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Motion of a Harmonic Oscillator with Sliding Friction

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Elementary mechanics courses usually discuss the motion of a harmonic oscillator with viscous friction. A variation of this problem with some amusing aspects is the problem of a harmonic oscillator acted on by sliding friction. This situation is actually more familiar to students and can readily be presented in a laboratory program. While the solution of this problem is straightforward, it does offer a challenge to students in an introductory course and may serve as a pedagogically useful special assignment.

The problem may be stated as follows: A block of mass m is attached to a spring with elastic constant k for both extension and compression and moves on a horizontal surface with a coefficient of friction μ. If the block is initially at a displacement x_0 from the equilibrium position and released from rest, find the position where the block ultimately comes to rest. Assume that the coefficients of static friction and sliding friction are equal.

The equation of motion for the system is

$$m\ddot{x} = -kx \pm \mu mg, \tag{1}$$

where the $\pm$ sign is used so that the friction force is opposite to the direction of the velocity of the block.

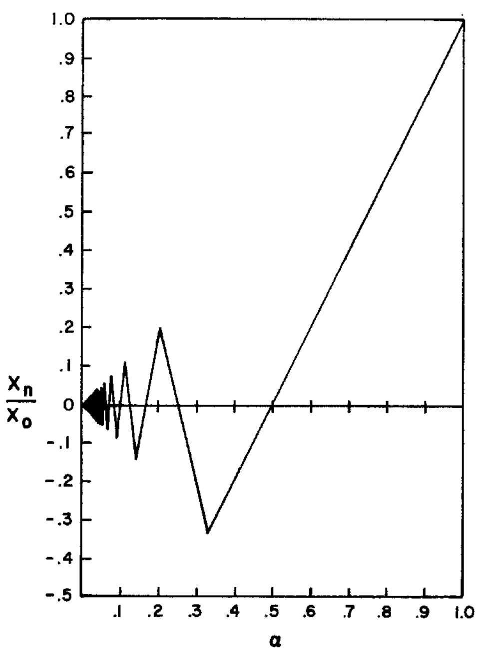

FIG. 1. Plot of the values of x_n versus α.

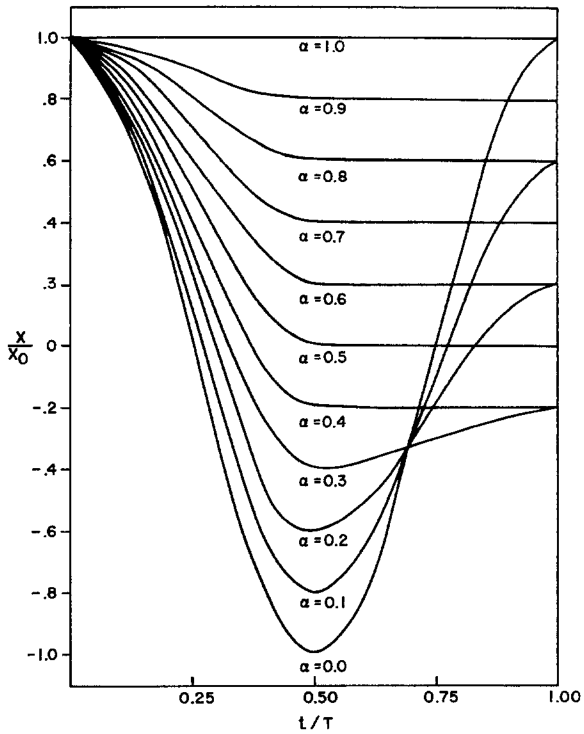

FIG. 2. Displacement of the oscillator as a function of time for several values of α.

We define the "critical" displacement

$$x_c = (\mu mg/kx_0)x_0 = \alpha x_0, \tag{2}$$

where the restoring force of the spring is equal in magnitude to the friction force.

During the first half-cycle of the motion, the loss of potential energy is equal to the work done against friction,

$$\tfrac{1}{2}kx_0^2 - \tfrac{1}{2}kx_1^2 = \mu mg(x_0 - x_1), \tag{3}$$

where x_1 is the displacement after one half-cycle.

Thus we have

$$x_1 = -x_0 + 2\alpha x_0. \tag{4}$$

Similarly, after each half-cycle

$$x_j = -x_{j-1} - (-1)^j 2\alpha x_0 \tag{5}$$

or

$$x_j = (-1)^j (1 - 2\alpha j)x_0. \tag{6}$$

The block comes to rest permanently at $x = x_n$ when

$$|x_n| \leq x_c < |x_{n-1}|. \tag{7}$$

Substituting into Eq. (6), we obtain

$$(1-\alpha)/2\alpha \leq n < (1+\alpha)/2\alpha. \tag{8}$$

Thus, the final position may be obtained directly from Eqs. (6) and (8). A graphical representation of our solution is given in Fig. 1, where x_n is plotted as a function of α.

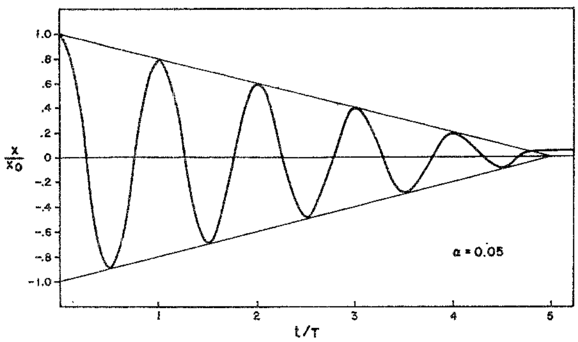

FIG. 3. Complete motion of the oscillator for $\alpha=0.05$.

From Eq. (8) and Fig. 1 we note that the damping of the oscillatory motion is determined by the magnitude of α. If $\alpha\geq 1$ (i.e., the initial displacement is smaller than the critical displacement) no motion takes place. If $\frac{1}{3}\leq\alpha<1$ the oscillator comes to rest without reversing its motion. If $\frac{1}{5}\leq\alpha<\frac{1}{3}$ the oscillator stops after reversing its motion once. For small values of α the number of oscillations becomes large, and the oscillator ultimately comes to rest close to the origin.

The complete solution of Eq. (1) may be obtained directly. Let $x^{(j)}$ be the displacement during the jth half-cycle. Then the motion is given by

$$x^{(j)}=x_0\{[1-(2j-1)\alpha]\cos(\omega t)-(-1)^j\alpha\}, \tag{9}$$

where $\omega=(k/m)^{1/2}=2\pi/T$ and T is the period of the motion.

A plot of Eq. (9) for $t\leq T$ is given in Fig. 2 for several values of α. We note that the limit $\alpha\to 0$ corresponds to undamped simple harmonic motion.

The displacement lies within a pair of straight lines with slopes $=\pm 4\alpha x_0/T$. (This is to be compared with the exponential envelopes in the case of viscous friction.) In order to illustrate this result we have plotted in Fig. 3 the complete motion of the oscillator for $\alpha=0.05$.

Since the period of oscillation is independent of the amplitude, the time elapsed during the complete motion is determined by the number of oscillations which take place. Thus if $\frac{1}{2}\leq\alpha<1$, $t=T/4$; if $\frac{1}{3}\leq\alpha<\frac{1}{2}$, $t=T/2$; and in general if $1/(n+1)\leq\alpha<1/n$, $t=nT/4$.

It is also of interest to examine the total energy loss of the system during the motion. The total distance traveled by the block is given by

$$s_n=x_0+\sum_{j=1}^{n-1}2\,|\,x_j\,|+|\,x_n\,|. \tag{10}$$

From Eq. (6) we obtain

$$s_n=2n(1-\alpha n)x_0. \tag{11}$$

Let the initial energy be $E_0=\frac{1}{2}kx_0^2$. Then the fractional energy loss is

$$\Delta E/E_0=(\mu m g s_n)/(\tfrac{1}{2}kx_0^2)=2\alpha s_n/x_0$$
$$=4\alpha n(1-\alpha n). \tag{12}$$

As a check, we note that

$$\Delta E/E_0=(\tfrac{1}{2}kx_0^2-\tfrac{1}{2}kx_n^2)/(\tfrac{1}{2}kx_0^2)$$
$$=1-x_n^2/x_0^2=1-(1-2\alpha n)^2$$
$$=4\alpha n(1-\alpha n).$$

A more complicated version of this problem may be presented by noting that the coefficient of static friction is not actually equal to the coefficient of sliding friction.

Motion of a Pulsed Rocket

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In elementary mechanics courses the motion of a rocket is a commonly discussed illustration of conservation of momentum and provides an excellent application of Newton's second law in the general form $\mathbf{F}=d\mathbf{p}/dt$.

In the absence of external forces the velocity of the rocket as a function of its (variable) mass is obtained directly by integrating

$$mdv/dt=v_0dm/dt \tag{1}$$

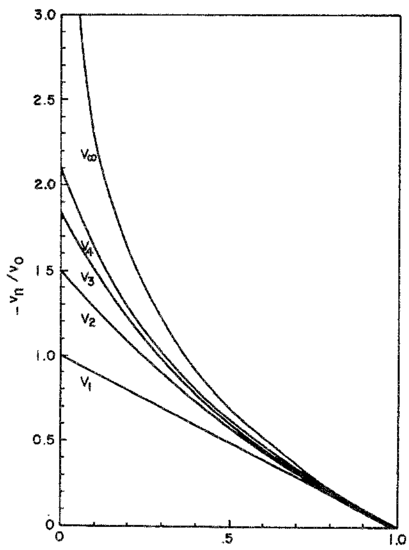

FIG. 1. Plots of v_n versus m for several values of n, where $m=m_0-n\mu$ and m_0=initial mass of rocket and fuel, μ=mass of each pulse of fuel ejected, n=total number of pulses, v_n=rocket velocity after n pulses of fuel have been ejected.