If at any point of the string the condition of being on the verge of slipping has not yet been reached, the corresponding portion of the string will not slip if T_2 is further infinitesimally increased. Therefore, a further infinitesimal increment of T_2 will make the entire string slip only if the condition of being on the verge of slipping is reached simultaneously at all points of the string. If the slipping condition could be reached at some portion of the string but not at the others, by a further infinitesimal increment of T_2 one portion of the string would start to slip before the others, which is impossible because the string is inextensible by hypothesis. In addition to these theoretical arguments, Eq. (24) leads to Eq. (26) below which is vindicated by experiment.[18]

Combining Eq. (24) with Eqs. (12) and (13) we find

$$\frac{dT}{d\theta} = \mu T. \tag{25}$$

It follows that

$$T(\theta) = T_1 e^{\mu\theta} \tag{26}$$

inasmuch as $T(0) = T_1$. Since $T_2 = T(\pi) = T_1 e^{\mu\pi}$, the friction coefficient is determined

$$\mu = \frac{1}{\pi}\ln\left(\frac{T_2}{T_1}\right). \tag{27}$$

By the way, the exponential growth of the tension explains why, if a rope is wound several times around a capstan, it takes an enormous force to make the rope slide on the capstan by pulling one end against a tiny force at the other end.[14,15]

Now the friction force *on the pulley* can be explicitly computed. From Eqs. (17) and (26) we have

$$\begin{aligned}\mathbf{F}_f^{pulley} &= -\mathbf{F}_f^{string} = -(T_1+T_2)\hat{\mathbf{y}} \\ &\quad + \int_0^\pi T(\theta)(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}})d\theta \\ &= -(T_1+T_2)\hat{\mathbf{y}} + \hat{\mathbf{x}}T_1\int_0^\pi e^{\mu\theta}\cos\theta d\theta \\ &\quad + \hat{\mathbf{y}}T_1\int_0^\pi e^{\mu\theta}\sin\theta d\theta\,. \end{aligned} \tag{28}$$

The integrals are elementary and can be easily found as the real and imaginary parts of the complex integral $\int e^{(\mu+i)\theta}d\theta = e^{(\mu+i)\theta}/(\mu+i)$

$$\begin{aligned}\int_0^\pi e^{\mu\theta}\cos\theta d\theta &= \frac{e^{\mu\theta}}{1+\mu^2}(\sin\theta + \mu\cos\theta)\Big|_0^\pi \\ &= -\frac{\mu}{1+\mu^2}(1+e^{\mu\pi}); \end{aligned} \tag{29}$$

$$\begin{aligned}\int_0^\pi e^{\mu\theta}\sin\theta d\theta &= \frac{e^{\mu\theta}}{1+\mu^2}(\mu\sin\theta - \cos\theta)\Big|_0^\pi \\ &= \frac{1}{1+\mu^2}(1+e^{\mu\pi}). \end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned}\mathbf{F}_f^{pulley} &= -(T_1+T_2)\hat{\mathbf{y}} - \frac{\mu T_1}{1+\mu^2}(1+e^{\mu\pi})\hat{\mathbf{x}} \\ &\quad + \frac{T_1}{1+\mu^2}(1+e^{\mu\pi})\hat{\mathbf{y}}. \end{aligned} \tag{31}$$

With the help of Eq. (27) and a little algebra this last result can be cast in the following form:

$$\mathbf{F}_f^{pulley} = -\frac{\mu}{1+\mu^2}(T_1+T_2)\hat{\mathbf{x}} - \frac{\mu^2}{1+\mu^2}(T_1+T_2)\hat{\mathbf{y}}. \tag{32}$$

Now we have a definite answer to our previous question: Eq. (32) yields the net friction force that prevents slippage of the string over the pulley, and it is worth noting that it has a horizontal component. Since the net friction force is the resultant of forces continuously applied along a curved path, it does not seem physically equivalent to a single force that would prevent the string from slipping.

From Eqs. (4) and (27), it follows that:

$$\frac{T_2}{T_1} = \frac{(4m_1+M)m_2}{(4m_2+M)m_1} = e^{\mu\pi}. \tag{33}$$

Solving this equation for M we find

$$M = \frac{4m_1m_2(e^{\mu\pi}-1)}{m_2 - m_1e^{\mu\pi}}. \tag{34}$$

Note that if $m_2 \le m_1e^{\mu\pi}$ then the string will never slip relative to the pulley no matter how large the pulley mass is. On the other hand, solving Eq. (33) for m_2 we get

$$m_2 = \frac{M m_1 e^{\mu\pi}}{M - 4m_1(e^{\mu\pi} - 1)}. \tag{35}$$

If $M \leq 4m_1(e^{\mu\pi} - 1)$ there is no positive solution for m_2. Therefore, two necessary conditions for the string to be on the verge of slipping over the pulley are

$$m_2 > m_1 e^{\mu\pi} \quad \text{and} \quad M > 4m_1(e^{\mu\pi} - 1). \tag{36}$$

The first requirement is expected from the force amplification effect brought about by a rope wrapped around a capstan.[14,15] It would be the only necessary condition if the pulley could not rotate or, equivalently, if its mass were infinite. The second requirement is not so obvious and stems from the fact that the pulley can freely turn on its axle.

So as to have an idea of the order of magnitude of the masses that are required for the string to be on the verge of sliding, let us assume that $m_1 = 1\,\text{kg}$, $m_2 = 3\,\text{kg}$ and $\mu = 0.3$. Then Eq. (34) gives $M \approx 43\,\text{kg}$, an appreciably large mass for the pulley.

VII. CONCLUDING REMARKS

We have argued that the usual textbook solution to a classic problem in rotational dynamics, Atwood's machine, relies on a faulty identification of forces on each object, since the tensions in the hanging parts of the string are identified as forces on the pulley. This may seem a small detail, but we believe it is an important one, since correctly identifying the forces on each of a system of interacting bodies is a fundamental step in solving a mechanics problem by means of Newton's laws. Such a shortcut also reinforces a common misconception between students to the effect that strings merely convey forces without affecting them.[11]

We have presented a consistent treatment of the problem by considering the normal and friction forces between each element of the string and the pulley, and have shown that the contact force exerted on the pulley by the entire segment of the string that touches the pulley gives rise to the net force and the net torque usually assumed without a convincing justification in the standard treatment given in the textbooks.

Although presenting the full mathematical treatment of the problem, as done here, is beyond the scope of an introductory physics course, we believe that attention should be called to the fact that the force on the pulley arises from contact forces exerted by the string and that a careful analysis gives the conjectured result (23) for the torque on the pulley. Another possibility is to consider the pulley together with the segment of the string that touches it as a single object with the same moment of inertia as that of the pulley, since the string is massless. As far as the angular acceleration of this object is concerned, one is allowed to disregard the internal forces between the aforesaid segment of the string and the pulley, and now T_1 and T_2 are actually external forces responsible for the net external torque on the pulley-string segment system.[13]

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[a)]Electronic mail: tro@if.uff.br
[b)]Electronic mail: nivaldo@if.uff.br

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[16]It has been noticed by Lubarda (Ref. 17) that Eq. (7) does not define n correctly if $\Delta\theta$ is finite, in particular n is not necessarily parallel to $\hat{\mathbf{r}}(\theta)$. Let $d\mathbf{n}/d\theta = \tilde{n}(\theta)\hat{\mathbf{r}}(\theta)$ be the normal force per unit angle, with $\tilde{n}(\theta)$ a continuous function. The normal force on the element of string that subtends the small but finite angle $\Delta\theta$ is $\mathbf{n} = \int_{\theta-\Delta\theta/2}^{\theta+\Delta\theta/2} \tilde{n}(\theta')\hat{\mathbf{r}}(\theta')d\theta'$. By the mean value theorem for integrals, $\mathbf{n} = (\tilde{n}(\theta_1)\cos\theta_1\,\hat{\mathbf{x}} + \tilde{n}(\theta_2)\sin\theta_2\,\hat{\mathbf{y}})\Delta\theta$ where the angles θ_1 and θ_2 belong to the closed interval $[\theta - \Delta\theta/2, \theta + \Delta\theta/2]$. Since $\tilde{n}(\theta_1)\cos\theta_1 = \tilde{n}(\theta)\cos\theta + \epsilon_1$ and $\tilde{n}(\theta_2)\sin\theta_2 = \tilde{n}(\theta)\sin\theta + \epsilon_2$, where both ϵ_1 and ϵ_2 tend to zero as $\Delta\theta \to 0$, it follows that $\mathbf{n} = \tilde{n}(\theta)\hat{\mathbf{r}}(\theta)\Delta\theta + \boldsymbol{\epsilon}\Delta\theta$, where $\boldsymbol{\epsilon} = \epsilon_1\hat{\mathbf{x}} + \epsilon_2\hat{\mathbf{y}}$. Inserting this into Eq. (8b), dividing by $\Delta\theta$ and letting $\Delta\theta \to 0$ one gets Eq. (13) because $\lim_{\Delta\theta\to 0} \boldsymbol{\epsilon}\cdot\hat{\mathbf{r}}(\theta) = 0$. The same argument applies to the tangential force $\mathbf{f}$ and the corresponding Eq. (12). All we have assumed is that the tangential and normal forces *per unit angle*, namely, $d\mathbf{f}/d\theta$ and $d\mathbf{n}/d\theta$, are well defined and continuous, which is consistent with Eqs. (12) and (13) that follow from the assumption. Equation (7) is the usual shortcut employed by physicists to avoid the rigorous but lengthy discussion above, which has no effect on the final result. Therefore, our derivation does not suffer from the drawback pointed out by Lubarda.

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