the paraboloid of revolution on which it must lie to displace its own mass of fluid and would come to rest at the axis. The same is true of a sphere at a free surface of rotating liquid, since this is simply a particular distribution of density with respect to Ψ.

On the other hand, if the sphere is sufficiently non-uniform in density, say by being weighted on one side, it is clearly possible for the total centri-fugal force on the sphere to be greater than that on displaced fluid of the same total mass, in which case the sphere moves outward on a paraboloid of revolution until it meets the wall of the vessel.
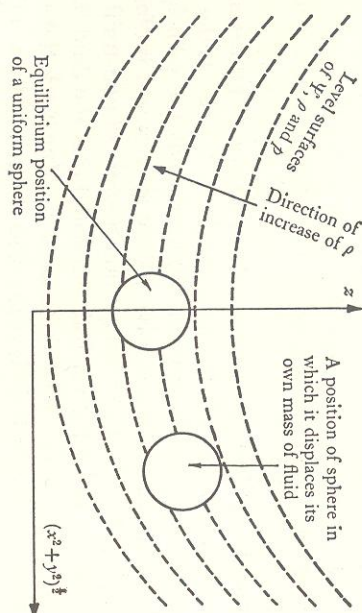

Figure 1.4.1. Non-uniform fluid at rest under the action of gravity and centrifugal force.

Fluid at rest under gravity

The case in which gravity is the only volume force acting on the fluid is both important and simple. Two extreme situations may be distinguished. In the first one, the mass of fluid concerned is large and isolated so that the gravitational attraction of other parts of the fluid provides the volume force on any element of the fluid, as in the case of a gaseous star. At the other extreme, the mass of fluid concerned is much smaller than that of neigh-bouring matter and the gravitational field is approximately uniform over the region occupied by the fluid.

In the case of a self-gravitating fluid, we have $\mathbf{F} = -\nabla\Psi$, where the gravitational potential Ψ is related to the distribution of density by the equation

$$\nabla^2\Psi = 4\pi G\rho, \qquad (1.4.6)$$

G being the constant of gravitation. On combining (1.4.6) with equation (1.4.3) for the pressure in a fluid at rest, we obtain

$$\nabla \cdot \left(\frac{\nabla p}{\rho}\right) = -4\pi G\rho. \qquad (1.4.7)$$

It is also necessary, as found earlier, that the level-surfaces of Ψ, ρ and p coincide. On expressing the differential operator in (1.4.7) in terms of curvi-linear co-ordinates (not necessarily orthogonal) such that the level-surfaces

of ρ coincide with one set of parametric surfaces, we see that the kinds of solution are severely restricted. Rigorous enumeration of the solutions is difficult, but the only possibilities seem to be solutions in which ρ and p are functions only of (i) one co-ordinate of a rectilinear system, or (ii) the radial co-ordinate of a cylindrical polar system, or (iii) the radial co-ordinate r of a spherical polar system, corresponding to symmetrical 'stars' in one, two or three dimensions.

In the last case, describing a spherically symmetrical distribution of density and pressure, (1.4.7) becomes

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi Gr^2\rho, \qquad (1.4.8)$$

and further progress cannot be made without information about the distri-bution of density. In real stars the density is in general not a function of p alone, but solutions of (1.4.8) corresponding to an assumed simple relation-ship between ρ and p are sometimes useful for comparison with more complicated models. If we assume for instance that

$$p \propto \rho^{1+1/n} \quad (n > 0),$$

it is possible to integrate (1.4.8) numerically for any value of n. Two analytical and representative solutions are also available. When $n = 0$, corresponding to a fluid of uniform density, ρ_0 say, we have

$$p = \tfrac{2}{3}\pi G\rho_0^2(a^2 - r^2),$$

where $r = a$ may be interpreted as the outer boundary of the star. When $n = 5$, it may be verified that

$$p = C\rho^{\frac{6}{5}} = \frac{27a^3C^{\frac{5}{2}}}{(2\pi G)^{\frac{3}{2}}(a^2+r^2)^3},$$

the pressure and density here are non-zero for all r and there is no definite outer boundary, but the total mass of the star is finite.

In the case of a uniform body force due to gravity, we have

$$\mathbf{F} = \mathbf{g}\,(= \text{const.}), \quad \Psi = -\mathbf{g}\cdot\mathbf{x}, \qquad (1.4.9)$$

and the equation for the pressure in a fluid at rest is

$$\nabla p = \rho\mathbf{g}. \qquad (1.4.10)$$

The three functions Ψ, ρ and p are constant on each horizontal plane normal to $\mathbf{g}$, and hence depend only on $\mathbf{g}\cdot\mathbf{x}$. If we choose the z-axis of a rectilinear co-ordinate system to be vertical (positive upwards) so that $\mathbf{g}\cdot\mathbf{x} = -gz$, (1.4.10) becomes

$$dp/dz = -g\rho(z). \qquad (1.4.11)$$

Again this is as much as we can deduce from the condition of mechanical equilibrium alone.