Ortogonalidad

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{ikx}\,dx &= \int_{-\infty}^{0} e^{ikx}\,dx + \int_{0}^{\infty} e^{ikx}\,dx \\
&= \int_{0}^{\infty} e^{-ikx}\,dx + \int_{0}^{\infty} e^{ikx}\,dx \\
&= \int_{0}^{\infty} (e^{ikx} + e^{-ikx})\,dx
\end{aligned}$$

Truco:

$$\int_{-\infty}^{\infty} e^{ikx}\,dx = \lim_{\epsilon \to 0^+} \int_{0}^{\infty} (e^{ikx} + e^{-ikx})\, e^{-x\epsilon}\,dx$$

Ortogonalidad

$$\int_{-\infty}^{\infty} e^{ikx}\,dx = \lim_{\epsilon\to 0^+}\int_0^{\infty}[e^{ix(k+i\epsilon)} + e^{-ix(k-i\epsilon)}]\,dx$$

$$= \lim_{\epsilon\to 0^+}\left[\frac{e^{ix(k+i\epsilon)}}{i(k+i\epsilon)} + \frac{e^{-ix(k-i\epsilon)}}{-i(k-i\epsilon)}\right]_0^{\infty}$$

$$= \lim_{\epsilon\to 0^+}\left[0 + 0 - \frac{1}{i(k+i\epsilon)} - \frac{1}{-i(k-i\epsilon)}\right]$$

$$= \lim_{\epsilon\to 0^+}\left[\frac{i}{k+i\epsilon} - \frac{i}{k-i\epsilon}\right]$$

$$= \lim_{\epsilon\to 0^+}\frac{2\epsilon}{k^2+\epsilon^2}$$

Ortogonalidad

$$2\pi\,\delta(k) = \int_{-\infty}^{\infty} e^{ikx}\,dx = \lim_{\epsilon\to 0^+} \frac{2\epsilon}{k^2+\epsilon^2} = \begin{cases} \infty & \text{si} \quad k=0 \\ 0 & \text{si} \quad k\neq 0 \end{cases} \tag{1}$$

Atención:

$$\int_{-\infty}^{\infty} \frac{2\epsilon}{k^2+\epsilon^2}\,dk = 2\arctan\left(\frac{k}{\epsilon}\right)\Bigg|_{-\infty}^{\infty} = 2\pi \tag{2}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \delta(k)\,dk = 1 \tag{3}$$