

When the fluid is of uniform density, we obtain from (1.4.11) the linear relation between pressure and height well-known in the study of hydrostatics:

$$p = p_0 - \rho g z. \quad (1.4.12)$$

In the case of the earth's atmosphere,  $\rho$  decreases with decrease of the pressure owing to the compressibility of the air, although thermal effects are usually present and no single functional relation between  $\rho$  and  $p$  is adequate. As a crude approximation one may put

$$p/\rho = \text{const.}, \quad = gH \text{ say,}$$

corresponding to Boyle's law for a perfect gas of uniform temperature and constitution (§ 1.7). The pressure in an atmosphere for which this relation holds is found from (1.4.11) to be

$$p = p_0 e^{-z/H},$$

where  $p_0$  is the pressure at ground level,  $z = 0$ . Thus both  $p$  and  $\rho$  diminish by a factor  $e^{-1}$  over a height interval  $H$ , and the constant  $H$  may be termed the 'scale-height' of the atmosphere. For air at  $0^\circ\text{C}$ ,  $H = 8.0\text{ km}$ . When the temperature is not uniform,  $p/\rho g$  may still be regarded as a local scale-height. Observed average values of the pressure, density and temperature at different heights in the atmosphere will be found in appendix 1 (b).

### Exercises

1. A closed vessel full of water is rotating with constant angular velocity  $\Omega$  about a horizontal axis. Show that the surfaces of equal pressure are circular cylinders whose common axis is at a height  $g/\Omega^2$  above the axis of rotation.
2. Obtain an expression for the pressure at the centre of a self-gravitating spherical star of which the density at distance  $r$  from the centre is

$$\rho = \rho_0(1 - \beta r^2).$$

Show that if the mean density be twice the surface density, the pressure at the centre is greater, by a factor  $\frac{1}{3}$ , than if the star had uniform density with the same total mass.

## 1.5. Classical thermodynamics

In our subsequent discussion of the dynamics of fluids we shall need to make use of some of the concepts of classical thermodynamics and of the relations between various thermodynamic quantities, such as temperature and internal energy. Classical thermodynamics is concerned, at any rate as the bulk of the subject stands, with equilibrium states of uniform matter, that is, with states in which all local mechanical, physical and thermal quantities are virtually independent of both position and time. Thermodynamical results may be applied directly to fluids at rest when their properties are uniform. Comparatively little is known of the thermodynamics

of non-equilibrium states. However, observation shows that results for equilibrium states are approximately valid for the non-equilibrium non-uniform states common in practical fluid dynamics; large though the departures from equilibrium in a moving fluid may appear to be, they are apparently small in their effect on thermodynamical relationships.

The purpose of this section is to recapitulate briefly the laws and results of equilibrium thermodynamics and to set down for future reference the relations that will be needed later. For a proper account of the fundamentals of the subject the reader should refer to one of the many text-books available.†

The concepts of thermodynamics are helpful to the student of fluid mechanics for the additional reason that in both subjects the objective is a set of results which apply to matter as generally as possible, without regard for the different molecular properties and mechanisms at work. Additional results may of course be obtained by taking into account any known molecular properties of a fluid, as proves to be possible for certain gases with the aid of kinetic theory (see § 1.7).

It is taken as a fact of experience that the state of a given mass of fluid in equilibrium (the word being used here and later to imply spatial as well as temporal uniformity) under the simplest possible conditions is specified uniquely by two parameters, which for convenience may be chosen as the specific volume  $v$  ( $= 1/\rho$ , where  $\rho$  is the density) and the pressure  $p$  as defined above. All other quantities describing the state of the fluid are thus functions of these two *parameters of state*. One of the most important of these quantities is the temperature. A mass of fluid in equilibrium has the same temperature as a test mass of fluid also in equilibrium if the two masses remain in equilibrium when placed in thermal contact (that is, when separated only by a wall allowing transmission of heat); and the second law of thermodynamics provides an absolute measure of the temperature of a fluid, as we shall note later. The relation between the temperature  $T$  and the two parameters of state, which we may write as

$$f(p, v, T) = 0, \quad (1.5.1)$$

thereby exhibiting formally the arbitrariness of the choice of the two parameters of state, is called an *equation of state*. For every quantity like temperature which describes the fluid, but excluding the two parameters of state of course, there is an equation of state.

Another important quantity describing the state of the fluid is the *internal energy* per unit mass,  $E$  say.‡ Work and heat are regarded as equivalent forms

† See, for instance, *Classical Thermodynamics*, by A. B. Pippard (Cambridge University Press, 1957).

‡ The usual practice in the literature of thermodynamics is to use a capital letter for the total amount of some extensive quantity like internal energy in the system under consideration, and a small letter for the amount per unit mass. Introduction of the latter quantity alone is sufficient in fluid dynamics, and the use of a capital letter for it is conventional.