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Another look at a damped physical pendulum

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NOTES AND DISCUSSIONS

Another look at a damped physical pendulum

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I. INTRODUCTION

Measuring the time dependence of the angular position of a physical pendulum is a common task in an undergraduate physics laboratory.[1] To model the behavior of the pendulum, students generally assume that the damping force is viscous and proportional to the velocity of the pendulum. This model produces a continuous function that can be compared to the measurements. Although this model describes the data reasonably well for a number of cycles, it often fails noticeably as the pendulum slows down. Several authors have modeled the time dependence assuming that the damping is due to a dry friction force.[2–6] This model produces a piece-wise continuous function, which probably accounts for its infrequent use in an undergraduate laboratory. The purpose of this note is to describe these two models and show how undergraduates can use them to analyze the motion of a physical pendulum.

II. EXPERIMENTAL METHOD

Figure 1 shows a familiar experimental setup that uses the shaft of a rotational-type variable resistor as a pivot for the pendulum.[6–8] The damping is mostly due to the pivot in this arrangement. The variable resistor serves as a transducer, relating the angular position to the resistance. In general, the latter is related to a voltage using a voltage divider. The angular position of the pendulum can be determined from a linear equation using measurements of transducer output for angles of 0° and 90°. Any one of several commercial data acquisition systems can be used to record the transducer voltage as a function of time.[9] Measurements for the analyses that follow were recorded with LabVIEW and a National Instruments data acquisition system.

III. MODELS

A. Frictional torque proportional to velocity

We use Newton's second law, the small angle approximation for $\sin\Theta$, and a frictional torque proportional to velocity, and write the equation of motion for the pendulum as[10,11]

$$\frac{d^2\Theta}{dt^2}+\frac{2}{\tau}\frac{d\Theta}{dt}+\omega^2\Theta=0, \tag{1}$$

where

$$\omega=\sqrt{\frac{mgx}{I}}. \tag{2}$$

The total mass of the pendulum is m, g is the acceleration due to the gravitational force, x is the distance from the center of gravity to the axis of rotation, I is the moment of inertia of the pendulum about the axis of rotation, τ is the damping constant, and Θ is the angular position of the pendulum (see Fig. 1). Equation (1) is solved by

$$\Theta=\Theta_o e^{-t/\tau}\cos\omega_1 t, \tag{3}$$

where Θ_o is the maximum angular position and $\omega_1=\omega\sqrt{1-1/\omega^2\tau^2}$. To compare Eq. (3) with experimental measurements, initial values of Θ_o and τ were obtained by visual inspection of the data and ω was calculated from Eq. (2). Adjustments of the parameters were then made for comparison with the measurements (see Fig. 2). Clearly, the amplitude of the motion falls off much more rapidly than Eq. (3) predicts.

B. Constant frictional torque

The constant frictional torque can be modeled by $C\,\mathrm{sgn}(d\Theta/dt)$, where C is a friction parameter and $\mathrm{sgn}(x)$ is the signum function defined as $\mathrm{sgn}(x)=1$ for $x>0$, $\mathrm{sgn}(x)=0$ for $x=0$, and $\mathrm{sgn}(x)=-1$ for $x<0$. In terms of the signum function, the equation of motion can be written as

$$\frac{d^2\Theta}{dt^2}+\omega^2\Theta=-C\,\mathrm{sgn}\left(\frac{d\Theta}{dt}\right). \tag{4}$$

The homogeneous part of Eq. (4) is solved by

$$\Theta(t)=A\sin\omega t+B\cos\omega t. \tag{5}$$

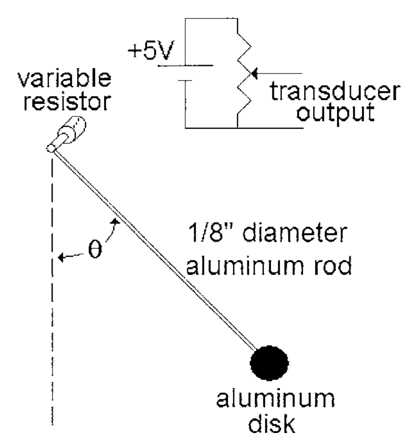

Fig. 1. The experimental arrangement for the physical pendulum.

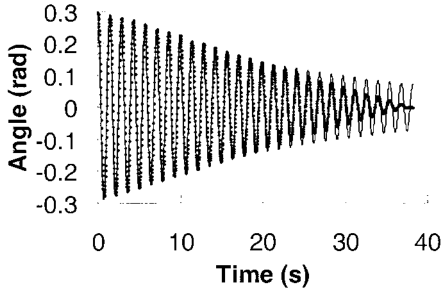

Fig. 2. Comparison of experimental measurements (solid points) for the physical pendulum with calculations (solid line) assuming a damping force proportional to velocity.

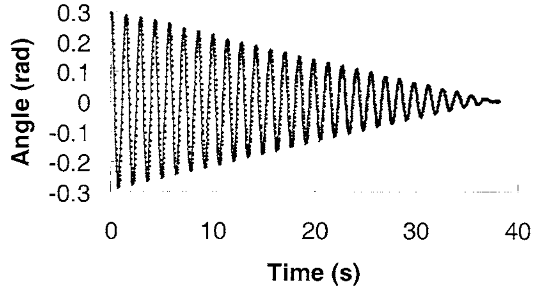

Fig. 3. Comparison of experimental measurements (solid points) for the physical pendulum with calculations (solid line) assuming a dry-friction damping force.

If the pendulum is released from a positive angle Θ_o, $d\Theta/dt$ is negative for the first half period and sgn($d\Theta/dt$) also is negative. The inhomogeneous part of Eq. (4) is solved by $\Theta(t)=C/\omega^2$, making the complete solution $\Theta(t)=A\sin\omega t+B\cos\omega t+C/\omega^2$. Satisfying the initial conditions of $\Theta=\Theta_o$ and $d\Theta/dt=0$ at $t=0$ requires $A=0$ and $B=\Theta_o-C/\omega^2$. The particular solution for the first half period is then

$$\Theta(t)=\left(\Theta_o-\frac{C}{\omega^2}\right)\cos\omega t+\frac{C}{\omega^2}. \tag{6}$$

The same technique for each half period yields the general result

$$\Theta(t)=\left(\Theta_o-(2n+1)\frac{C}{\omega^2}\right)\cos\omega t+(-1)^n\frac{C}{\omega^2} \tag{7}$$

for $n\pi/\omega\leqslant t\leqslant(n+1)\pi/\omega$ and $n=0,1,2,\ldots$

The period of the oscillations is unchanged in the constant torque model so that the time for a half period is π/ω. By dividing each successive half period into several equal time segments, the angular position as a function of time for each half period can be calculated using Eq. (7). Plots constructed from a spreadsheet using $\Theta_o=0.296$ rad, $C=0.0555\ \mathrm{s}^{-2}$, and $\omega=4.447$ rad/s, and the experimental measurements are displayed in Fig. 3. Clearly, the constant frictional torque describes the damping much better than the velocity dependent damping force.

IV. SUMMARY

A calculus-based physics course does not ordinarily discuss damped harmonic motion with a constant damping force. Nevertheless, the model developed in this note is well within the expertise of a student who is familiar with damped harmonic motion and who has some experience with solving ordinary differential equations. Calculations can be performed easily with a spreadsheet and compared with measurements that have been taken with a data acquisition system. Although we have applied the model to the motion of a physical pendulum where the damping is due to sliding friction in a variable resistor, students can apply the model to measurements made on other damped harmonic oscillators.

ACKNOWLEDGMENTS

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Erratum: "Orbits of central force law potentials" [Am. J. Phys. 73 (1), 40–44 (2005)]

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In the paragraph before Eq. (1) the expression "the radial acceleration $\ddot{r}=mv_0^2/r_0=r_0\omega_0^2$" should be replaced by "the only acceleration that is present is the centripetal acceleration $a_c=v_0^2/r_0=r_0\omega_0^2$." We are grateful to Martin S. Tiersten and Dubravko Klabucar for pointing out this error.

BIG BANG

Don't speak to me, lightning: I'll hide
where you can't find me, under my bed,
as the hair on my arms rises, the dust-
bunnies cling; till you shake the floor, slow-voiced.

Astronomers stare. Their eyes are so big,
they can see so long ago. A cracked egg-
shell of heat-lightning frames the oldest sky
in all directions. Thunder, pass me by.

Lee Rudolph
Department of Mathematics
Clark University

"Big Bang" will appear in
A Woman and a Man, Ice-Fishing
(Texas Review Press, 2005),
winner of the 2004 X. J. Kennedy Poetry
Award.

