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Teaching Faraday's law of electromagnetic induction in an introductory physics course

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Teaching Faraday's law of electromagnetic induction in introductory physics courses is challenging. We discuss some inaccuracies in describing a moving conductor in the context of electromagnetic induction. Among them is the use of the ambiguous term "area change" and the unclear relation between Faraday's law and Maxwell's equation for the electric field circulation. We advocate the use of an expression for Faraday's law that shows explicitly the contribution of the time variation of the magnetic field and the action of the Lorentz force, which are usually taught separately. This expression may help students' understanding of Faraday's law and lead to improved problem solving skills. © *2006 American Association of Physics Teachers.*
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I. INTRODUCTION

Faraday's law of electromagnetic induction is given by

$$\mathcal{E} = -\frac{d\Phi}{dt}, \tag{1a}$$

where $\mathcal{E}$ represents the electromotive force (emf) induced in a circuit and Φ is the magnetic flux through the circuit of area A,

$$\Phi = \int\int_A \mathbf{B}\cdot d\mathbf{A}. \tag{1b}$$

Although Eq. (1) describes simple laboratory settings, it presents a conceptual challenge for students and teachers of introductory physics. We emphasize that the integral form of Faraday's law in Eq. (1) includes all cases, including transformer emf (Ref. 1) and motional emf. In the latter case, Faraday's law includes not only closed circuits but also open circuits and those with moving segments (segments in motion relative to the other parts of the circuit), such as the Faraday disc and unipolar generator.

In this paper we comment on some remarks given in the Feynman Lectures[2] and show that the integral form of Faraday's law explains the cases of motional emf that were presented as problematic. Faraday's disc was mentioned as an example of the failure of the "flux rule," in which the emf of induction is created despite an "unchanged circuit." Two rotating plates, touching at a point and creating a closed circuit located in a magnetic field, was given as an example of the creation of an insignificant emf following a big change of the linked magnetic flux.

Faraday's law provides a good opportunity to illustrate Einstein's relativistic perspective on electromagnetic induction.[3] In connection with motional emf, the idea of area change and change of orientation used in many textbooks,[4] should be refined to reduce confusion. We illustrate our discussion with several examples that might be useful in teaching electromagnetic induction.

II. FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Most introductory university-level texts present electromagnetic induction starting with transformer emf in geometrically linear[5] circuits.[6] Equation (1) is then presented. Motional emf is considered subsequently, and it is shown that Eq. (1) also accounts for it. Alternatively, we can consider a change of magnetic flux through a conducting loop (see Fig. 1) in the inertial frame of reference where the circuit is moving. The complete derivative of the magnetic flux through the surface A is

$$\frac{d\Phi}{dt} = \left(\frac{\partial\Phi}{\partial t}\right)_{\mathbf{v}=0} + \left(\frac{\partial\Phi}{\partial t}\right)_{\mathbf{B}=\text{const}}. \tag{2}$$

In standard notation (see Appendix):

$$\frac{d\Phi}{dt} = \int\int_A \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{A} - \oint_L [\mathbf{v}\times\mathbf{B}]\cdot d\mathbf{L}. \tag{3}$$

By using the form of Faraday's law in Eq. (1), we obtain[7]

$$\mathcal{E} = -\int\int_A \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{A} + \oint_L [\mathbf{v}\times\mathbf{B}]\cdot d\mathbf{L}. \tag{4}$$

The form of the emf in Eq. (4) relates two phenomena. The first term,

$$\mathcal{E}_{\text{transformer}} = -\int\int_A \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{A}, \tag{5}$$

accounts for the motionless case of transformer emf, termed by Faraday "volta-electric induction"[8] (note the partial derivative) and corresponds (after stating the validity for any path of integration) to Maxwell's equation for the curl of the electric field,

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{6}$$

The second term,

$$\mathcal{E}_{\text{motional}} = \oint_L [\mathbf{v}\times\mathbf{B}]\cdot d\mathbf{L}, \tag{7}$$

represents motional emf, termed by Faraday as "magneto-electric induction,"[9] and arises from the definition of the complete derivative and the Maxwell equation, $\mathbf{B}\oiint d\mathbf{A}=0$. It is immediately recognized that the integrand of Eq. (7) gives the Lorentz force

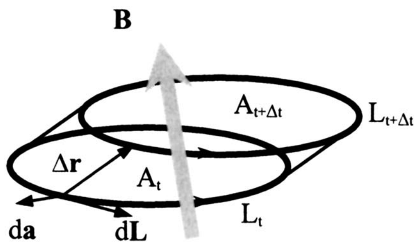

Fig. 1. A closed loop in a magnetic field **B**. L_t and $L_{t+\Delta t}$ represent two positions of the loop with areas A_t and $A_{t+\Delta t}$, respectively.

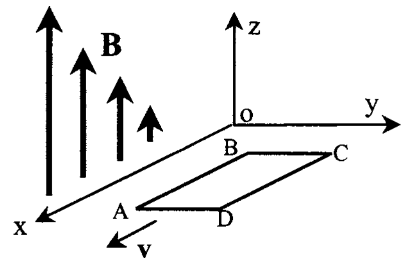

Fig. 2. A rectangular conducting loop *ABCD* (sides *b* and *d*) moves along the *x* axis in a plane through a magnetic field with a linearly increasing intensity **B** $(0,0,B_0x)$.

$$\mathbf{F}_m = q[\mathbf{v} \times \mathbf{B}], \tag{8}$$

acting on a charge q that moves with the speed of the circuit. Thus, the Lorentz force naturally follows from the definition of the rate of flux change $d\Phi/dt$ as a complete derivative.

There are cases where Faraday's law of induction is applied not to circuits, but to extended bodies, such as a Faraday disc. In this case, according to the derivation in the Appendix, the path of integration in the Lorentz term should reflect the motion of the material of the conductor that closes the circuit, which might be in motion relative to other parts of the circuit.[10] This understanding is important for calculations of $(\partial\Phi/\partial t)_{\mathbf{B}=\text{const}}$ as shown in the following examples.

In Ref. 2 it is suggested that the integral form of Faraday's law, Eq. (1), can fail to account for electromagnetic induction. Two examples of such failures are given, and it is stated that these exceptions demonstrate the superiority of the differential laws, Eq. (6) and Eq. (8), over the integral form. This point is not mentioned by other introductory physics texts.[11] However, Faraday's law in its integral form is indispensable in introductory physics courses in situations such as the electric generator for which Eq. (6) is obviously not practical. Also, partial derivative equations such as in Eq. (6) are beyond the scope of the introductory course and the equivalence of Faraday's law with Maxwell's equation regarding the circulation of the electric field requires knowledge of field transformations.[12]

In addition, the composite nature of Faraday's law should attract the attention of the student:

> "We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of *two different phenomena*. Usually such a beautiful generalization is found to stem from a single *deep underlying principle*. Nevertheless, in this case there does not appear any such profound implication. We have to understand the rule as the combined effects of two quite separate phenomena."[13] (Italics in the original.)

By considering the two different contributions to the electromagnetic induction, a teacher can discuss why this interpretation was challenged by Einstein in his seminal paper of 1905.[3]

III. A RELATIVISTIC PERSPECTIVE IN THE INTRODUCTORY COURSE

Electromagnetism is not consistent with classical Newtonian mechanics, but we can present relativistic concepts in a qualitatively correct way even in an introductory course.[14,15] In this context it is important to demonstrate by a simple example that the distinction between the two types of emf's is not absolute. The motional emf's detected by an inertial observer may appear as a transformer emf to another observer. It is sufficient to use approximations valid for $v/c \ll 1$ in which the theory of relativity allows the magnetic field to be observer independent.[16]

Consider the rectangular conducting loop *ABCD* (sides *b* and *d*) (see Fig. 2) moving in the *x* direction with a constant velocity **v** through the magnetic field **B**, as observed in the laboratory reference frame S_L. The magnetic field increases linearly in the z direction with magnitude $\mathbf{B}_L=(0,0,B_0x)$ where B_0 is a constant. The location of the front and rear sides of the loop at time t is $x_1=vt$ and $x_2=b+vt$, respectively. The magnitude of the magnetic field at these locations is $\mathbf{B}_{L,1}=(0,0,B_0vt)$ and $\mathbf{B}_{L,2}=(0,0,B_0b+B_0vt)$. The observer S_F moving with the loop measures the magnetic field through the loop which changes with time.

The two observers explain the phenomenon of electromagnetic induction as follows. Observer S_L observes the moving conducting frame and accounts for the resulting motional emf using Eq. (7) and finds

$$\mathcal{E}_{\text{motional}} = \oint_{ABCD} [\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{s} = -vB_0A, \tag{9}$$

where $A=bd$, the area of the loop *ABCD*. Observer S_F sees no motion, records the different magnetic field $B_z=B_0(x_F+vt)$, and accounts for the transformer emf using Eq. (5):

$$\mathcal{E}_{\text{transformer}} = -\int\int_A \frac{\partial \mathbf{B}_E}{\partial t} \cdot d\mathbf{A} = -vB_0A, \tag{10}$$

because $\partial\mathbf{B}_F/\partial t=(0,0,B_0v)$. The equality of Eqs. (9) and (10) demonstrates the relativity of the emf as either transformer (the interpretation of S_F) or motional (the interpretation of S_L).

This example unifies the two types of electromagnetic induction similar to the unification of the electric and magnetic fields when considering the force exerted on an electric charge as interpreted by different inertial observers. Just as the identification of the force as magnetic or electric changes with a change in reference frame, the identification of the type of electromagnetic induction can change while preserving the total emf as an invariant. For the observer for whom the frame is motionless, the only contribution is

$$\mathcal{E}_{\text{transformer}} = \oint_L \mathbf{E}_F \cdot d\mathbf{L} = -\left(\frac{\partial \Phi}{\partial t}\right)_{v=0}. \tag{11}$$

Because of the weak approximation $(v/c \ll 1)$ the electric fields, as observed by the two observers are related by[17]

$$\mathbf{E}_F = \mathbf{E}_L + \mathbf{v} \times \mathbf{B}, \tag{12}$$

and the emf in Eq. (11) can be expressed by using the fields in the laboratory frame:

$$\mathcal{E}_{\text{transformer}} = \oint_L (\mathbf{E}_L + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}, \tag{13}$$

which leads to Eq. (4). Here the students will arrive at the understanding of the "single deep underlying principle" which unifies the "two different phenomena," the relativistic nature of the electromagnetic field.

IV. AREA AND PATH CHOICE

The use of relativity is optional. Whether or not the treatment is relativistic, there are other conceptual problems with the application of Faraday's law in the form of Eq. (1). The form in Eq. (4) has pedagogical advantages that reinforce the discussion in introductory texts. It is common to consider a uniform field **B** and derive the following:[18]

$$\mathcal{E}_{\text{induction}} = -\frac{d\Phi}{dt} = -\frac{d(\mathbf{B}\cdot\mathbf{A})}{dt} = -\mathbf{A}\cdot\frac{d\mathbf{B}}{dt} - \mathbf{B}\cdot\frac{d\mathbf{A}}{dt}, \tag{14}$$

where **A** represents the area enclosed by a circuit, and the second term represents the change of area and the change of orientation. To clarify the meaning of these terms, we relate Eqs. (4) and (14). For a uniform magnetic field the second term of Eq. (4) becomes

$$\mathcal{E}_{\text{motional}} = \oint_L [\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{L} = -B\frac{dA_m}{dt}, \tag{15}$$

with

$$\frac{dA_m}{dt} \equiv \oint_L v_\perp dL. \tag{16}$$

The definition of A_m is not unique and only the change of area dA_m is physically meaningful ($v_\perp$ represents the velocity component perpendicular to the element of the moving conductor and the subscript m in A_m emphasizes the relation to the charges in motion). The integration in Eq. (15) accumulates the effect of the Lorentz force and Eq. (16) reflects the area swept out by the movement of the points on the conductor.[19] In successful uses of area change in the determination of the emf, the area is given by Eq. (16); in contrast, if the area is not given by Eq. (16), Faraday's law appears to fail as we will show in the following examples.

To explain Faraday's disc generator (a conducting disc rotating between the poles of a permanent magnet with the disc at right angles to the magnetic field), we can apply Eq. (7) and sum the action of the magnetic force on the charges moving with the disc and located in the segment oc (see Fig. 3) closing the circuit:

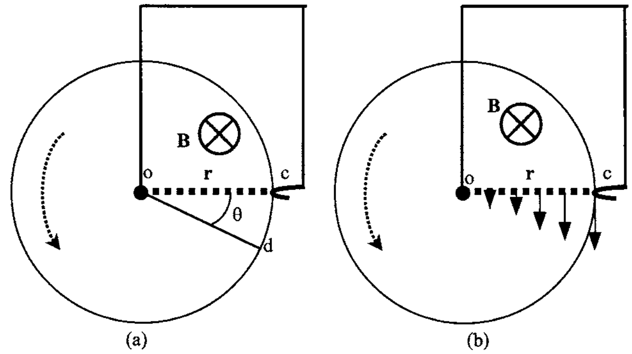

Fig. 3. The external part of the Faraday disc generator is connected to the terminals o and c. (a) The induced emf in the rotating disc is related to the rate of change of the area ocd (swept out by the moving radius) giving the change of the circuit, including the external part which does not change. (b) The same emf can be calculated using the velocities of the charges along the radius r (drift velocities are neglected).

$$\mathcal{E}_{\text{motional}} = \oint_L [\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{L} = \int_{OC} \omega r B dr = \frac{1}{2}\omega R^2 B, \tag{17}$$

where ω is the angular velocity and R is the radius of the disk. Alternatively, we can obtain the same result from the area change of the disc sector ocd [see Fig. 3(a)].[20] The result obtained in this way is consistent with the definition of A_m [see Fig. 3(b)] suggesting the appropriate area to be addressed. Indeed, the flux change is not obvious, whereas the explanation of the emf by the Lorentz force is straightforward.

In this regard it is interesting and educational to discuss with students the reasoning used in *The Feynman Lectures*:

> "As disc rotates, the "circuit," in the sense of the place in space where the currents are, is always the same … Although the flux trough the "circuit" is constant, there is still an emf … Clearly, here is a case where $\mathbf{v}\times\mathbf{B}$ force in the moving disc gives rise to an emf which cannot be equated to a change of flux."[21]

The last sentence of the quote contradicts the expression for the change of flux given in Eq. (3).

The same approach resolves the puzzle of the rotating plates discussed in *The Feynman Lectures*. Two metal plates with slightly curved edges (see Fig. 4) are placed in a uni-

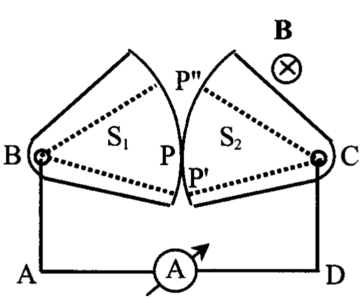

Fig. 4. Feynman's rotating conducting plates in a magnetic field **B**. The circuit $ABPCD$ is closed through the point of contact P which is moving from P' to P''. The area change of the circuit is shown by the dotted lines (sectors S_1 and S_2).

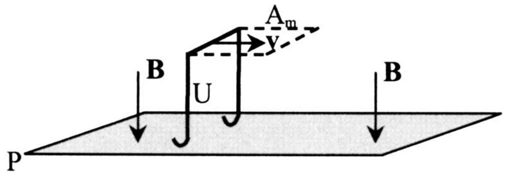

Fig. 5. The conducting frame U slides over the conducting plate P in a magnetic field **B**. An induction emf causes current in the circuit. The area change A_m, shown by the dashed line, is relevant for the emf of induction, even though the area of the circuit does not change.

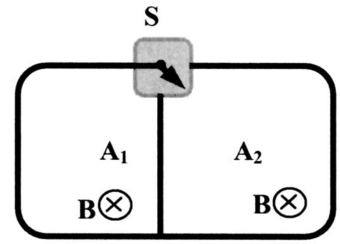

Fig. 6. A two-loop circuit in a magnetic field **B**. The switch S can change the area of the closed circuit from A_1 to A_1+A_2 and back at any frequency, but almost no emf is induced in the circuit.

form magnetic field perpendicular to their surfaces. The plates make contact at a single point P comprising a complete circuit $ABPCD$. When the plates are rocked, the point of contact moves from P' to P''. We can imagine the circuit completed through the plates on the dashed lines connecting points B, P, and C, and might consider the area change of the circuit as caused by the movement of these lines: the area S_1+S_2 (between the dashed lines to the positions P' and P''). It is stated that there is "a somewhat unusual situation in which the flux through a circuit (again in the sense of the place where the current is) changes but where there is no emf."[21] However, although the area S_1+S_2 does represent the change of the circuit area, it does not reflect the velocities of the material points of the plates, as required by Eq. (3) for the flux change. This mismatch occurs because the point P is not a physical object. In fact, the material points of the plates move in the magnetic field with smaller velocities (and in the opposite direction) than does point P.[22] This difference explains the fact stated by the authors of Ref. 2 that the Lorentz force causing the emf causes it to be small. Strictly speaking, as in Faraday's disc, the induced currents in the rotating plates (as in any extended conductor) cannot be reduced to a line of current and any treatment introducing a straight path of integration remains approximate. However, the qualitative argument explaining the low magnitude of the emf suffices for an introductory course.

The treatment of open and composite circuits using Eq. (1) might challenge students who look for an area change. To find the latter they should create an imaginary area that reflects the movement. The valid choice is provided only by A_m as defined by Eq. (16). This area may have nothing to do with the area of the circuit in which the electrical current is induced (see Fig. 5).

We note another important point about the path of integration. Unlike many texts, in the Berkeley series we find the following definition of Faraday's law:

> "If C is some closed curve, stationary in coordinates x,y,z, if S is a surface spanning C, and if $B(x,y,z,t)$ is the magnetic field measured in x,y,z at any time t, then … "[23] [Eq. (1) follows].

The terms *curve* and *stationary* are seldom used by other authors and were added to justify the statement made later in the text that the integral form of Faraday's law is equivalent to the differential form of Maxwell's equation (6). For such an equivalence to be valid, the path of integration is arbitrary and stationary.

Obviously, it is preferable to explain the relation of Maxwell's equation to the curl of the electric field in its integral form:

$$\oint_L \mathbf{E}\cdot d\mathbf{L} = -\int\int_A \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{A}, \qquad (18)$$

with the first term in Eq. (4). As mentioned, a complete understanding of the relation between Eq. (1) and Eq. (18) can be obtained only by considering special relativity, which introduces field transformations that show, using Eqs. (11) and (12), how Maxwell's equation for a stationary observer, Eq. (18), produces Faraday's law, Eq. (4), which includes the motion of a conducting loop.

V. EXAMPLES

In the following we give some simple but conceptually rich examples that can be usefully analyzed by students.

(1) Consider the circuit of Fig. 6. Switch S establishes a closed circuit either of area A_1 or A_1+A_2, without the significant movement of a conductor during the change of the position of the switch. Although the rate of change of the circuit area threaded by a magnetic field can be arbitrarily large, practically no emf is induced because such a change is not accompanied by a corresponding movement of a conductor. Equation (4) gives a null result, whereas finding the area change might lead to confusion.

(2) An electromagnetic generator is often explained by the change in the circuit orientation of the magnetic field, as implied by Eq. (14). Figure 7 shows two arrangements involving identical changes of circuit orientation which lead to the creation of different emfs. This asymmetry is caused by the difference in the movement of the charge carriers (differ-

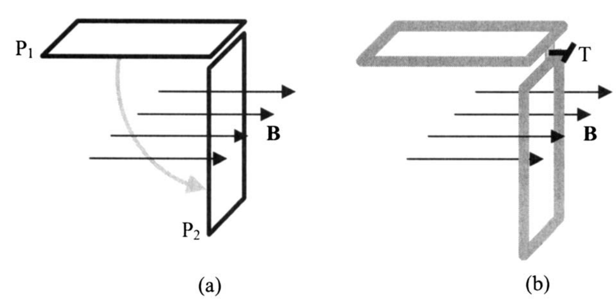

Fig. 7. (a) A solid conducting frame placed in a horizontal magnetic field **B** changing its orientation from horizontal, P_1 to vertical P_2. As a result, emf of induction is created in the circuit. (b) Two rectangular glass tubes are placed at right angles in a horizontal magnetic field **B**. The flow of conducting fluid from the horizontal tube to the vertical tube causes the change of the circuit orientation as in (a), but there is no emf of induction (neglecting Hall voltage across the liquid).

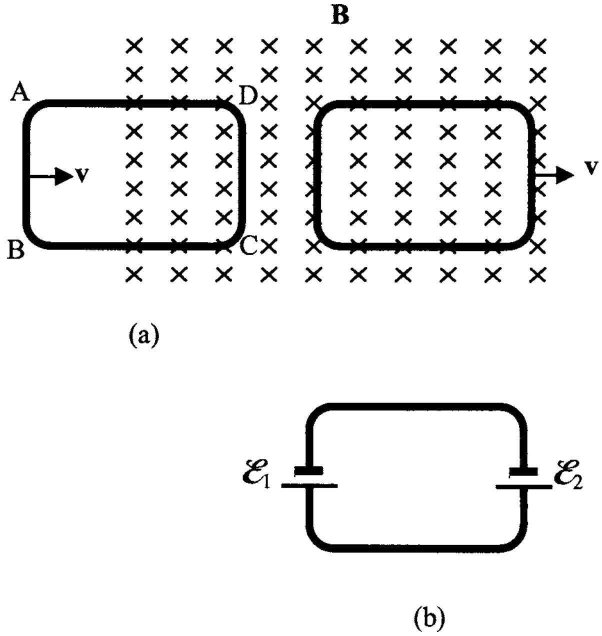

Fig. 8. (a) A closed conducting loop *ABCD* moves with velocity **v** toward and through the area with a uniform magnetic field **B**. In the left position, motional emf is created along the front side *CD*, causing electrical current in the loop. In the second position motional emf is created in the front and rear sides. There is a potential difference in the loop between the top and bottom sides, but no current (neglecting the transients) because the net emf around the loop is zero. (b) Electrical circuit equivalent to the frame *ABCD* in its second position, entirely within the magnetic field.

ent A_m) in the two cases. Equation (4) yields the correct result in both cases, whereas considering the change in the circuit orientation might lead to confusion.

(3) A conducting frame is pulled at a constant velocity through a magnetic field localized in a rectangular area [see Fig. 8(a)]. The usual explanation states that a current is induced and thus there is an induced emf in the loop following the flux change through the frame as long as the frame enters into the magnetic field (or leaves it). When the entire frame is in the area, the area change argument implies that there is no current. Although $\oint_L \mathbf{E}\cdot d\mathbf{L}=0$, an induced emf is created in the frame (the Hall effect). The failure to recognize the nonzero effect of the induced voltage in the frame, thus missing the important physical phenomenon, can be prevented if we use Eq. (4) instead and integrate around the circuit. Such a treatment can reveal the equal emfs created in the front (*CD*) and in the rear (*AB*) sides of the frame. This situation is equivalent to a circuit with two identical batteries connected with opposite polarity. Although no current is produced, there are voltage differences in parts of the circuit [see Fig. 8(b)].

(4) An especially interesting case of a commutating magnet was presented by Cohn[24] and might be considered to be violating Faraday's law. The circuit consists of a pair of spring clips and moves across the body of a magnet [see Fig. 9(a)]. When the loop escapes the magnet, the clips rub over the magnet and the body of the magnet becomes a part of the circuit. The use of Faraday's law in the form of Eq. (1) is misleading and predicts a nonzero emf, because the magnetic flux through the loop decreases. However, no emf is created. Faraday's law in the form of Eq. (4) is again more useful. The magnetic field **B** does not change in time and no charge carrier moves through the magnetic field. The segment of the magnet closing the circuit, which changes at each instant, is at rest relative to the magnet.[25] Note that if the loop were pulled out of the magnet between the magnetic poles [Fig. 9(b)], a motional emf would be created in the loop.

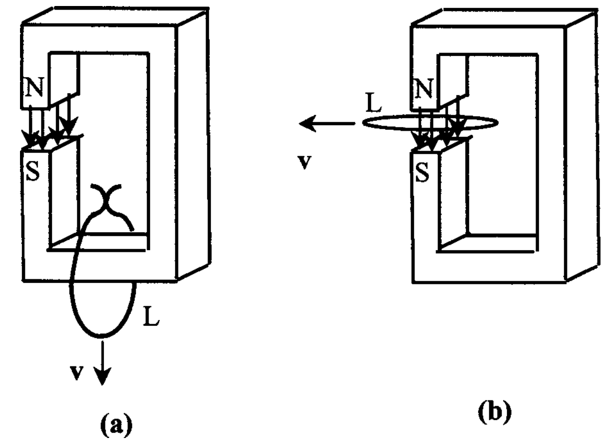

Fig. 9. (a) Clips allow a conducting loop *L* to escape the magnet. During the escape, the body of the magnet (a conductor) becomes a segment of the closed circuit. Although the flux of the magnetic field through the circuit *L* decreases, no induction emf is created. (b) The conducting loop *L* escapes the magnet between the magnet's poles. Induction emf is created in the loop.

(5) Consider an isolated conducting wire in the form of the twisted loop placed in a homogeneous magnetic field **B** at right angles to the plane of the circuit (see Fig. 10). Any temporal change of the intensity of the magnetic field will not cause the creation of an induction emf in the circuit in contrast to reasoning solely on Faraday's law in the form of Eq. (1). An application of Eq. (4), especially when considering the area integral, can lead students to learn that the circuit is, equivalent to a simple circuit of the kind shown in Fig. 8(b), providing a null result due to the conflict in the polarities of the sources incorporated in each half-loop. This reasoning is well known to practitioners needing to produce noninductive coils.

VI. IMPLICATIONS FOR TEACHING

We recommend that Eq. (4) be used in introductory courses to clarify the meaning of Faraday's law of electromagnetic induction, which is usually initially expressed in the form of Eq. (1). Clarifying to the students the expression of the magnetic flux derivative, Eq. (3), introduces two types of emfs, explicit in Eq. (4): transformer emf (corresponding to Maxwell's equation of electric field circulation) and motional emf (caused by the Lorentz force). It is desirable to discuss by a simple example that the distinction between the

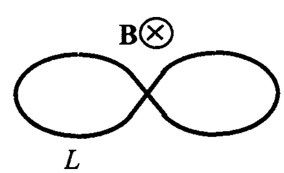

Fig. 10. A twisted conducting circuit *L* is placed in a magnetic field **B**. Wires cross without electrical contact.

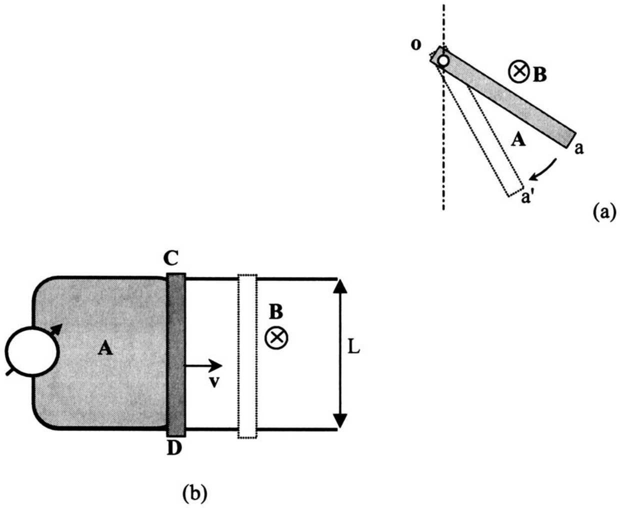

Fig. 11. (a) A conducting rod L swings around the axis O in a magnetic field $\mathbf{B}$. (b) A conducting rod CD slides with velocity $\mathbf{v}$ over the stationary U-shaped conductor in a magnetic field $\mathbf{B}$.

two contributions is not absolute and varies for different inertial observers. This approach may reveal to the students the deep meaning of Feynman's words:

> "The flux rule … applies whether the flux changes because the field changes or because the circuit moves (or both). The two possibilities—'circuit move' or 'field changes'—are not distinguished in the statement of the rule. Yet in our explanation of the rule we have used two completely distinct laws for the two cases."[26]

Introducing the area A_m in Eq. (16) can guide the appropriate choice of the path of integration and correctly incorporates the relevant motion of the material segments of the loop. Such an approach can be useful, especially when considering open circuits and looking for the area for the magnetic flux "passing through" a circuit [see Fig. 11(a)]. The same approach helps to address compound circuits incorporating elements in relative motion and circuits with a more complex topology.

If the teacher has a choice of reasoning either by area (orientation) change or by the Lorentz force, it is important to note that the latter is more fundamental. For example, the example of a rod CD sliding on a U-shape conductor in a magnetic field [see Fig. 11(b)] is often explained by using an area change argument.[27] The explanation using the Lorentz force might be mentioned as secondary, or even as an alternative to Faraday's law.[28] Students' intuition regarding the creation of motional emf could benefit from understanding the Faraday-Maxwell metaphor of cutting lines of magnetic force as the cause for electromagnetic induction.[29] Maxwell used it to address a carriage sliding along the rails through the magnetic field of the Earth, with its wheels and axle comprising a closed circuit.[30]

Emphasizing subtleties such as the distinction between complete and partial derivatives in the presentation of the law of induction, explaining the choice of terms for the path of integration (loop, contour, circuit, path), careful elaboration of the relation between Maxwell's equation, in the form of Eqs. (6) and (18), and Faraday's law, Eq. (1), can all be useful for students' understanding of electromagnetic induction.

ACKNOWLEDGMENT

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APPENDIX: DERIVATION OF THE EXPRESSION FOR THE FLUX CHANGE

Here we reproduce the derivation of Eq. (2) for the complete time derivative of the flux through a conducting circuit.[31] We consider a cylindrical surface created by a loop as moves from its position L_t to a $L_{t+\Delta t}$ in the space containing the magnetic field $\mathbf{B}$ (see Fig. 1). To first order, the rate of flux change results in

$$\frac{\Delta\Phi}{\Delta t}=\frac{\int\int_{A(t+\Delta t)}\mathbf{B}(t+\Delta t)d\mathbf{A}-\int\int_{A(t)}\mathbf{B}(t)d\mathbf{A}}{\Delta t}$$
$$=\frac{\int\int_{A(t+\Delta t)}\mathbf{B}(t)d\mathbf{A}-\int\int_{A(t)}\mathbf{B}(t)d\mathbf{A}}{\Delta t}$$
$$+\frac{\int\int_{A(t)}\frac{\partial\mathbf{B}(t)}{\partial t}\Delta t\cdot d\mathbf{A}}{\Delta t}=\left(\frac{\Delta\Phi}{\Delta t}\right)_1+\left(\frac{\Delta\Phi}{\Delta t}\right)_2. \tag{A1}$$

In the absence of sources for magnetic field we obtain

$$\oiint_A \mathbf{B}\cdot d\mathbf{A}=\int\int_{A(t+\Delta t)}\mathbf{B}\cdot d\mathbf{A}-\int\int_{A(t)}\mathbf{B}\cdot d\mathbf{A}$$
$$+\int\int_{\text{Side}}\mathbf{B}\cdot d\mathbf{A}=0. \tag{A2}$$

The first term of Eq. (A1) is

$$\left(\frac{\Delta\Phi}{\Delta t}\right)_1=\frac{1}{\Delta t}\left(\int\int_{A(t+\Delta t)}\mathbf{B}\cdot d\mathbf{A}-\int\int_{A(t)}\mathbf{B}\cdot d\mathbf{A}\right)$$
$$=-\frac{1}{\Delta t}\int\int_{\text{Side}}\mathbf{B}\cdot d\mathbf{A}. \tag{A3}$$

We can further develop the expression for the flux through the side surface using the vectors shown in Fig. 1 and $\Delta\mathbf{r}=\mathbf{v}\Delta t$:

$$\int\int_{\text{Side}}\mathbf{B}\cdot d\mathbf{A}=\oint_{\mathbf{L}}\mathbf{B}\cdot[d\mathbf{L}\times\Delta\mathbf{r}]=\oint_L\mathbf{B}\cdot[d\mathbf{L}\times\mathbf{v}]\Delta t$$
$$=\Delta t\oint_L[\mathbf{v}\times\mathbf{B}]\cdot d\mathbf{L}, \tag{A4}$$

and thus obtain

$$\left(\frac{\Delta\Phi}{\Delta t}\right)_1=-\oint_L[\mathbf{v}\times\mathbf{B}]\cdot d\mathbf{L}\rightarrow\left(\frac{\partial\Phi}{\partial t}\right)_{\mathbf{B}=\text{const}}. \tag{A5}$$

For the second term of Eq. (A1), we obtain within the same approximation,

$$\left(\frac{\Delta\Phi}{\Delta t}\right)_2 = \frac{\int\int_{A(t)} \frac{\partial \mathbf{B}(t)}{\partial(t)} \Delta t \cdot d\mathbf{A}}{\Delta t} = \int\int_{A(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \rightarrow \left(\frac{\partial\Phi}{\partial t}\right)_{\mathbf{v}=0}. \tag{A6}$$

Thus, Eqs. (A5) and (A6) yield the complete derivative of the magnetic flux in the form of Eq. (3).

This derivation uses only elementary calculus. Although this derivation employs a rigid loop, it demonstrates the origin of the resultant expression of Faraday's law and enables us to consider whether the result holds in more sophisticated cases of open, compound and twisted circuits.

[1] *Transformer* emf is a less common term denoting the induction emf caused by changes of intensity of the magnetic field. The *motional* emf is caused by the motion of a conductor in a magnetic field.

[2] R. Feynman, R. B. Leighton, and M. Sands, *Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. 2, pp. 17-1–17-1-3.

[3] A. Einstein, "On the electrodynamics of moving bodies," Ann. Phys. **17**, 891–921 (1905). English translation in A. Einstein, *The Principle of Relativity* (Dover, New York, 1952), pp. 37–65.

[4] We surveyed about thirty textbooks on university-level introductory physics course, all published in English during the last 10 years.

[5] By "geometrically linear" circuits we mean simple electrical circuits treated in introductory courses including elements such as simple resistors and coils, connected in series or parallel or otherwise.

[6] See, for example, H. D. Young and R. A. Freedman, *University Physics* (Addison-Wesley, San Francisco, 2004), pp. 1106 and 1120.

[7] In an algebra-based course, we can express Faraday's law using finite time steps: $\mathcal{E}_{\text{induction}}=-(\Delta B/\Delta t)A-vBl\sin(v,B)$. See, for example, J. Touger, *Introductory Physics* (Wiley, New York, 2006).

[8] M. Faraday, *Experimental Researches in Electricity* (Britannica Great Books, Chicago, 1832/1978), First Series, pp. 265–285.

[9] See Ref. 7, Second Series, pp. 286–302.

[10] I. E. Tamm, *Fundamentals of the Theory of Electricity* (Mir, Moscow, 1979), Sec. 112.

[11] The scope of introductory physics courses varies among universities and usually does not include Maxwell's equations in differential form. Our treatment remains applicable because it addresses shortcomings in the application of the integral form of the law of induction.

[12] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960), Sec. 63.

[13] See Ref. 2, p. 17-2.

[14] R. Chabay and R. Sherwood, *Electric & Magnetic Interactions* (Wiley, New York, 1995).

[15] I. Galili and D. Kaplan, "Changing approach in teaching electromagnetism in a conceptually oriented introductory physics course," Am. J. Phys. **65**(7), 657–668 (1997).

[16] The weak relativistic approximation can be developed if force invariance is allowed. See for example, Ref. 15.

[17] The index F denotes the frame of reference of the moving rectangular frame.

[18] See, for example, H. Benson, *University Physics* (Wiley, New York, 1996).

[19] This definition justifies neglecting the drift velocities of the charges free to move.

[20] See, for example, Ref. 9.

[21] Reference 2, p. 17-3.

[22] This argument is qualitative. Similar to the velocities of the wheel of a moving car near the point of touching the road, which is not the velocity of the car relative to the road, the velocities of the material of the rocking plates are not equal to the velocity of the touching point P. See also F. Munlay, "Challenges to Faraday's flux rule," Am. J. Phys. **72**(12), 1478–1483 (2004).

[23] E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985), p. 272.

[24] G. I. Cohn, "Electromagnetic induction," Electr. Eng. **68**(5), 441–447 (1949) (thanks to Bruce Sherwood). P. J. Scanlon, R. N. Henriksen, and J. R. Allen, "Approaches to electromagnetic induction," Am. J. Phys. **37**(7), 698–708 (1969) ascribed an equivalent case to F. A. Kaempffer, *Elements of Physics* (Blaisdell, Waltham, MA, 1967), p. 164.

[25] This statement is valid only for translational movement. For rotation (unipolar generator) the argument about the absence of relative motion for the absence of emf does not hold (see Ref. 9).

[26] See Ref. 2, p. 17-1.

[27] See, for example, D. Halliday and R. Resnick, *Fundamentals of Physics*, 3rd ed. (Wiley, New York, 1988), p. 745.

[28] For example, "We can obtain the same relation [emf of induction] in another way without the use of Faraday's law," D. C. Giancoli, *Physics for Scientists and Engineers*, 2nd ed. (Prentice Hall, Englewood Cliffs, NJ, 1988), p. 677.

[29] This account is equivalent to using the Lorentz force, but needs additional and somewhat artificial assumptions for the case of magnet rotation (as in unipolar induction).

[30] J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Dover, New York, 1954), Vol. 2, Chap. III, pp. 179–189.

[31] For example, a similar derivation can be found in W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1962), pp. 160–163.