The force on a magnetic dipole

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The classical magnetic force on a magnetic dipole depends upon the model for the dipole. The usual electric current loop model for a magnetic dipole leads to the force $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$ on a magnetic dipole $\mathbf{m}$ in a magnetic field $\mathbf{B}$. The separated magnetic charge model for a magnetic dipole leads to the force $\mathbf{F} = (\mathbf{m}\cdot\nabla)\mathbf{B}$ on a magnetic dipole. The latter expression is analogous to the force experienced by an electric dipole in an electric field. Here, some elementary examples are given where the force expressions yield entirely different forces on a magnetic dipole. Electromagnetism textbooks usually do not emphasize the difference between these force expressions; however, occasionally the difference is important for understanding experimental results. In the 1930s and 1940s the difference in force expressions was involved in a determination of the nature of the neutron dipole moment. At present, in the 1980s, the difference in the force expressions is central to a controversy over an experiment to test the proposed Aharonov–Casher effect.

I. INTRODUCTION

It is emphasized in electromagnetism textbooks that many of the results of classical magnetostatics take the same form as in electrostatics.[1] For example, the magnetic field of a magnetic dipole takes the same form as the electric field of an electric dipole, even though the models for the dipoles are not at all analogous. Continuing in this vein, the magnetic force on a magnetic dipole is often treated[1] as though it took the same form as the electric force on an electric dipole. However, the force expressions are actually quite different. Recently, a controversy in the research literature related to an ongoing experiment has arisen precisely because of a failure to distinguish clearly between these two force expressions for a magnetic dipole. Accordingly, in this article, we wish to discuss this difference in force expressions. We will consider two different magnetic dipole models leading to two different classical force expressions for magnetic dipoles.

II. TWO MODELS FOR MAGNETIC DIPOLES

There are two natural models for a magnetic dipole. The first, and usual model, is the small electric current loop description that appears in all the classical electromagnetism textbooks. Here, in simplest form, a small current loop of area $\mathbf{A}$ and current I gives a magnetic dipole $\mathbf{m} = \mathbf{A}I$. The orientation of the dipole is given by the right-hand rule connected with the current flow.

The second model for a magnetic dipole is the magnetic charge analog of the usual electric dipole. Here, in simplest form, two magnetic monopoles $\pm g$ are separated by a small distance $\mathbf{l}$, giving a magnetic dipole moment $\mathbf{m} = g\mathbf{l}$. The displacement vector $\mathbf{l}$ points from the minus to the plus magnetic charge.

Both these models for magnetic dipoles lead to the same dipole magnetic field pattern

$$\mathbf{B} = (\mu_0/4\pi r^3)(3\hat{n}\hat{n}\cdot\mathbf{m} - \mathbf{m}), \quad r > 0, \tag{1}$$

far from the dipole or in the limit of vanishingly small spatial dimensions for the dipole. However, although the magnetic field pattern (1) is common to the two magnetic dipole models,[2] the force experienced by the magnetic dipole reflects the structure of the dipole, even in the point dipole limit. Corresponding to the two different models for magnetic dipoles there are two different forces on the dipoles. The derivations for the force expressions correspond to those usually given in classical electromagnetism texts for the electric current loop model of a magnetic dipole and for the separated charge description of an electric dipole.

III. FORCE ON THE ELECTRIC CURRENT MODEL FOR A MAGNETIC DIPOLE

The force on the usual magnetic dipole modeled as a distribution of electric current is derived in the textbook by Jackson.[3] (An alternative elementary derivation involving not an arbitrary current distribution but an infinitesimal rectangular current loop is suggested in some textbooks.[4]) The force on a small current distribution is obtained by expanding the external magnetic field in a Taylor series about the center of the distribution at $\mathbf{r} = \mathbf{a} = 0$,

$$\mathbf{F} = \int \mathbf{J}(\mathbf{r})\times\mathbf{B}(\mathbf{r})\,d^3r$$

$$= \int \mathbf{J}(\mathbf{r})\times[\mathbf{B}(\mathbf{a}) + (\mathbf{r}\cdot\nabla_\mathbf{a})\mathbf{B}(\mathbf{a}) + \cdots]_{\mathbf{a}=0}\,d^3r. \tag{2}$$

The notations $\nabla_\mathbf{a}$ and $\mathbf{B}(\mathbf{a})$ have been used to emphasize the independence of these expressions from the integral in $\mathbf{r}$.

For current distributions that do not extend to spatial infinity, it is possible to simplify the force expression (2). We use the identity

$$\nabla\cdot(fg\mathbf{J}) = g\mathbf{J}\cdot\nabla f + f\mathbf{J}\cdot\nabla g + fg\nabla\cdot\mathbf{J}, \tag{3}$$

the charge continuity equation

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}, \tag{4}$$

and the divergence theorem to obtain

$$0 = \int\left(f\mathbf{J}\times\nabla g + g\mathbf{J}\cdot\nabla f - fg\frac{\partial\rho}{\partial t}\right)d^3r. \tag{5}$$

We will assume that the electric charge distribution $\rho(\mathbf{r},t)$ of the current distribution vanishes in its own rest frame,

$\rho(\mathbf{r},t) = 0$. Then the expression (5) simplifies to

$$0 = \int (f\mathbf{J}\cdot\nabla g + g\mathbf{J}\cdot\nabla f)\, d^3r. \quad (6)$$

For $f = 1$ and $g = x_i$, this gives

$$\int J_i(\mathbf{r})\, d^3r = 0 \quad (7)$$

and, hence,

$$\left(\int \mathbf{J}(\mathbf{r})\, d^3r\right) \times \mathbf{B}(0) = 0, \quad (8)$$

the first term in the integral (2) vanishes.

The second term in the integral (2) involves

$$F_i = \sum_{j,k,l=1}^{3} \epsilon_{ijk} \left(\int J_j(\mathbf{r}) x_l\, d^3r\right) \left(\frac{\partial}{\partial a_l} B_k(\mathbf{a})\right)_{\mathbf{a}=0}, \quad (9)$$

where ϵ_{ijk} is the completely antisymmetric tensor on three indices with $\epsilon_{123} = 1$. Then Eq. (6) with $f = x_j$ and $g = x_l$ tells us

$$\int J_j x_l\, d^3r = -\int J_l x_j\, d^3r, \quad (10)$$

so that F_i can be rewritten as

$$\begin{aligned} F_i &= \sum_{j,k,l=1}^{3} \epsilon_{ijk} \left(\frac{1}{2}\int (x_l J_j - x_j J_l)\, d^3r\right) \times \left(\frac{\partial}{\partial a_l} B_k(\mathbf{a})\right)_{\mathbf{a}=0} \\ &= \sum_{j,k,l,m=1}^{3} \epsilon_{ijk}\epsilon_{mlj} \left(\frac{1}{2}\int (\mathbf{r}\times\mathbf{J})_m\, d^3r\right) \times \left(\frac{\partial}{\partial a_l} B_k(\mathbf{a})\right)_{\mathbf{a}=0}. \end{aligned} \quad (11)$$

The integral $\frac{1}{2}\int \mathbf{r}\times\mathbf{J}\, d^3r$ is the magnetic dipole moment

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r}\times\mathbf{J}\, d^3r. \quad (12)$$

(For the case of a planar current loop of area **A** and current I, the integral is exactly $\mathbf{m} = \mathbf{A}I$.) Now from the identity

$$\sum_{j=1}^{3} \epsilon_{ijk}\epsilon_{mlj} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl}, \quad (13)$$

Eq. (11) becomes

$$\begin{aligned} F_i &= \sum_{k=1}^{3} \left[m_k \left(\frac{\partial}{\partial a_i} B_k(\mathbf{a})\right)_{\mathbf{a}=0} - m_i \left(\frac{\partial}{\partial a_k} B_k(\mathbf{a})\right)_{\mathbf{a}=0} \right] \\ &= \mathbf{m}\cdot\left(\frac{\partial}{\partial a_i}\mathbf{B}(\mathbf{a})\right)_{\mathbf{a}=0} - m_i(\nabla_{\mathbf{a}}\cdot\mathbf{B})_{\mathbf{a}=0}. \end{aligned} \quad (14)$$

Since $\nabla\cdot\mathbf{B} = 0$ and **m** is independent of the spatial derivative, this gives the result[5]

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \quad (15)$$

for the force on the electric current model of a magnetic dipole.

IV. FORCE ON THE MAGNETIC CHARGE MODEL FOR A MAGNETIC DIPOLE

The magnetic force on the magnetic charge model for a magnetic dipole follows the usual derivation for the electric force on an electric dipole. The magnetic force density **f** for a magnetic charge density $\eta(\mathbf{r},t)$ is $f = \eta\mathbf{B}$, analogous to the electric force density $\mathbf{f} = \rho\mathbf{E}$ for an electric charge density $\rho(\mathbf{r},t)$ in an electric field **E**. The force on a static distribution of magnetic charge $\eta(\mathbf{r})$ is

$$\mathbf{F} = \int \eta(\mathbf{r})\mathbf{B}(\mathbf{r})\, d^3r. \quad (16)$$

We expand the magnetic field as a Taylor series about the center of the charge distribution taken as $\mathbf{r} = \mathbf{a} = 0$,

$$\mathbf{F} = \int \eta(\mathbf{r})\, [\mathbf{B}(\mathbf{a}) + \mathbf{r}\cdot\nabla_{\mathbf{a}}\mathbf{B}(\mathbf{a}) + \cdots]_{\mathbf{a}=0}\, d^3r. \quad (17)$$

The integral of the first term on the right-hand side corresponds to the total magnetic charge of the distribution. We assume that the total magnetic charge of the distribution vanishes so that

$$\int \eta(\mathbf{r})\, d^3r = 0. \quad (18)$$

The integral of the second term involves the magnetic dipole moment

$$\mathbf{m} = \int \mathbf{r}\eta(\mathbf{r})\, d^3r. \quad (19)$$

(In the case of two-point magnetic monopoles $\pm g$ separated by a distance **l**, this integral is exactly $\mathbf{m} = g\mathbf{l}$.) Then, from (17) and (19), the force is

$$\mathbf{F} = (\mathbf{m}\cdot\nabla)\mathbf{B} \quad (20)$$

for the magnetic charge model of a magnetic dipole.

V. THE DIFFERENCE BETWEEN THE TWO FORCE EXPRESSIONS

Although the force expressions $\mathbf{F} = \nabla[\mathbf{m}\cdot\mathbf{B}(\mathbf{r})]$ and $\mathbf{F} = (\mathbf{m}\cdot\nabla)\mathbf{B}(\mathbf{r})$ corresponding to the different magnetic dipole models are different, many textbooks treat them as though they were the same. We can find how the expressions differ by using the vector identity

$$\begin{aligned} \nabla(\mathbf{m}\cdot\mathbf{B}) &= \mathbf{m}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{m}) \\ &\quad + (\mathbf{m}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{m} \end{aligned} \quad (21)$$

together with the fact that **m** has no **r** dependence. Then we obtain

$$\nabla(\mathbf{m}\cdot\mathbf{B}) = (\mathbf{m}\cdot\nabla)\mathbf{B} + \mathbf{m}\times(\nabla\times\mathbf{B}). \quad (22)$$

Comparing Eqs. (15), (20), and (22), it is clear that the two different force expressions for the two different magnetic dipole models agree if and only if

$$\mathbf{m}\times(\nabla\times\mathbf{B}) = 0. \quad (23)$$

For virtually all the examples in the textbooks $\nabla\times\mathbf{B} = 0$ and so no error is made in using the formulas interchangeably. However, the force expressions can differ when $\nabla\times\mathbf{B} \neq 0$. This occurs in steady-state magnetostatics where $\mathbf{J} \neq 0$ or where the displacement current $\partial\mathbf{E}/\partial t \neq 0$.

VI. EXAMPLES OF FORCES ON MAGNETIC DIPOLES

Since electromagnetism textbooks give examples of forces on magnetic dipoles where the force expressions agree, here we will give examples only for the situations where the force equations (15) and (20) for the two magnetic dipole models actually differ.

As a first example, we consider a situation with rectangular symmetry. Assume that a uniform current density $\mathbf{J}=\hat{k}J_0$ along the z axis exists in the region $|x|<a$, $-\infty<y<\infty$,

$$\mathbf{J}(\mathbf{r})=\begin{cases}\hat{k}J_0, & |x|\leqslant a,\\ 0, & |x|>a.\end{cases} \tag{24}$$

Then, using Ampere's law, the associated magnetic field is in the y direction,

$$\mathbf{B}=\begin{cases}\hat{j}\mu_0 J_0 x, & |x|\leqslant a,\\ \hat{j}\mu_0 J_0 a(x/|x|), & |x|>a.\end{cases} \tag{25}$$

If a magnetic moment $\mathbf{m}$ is oriented in the x direction, $\mathbf{m}=\hat{i}m$, within the region $|x|<a$, then there is no force on the electric current loop model for a magnetic dipole,

$$\mathbf{F}=\nabla(m\hat{i}\cdot\mathbf{B})=0. \tag{26}$$

In contrast, there is a force on the magnetic charge model for a magnetic dipole,

$$\mathbf{F}=(m\hat{i}\cdot\nabla)\mathbf{B}=\hat{j}m\mu_0 J_0,\quad |x|<a. \tag{27}$$

On the other hand, if the magnetic moment $\mathbf{m}$ is oriented in the y direction, the situation is reversed. Now the electric current loop model experiences a force

$$\mathbf{F}=\nabla(m\hat{j}\cdot\mathbf{B})=\hat{i}m\mu_0 J_0,\quad |x|<a, \tag{28}$$

and the magnetic charge model experiences no force

$$\mathbf{F}=(m\hat{j}\cdot\nabla)\mathbf{B}=0. \tag{29}$$

The basis for the discrepancy in the forces is easily understood by going back to the magnetic dipole models using a rectangular current loop or two separated point monopoles and seeing just where the forces balance exactly and where they do not.

Situations with cylindrical symmetry can also be treated. If a uniform current density $\mathbf{J}=\hat{k}J_0$ flows in a cylindrical region of space,

$$\mathbf{J}=\begin{cases}\hat{k}J_0, & \sqrt{x^2+y^2}\leqslant a,\\ 0, & \sqrt{x^2+y^2}>a,\end{cases} \tag{30}$$

then Ampere's law gives a magnetic field

$$\mathbf{B}=\begin{cases}(\mu_0 J_0/2)(\hat{j}x-\hat{i}y), & \sqrt{x^2+y^2}\leqslant a,\\ (\mu_0 J_0 a^2/2)[(\hat{j}x-\hat{i}y)/(x^2+y^2)], & \sqrt{x^2+y^2}>a.\end{cases} \tag{31}$$

Since this situation has azimuthal symmetry, we can take the magnetic dipole in the x-z plane when evaluating the forces. Thus for a magnetic dipole in the radial direction $\mathbf{m}=\hat{i}m$, at x, $0<x<a$, the electric current loop model gives a force

$$\mathbf{F}=\nabla(m\hat{i}\cdot\mathbf{B})_{y=0}=-\hat{j}(m\mu_0 J_0/2)=-\hat{\varphi}(m\mu_0 J_0/2), \tag{32}$$

whereas the magnetic charge model gives a force

$$\mathbf{F}=(m\hat{i}\cdot\nabla)\mathbf{B}|_{y=0}=\hat{j}(m\mu_0 J_0/2)=\hat{\varphi}(m\mu_0 J_0/2), \tag{33}$$

which is in the opposite direction. Similarly, a magnetic dipole in the azimuthal direction $\mathbf{m}=\hat{j}m$ gives opposite forces $\pm\hat{r}m\mu_0 J_0/2$ in the radial direction. In performing the calculations above, rectangular coordinates were intentionally retained until the last step despite the cylindrical symmetry. This was done because the magnetic dipole $\mathbf{m}$ cannot be regarded as a function of the field coordinate $\mathbf{r}$ when taking the gradient in $\mathbf{F}=\nabla[\mathbf{m}\cdot\mathbf{B}(\mathbf{r})]$. If we were to write $\mathbf{m}=\hat{r}m$ or $\mathbf{m}=\hat{\varphi}m$ before taking the gradient, we could easily obtain erroneous results.

Another example of a discrepancy between the forces is furnished by considering a magnetic dipole between the plates of a parallel-plate capacitor that is being charged. In the region between the plates, $\nabla\times\mathbf{B}\neq 0$ and hence the forces on the two different magnetic dipole models are not equal. If the capacitor plates are circular of radius R and separated by a small distance d, $d\ll R$, then a current I to the capacitor leads to a displacement current in the region between the plates giving

$$\mathbf{B}=\hat{\varphi}(\mu_0 Ir/2\pi R^2),\quad r<R. \tag{34}$$

This expression for $\mathbf{B}$ has the same functional dependence $\hat{\varphi}r=\hat{j}x-\hat{i}y$ as in Eq. (31), and hence leads to forces of the same form as found above for a cylindrical current density. As we saw in Eqs. (32) and (33), the forces are different for the two different models of magnetic dipoles.

VII. THE NATURE OF THE NEUTRON MAGNETIC MOMENT

Interest in the two alternative models for magnetic dipoles is not merely pedagogical. In this article, we mention two research controversies related to the alternative magnetic dipole models and their associated forces.

The first controversy[6] involved the magnetic moment of the neutron. In 1936, Bloch[7] calculated the scattering of neutrons from ferromagnetic materials while using the magnetic charge model for the neutron magnetic dipole moment. The scattering was recalculated by Schwinger[8] using the electric current model for the neutron magnetic moment. Finally, in 1951, two experimental groups reported results[9,10] in agreement with the electric current model and in disagreement with the magnetic charge model. The electric current model is now the usual model for the neutron dipole moment.[11]

One of the two experiments[10] reported in 1951 involved the glancing reflection of neutrons from the plane surface of a ferromagnetic material magnetized parallel to the surface. The different predictions for the magnetic force contribution for the reflected neutrons correspond to our Eqs. (15) and (20). The situation is essentially that discussed in our example above involving rectangular symmetry and a current flowing in the z direction in the region $-a<x<a$. The force expression $\mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B})$ predicts a repulsive or attractive magnetic force on a neutron polarized parallel or antiparallel to the magnetic z axis of the iron surface since $\mathbf{m}\cdot\mathbf{B}$ has a gradient normal to the surface as $\mathbf{m}\cdot\mathbf{B}$ increases by $\mathbf{m}\cdot(\mu_0\mathbf{M})$, where $\mathbf{M}$ is the magnetization inside the material. However, the force expression $\mathbf{F}=(\mathbf{m}\cdot\nabla)\mathbf{B}$ predicts zero magnetic force on a neutron polarized parallel to the iron surface since $\mathbf{B}$ does not change in the direction parallel to the surface. The experimental work confirmed the existence of the magnetic force contribution and hence established the electric current model for the neutron dipole moment.[11]

VIII. THE PROPOSED AHARONOV–CASHER EFFECT

The failure to distinguish clearly between the two different force expressions for magnetic dipoles has also led to a recent controversy regarding an ongoing experiment. In 1984, Aharonov and Casher[12] proposed the following experiment. Consider a magnetic moment **m**, oriented parallel to the z axis, which passes an electric line charge of charge λ per unit length. The line charge is oriented along the z axis and the magnetic moment **m** moving with velocity $\mathbf{v} = \hat{j}v$ parallel to the y axis can pass on either side of the line charge along $x = \pm d$. Aharonov and Casher[12] proposed that there is a quantum interference pattern shift proportional to the product $m\lambda$ of the magnetic moment m and the charge per unit length λ for particle beams that pass around opposite sides of the line charge. An experiment to detect the interference effect is currently under way by a University of Missouri–University of Melbourne group[13] using a beam of neutrons and a Bonse–Hart perfect silicon crystal interferometer.

The Aharonov–Casher proposal was suggested as having a "duality"[12] with the famous Aharonov–Bohm effect, where one is said to find quantum interference pattern shifts in the absence of classical forces. In the proposed Aharonov–Casher effect, the interference pattern was claimed to change due to the line charge λ despite the fact that the magnetic moment **m** experienced no classical force due to the line charge. The suggestion that no classical force acted on the magnetic dipole was made based upon an *ad hoc* Lagrangian proposed by Aharonov and Casher.[12]

From the discussion above we see that there are two possible force expressions for a magnetic dipole; which force is appropriate depends upon the model used for the magnetic dipole. When one derives forces from an *ad hoc* Lagrangian, it is not clear whether one has used the electric current model for a magnetic dipole or the magnetic charge model. However, one can easily calculate the classical electromagnetic forces for the two magnetic dipole models by using the force expressions (15) and (20).

The electric line charge λ produces an electrostatic field in its own rest frame,

$$\mathbf{E} = \hat{r}'\lambda/2\pi\epsilon_0 r', \tag{35}$$

where $\mathbf{r}' = \hat{i}x' + \hat{j}y'$ is the displacement from the line charge oriented along the z' axis. In an inertial frame in which the line charge is moving with velocity $-\mathbf{v} = -\hat{j}v$, the line charge λ located instantaneously at $x = 0$, $y = 0$ produces a magnetic field (to first order in v/c),[14]

$$\mathbf{B} \cong -\frac{v}{c^2}\hat{j}\times\mathbf{E} = \frac{-\mu_0 v}{2\pi}\hat{j}\times\left(\frac{\hat{i}x + \hat{j}y}{x^2 + y^2}\right)\lambda = \frac{\hat{k}\mu_0 x v\lambda}{2\pi(x^2 + y^2)}, \tag{36}$$

where $\mu_0\epsilon_0 = c^{-2}$. A magnetic dipole $\mathbf{m} = \hat{k}m$ oriented along the z direction and at rest at the point (x,y,z) will experience a force in the electric current model for the magnetic dipole

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{\mu_0 m x v\lambda}{2\pi(x^2 + y^2)} = \frac{\mu_0 m v\lambda}{2\pi}\frac{\hat{i}(y^2 - x^2) - \hat{j}2xy}{(x^2 + y^2)^2}. \tag{37}$$

On the other hand, in the magnetic charge model for the magnetic dipole, the force is calculated as

$$\mathbf{F} = (\mathbf{m}\cdot\nabla)\mathbf{B} = m\frac{\partial}{\partial z}\left(\frac{\hat{k}\mu_0 x v\lambda}{2\pi(x^2 + y^2)}\right) = 0. \tag{38}$$

This second vanishing force agrees with the claim by Aharonov and Casher using their *ad hoc* Lagrangian.

Thus here in this situation under experimental investigation the two different models lead to two different forces. The usual model for magnetic dipoles in atomic and nuclear physics is the electric current model. Using this model there is a force on the magnetic dipole, contrary to the claim of Aharonov and Casher. Indeed using the electric current model for a magnetic dipole, the effect predicted by Aharonov and Casher can be understood in detail[15] as an effect based upon classical electromagnetic forces.

IX. CLOSING SUMMARY

Although classical electrostatics and classical magnetostatics have many similarities in their formal structure, they also have important differences. Electrostatics is based upon the existence of electric charges while magnetostatics is usually based upon electric currents with $\nabla\cdot\mathbf{B} = 0$ rather than upon magnetic charge distributions with $\nabla\cdot\mathbf{B} \neq 0$. In electromagnetism textbooks, it is emphasized that the electric current model for a magnetic dipole gives rise to precisely the same form of magnetic field as an electric charge distribution model for an electric dipole gives for the electric field. However, it is not emphasized that the magnetic force experienced by the electric current model for a magnetic dipole has a form distinctly different from the electric force experienced by an electric charge distribution model for an electric dipole.

In this article, we have emphasized the differences in forces experienced by magnetic dipoles depending upon the detailed model for the dipole. The usual electric current model for a magnetic dipole **m** leads to a force $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$. A magnetic charge model for a magnetic dipole **m** leads to a force $\mathbf{F} = (\mathbf{m}\cdot\nabla)\mathbf{B}$; this is of the same form as for the electric force on an electric dipole. Here, we have given some elementary examples where these force expressions give different answers for the force on a magnetic dipole. The difference in the force expressions is relevant in the proposed Aharonov–Casher effect, which is currently under experimental investigation.

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this assistance and for his continued kindness in connection with the Aharonov–Casher proposal.

[1]See, for example, the remarks by D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1981), p. 221.
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[3]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., p. 185.
[4]See, for example, Ref. 1, p. 222, Problem 4. Note that this problem requires $\nabla \times \mathbf{B} = 0$ for the stated solution.
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Perfect disturbing measurements

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The Wigner–Araki–Yanase theorem is often interpreted as meaning that there must be an error in measurement under certain conditions (when the commutator C between an additive conserved quantity and the discrete-spectrumed measured quantity does not vanish), and that the error may only be reduced by increasing the size of the apparatus. By explicit example, it is shown that it *is* possible to have a perfectly accurate measurement if the system being measured is disturbed, regardless of the size of the apparatus. Also illustrated is how the error in an imperfectly accurate measurement may be reduced by decreasing the magnitude of C, without affecting the size of the apparatus.

I. INTRODUCTION

The quantum theory of measurement concerns the interaction between an apparatus and a system being measured, which results in the apparatus changing from its initial state to a new state determined by a property of the system. In an "ideal" measurement, the apparatus records the system's property with perfect accuracy and the system state is not disturbed: We shall call this a *perfect nondisturbing measurement*. However, in certain very common circumstances (to be detailed below), Wigner[1] and Araki and Yanase[2] (WAY theorem) proved that it is impossible to have a perfect nondisturbing measurement. Following this, Araki and Yanase and subsequent authors (Yanase,[3] Ghirardi *et al.*[4]) studied *imperfect nondisturbing measurements*, where the apparatus does not measure the system property with perfect accuracy, but when the measurement is correct the system is not disturbed. They showed how the error can be made arbitrarily small if the apparatus is made arbitrarily large.

Wigner himself wrote,[5] "It can even be shown that no observable which does not commute with the additive conserved quantities. . .can be measured precisely. . .," and indeed the WAY theorem is often interpreted as meaning that a perfect measurement is impossible, in spite of the fact that the theorem only proves that a perfect nondisturbing measurement is impossible. However, in an article dealing with various aspects of the WAY theorem, Stein and Shimony[6] suggested the existence of *perfect disturbing measurements*. We believe that this possibility of a perfectly accurate measurement that disturbs the system, as a means of "getting around" the WAY theorem, is not as well known as it should be. We shall define a perfect disturbing measurement somewhat more narrowly than the category introduced by Stein and Shimony, and show by two simple examples that perfect *totally* disturbing measurements (in which initially orthogonal system state vectors end up parallel) and perfect *partially* disturbing measurements (in which initially orthogonal system state vectors end up nonorthogonal, but nonparallel) are possible.

