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The geometrical nature and some properties of the capacitance coefficients based on Laplace's equation

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The fact that the capacitance coefficients for a set of conductors are geometrical factors is derived in most electricity and magnetism textbooks. We present an alternative derivation based on Laplace's equation that is accessible to students in an intermediate course on electricity and magnetism. The properties of Laplace's equation permits us to determine many properties of the capacitance matrix. Some examples are given to illustrate the usefulness of these properties.

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I. INTRODUCTION

The fact that capacitance is a geometrical factor is an important property in courses on electricity and magnetism.[1,2] Derivations of this property are usually based on the principle of superposition[1,2] and the Green function formalism.[3,4] Nevertheless, such derivations are not convenient for calculations. Alternative techniques to calculate the capacitance coefficients based on the Green function formalism[5] and other methods[6–8] have been developed.

In this paper we give a simple proof of the geometrical nature of the capacitance coefficients based on Laplace's equation. Our approach permits us to demonstrate many properties of the capacitance matrix. The method is illustrated by reproducing some well known results, and applications in complex situations are suggested.

II. CAPACITANCE COEFFICIENTS

We consider a system of N internal conductors and an external conductor that encloses them. The potential on each internal conductor is denoted by φ_i, $i=1,2,\dots,N$. The surface of the external conductor is denoted by S_{N+1} and its potential is denoted by φ_{N+1} (see Fig. 1). One reason to introduce the external conductor is that it provides a closed boundary to ensure the uniqueness of the solutions. In addition, many capacitors contain an enclosing conductor as for the case of spherical concentric shells. We shall see that the case in which there is no external conductor can be obtained in the appropriate limit.

The surface charge density σ on an electrostatic conductor is given by[1,2]

$$\sigma_i = \varepsilon_0 \mathbf{E}\cdot\mathbf{n}_i = -\varepsilon_0 \nabla\phi\cdot\mathbf{n}_i \quad (i=1,\dots,N+1), \tag{1}$$

where $\mathbf{n}_i$ is a unit vector normal to the surface S_i pointing outward with respect to the conductor (see Fig. 1); $\mathbf{E}$ and ϕ denote the electrostatic field and potential, respectively. The charge on each conductor is given by

$$Q_i = \oint_{S_i} \sigma_i\, dS = -\varepsilon_0 \oint_{S_i} \nabla\phi\cdot\mathbf{n}_i\, dS. \tag{2}$$

The surface S_i encloses the conductor i and is arbitrarily near and locally parallel to the real surface of the conductor (see Fig. 1).[9] We define the total surface S_T as

$$S_T = S_1 + \cdots + S_N + S_{N+1}. \tag{3}$$

The volume V_{S_T} defined by the surface S_T is the one delimited by the external surface S_{N+1} and the N internal surfaces S_i. The potential ϕ in such a volume must satisfy Laplace's equation with the boundary conditions

$$\phi(S_i) = \varphi_i \quad (i=1,\dots,N+1). \tag{4}$$

Because of the linearity of Laplace's equation, the solution for ϕ can be parameterized as

$$\phi = \sum_{j=1}^{N+1} \varphi_j f_j, \tag{5}$$

where the f_j are functions that satisfy Laplace's equation in the volume V_{S_T} with the boundary conditions

$$\nabla^2 f_j = 0, \quad f_j(S_i) = \delta_{ij} \quad (i,j=1,\dots,N+1). \tag{6}$$

The solutions for f_j ensure that ϕ is the solution of Laplace's equation with the boundary conditions in Eq. (4). The uniqueness theorem also ensures that the solution for each f_j is unique (as is the solution for ϕ). The boundary conditions (6) indicate that the f_j functions depend only on the geometry.

If we apply the gradient operator in Eq. (5) and substitute the result into Eq. (2), we obtain

$$Q_i = \sum_{j=1}^{N+1} C_{ij}\varphi_j \tag{7a}$$

$$C_{ij} \equiv -\varepsilon_0 \oint_{S_i} \nabla f_j\cdot\mathbf{n}_i\, dS, \tag{7b}$$

which shows that the C_{ij} factors are exclusively geometric. The symmetry of the associated C_{ij} matrix can be obtained by purely geometrical arguments. We start from the definition of C_{ij} in Eq. (7b) and find

$$C_{ij} = -\varepsilon_0 \oint_{S_i} \nabla f_j\cdot\mathbf{n}_i\, dS = \varepsilon_0 \oint_{S_T} f_i \nabla f_j\cdot(-\mathbf{n}_i)\, dS, \tag{8}$$

where we have used the fact that $f_i=1$ on the surface S_i and zero on the other surfaces. From Gauss's theorem we obtain

$$C_{ij} = \varepsilon_0 \int_{V_{S_T}} \nabla\cdot(f_i\nabla f_j)\, dV \tag{9a}$$

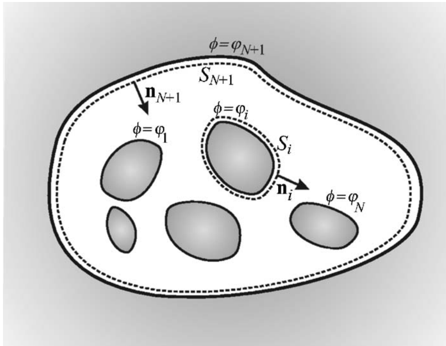

Fig. 1. A system consisting of N internal conductors with conductor $N+1$ enclosing them. The normals $\mathbf{n}_i$ with $i=1,\ldots,N+1$ point outward with respect to the conductors and inward with respect to the volume V_{S_T} (defined by the region in white). The surfaces S_i with $i=1,\ldots,N$ are slightly bigger than the ones corresponding to the conductors. In contrast, the surface S_{N+1} is slightly smaller than the surface of the external conductor.

$$=\varepsilon_0\int_{V_{S_T}}[\nabla f_i\cdot\nabla f_j+f_i\nabla^2 f_j]\,dV. \tag{9b}$$

Because $\nabla^2 f_j=0$ in V_{S_T}, it follows that

$$C_{ij}=\varepsilon_0\int_{V_{S_T}}\nabla f_i\cdot\nabla f_j\,dV. \tag{10}$$

Equation (10) implies that C_{ij} is symmetric,[10] that is,

$$C_{ij}=C_{ji}. \tag{11}$$

For certain configuration of conductors, consider two sets of charges and potentials $\{Q_i,\varphi_i\}$ and $\{Q_i',\varphi_i'\}$. From Eqs. (7) and (11) we have that

$$\sum_{i=1}^{N+1}Q_i\varphi_i'=\sum_{i=1}^{N+1}\left(\sum_{j=1}^{N+1}C_{ij}\varphi_j\right)\varphi_i' \tag{12a}$$

$$=\sum_{j=1}^{N+1}\left(\sum_{i=1}^{N+1}C_{ji}\varphi_i'\right)\varphi_j, \tag{12b}$$

which implies that

$$\sum_{i=1}^{N+1}Q_i\varphi_i'=\sum_{j=1}^{N+1}Q_j'\varphi_j. \tag{13}$$

Equation (13) is known as the reciprocity theorem.[1]

When one or more of the N internal conductors has an empty cavity, it is well known that there is no charge induced on the surface of the cavity[1,2] (let us call it $S_{i,c}$). Consequently, although $S_{i,c}$ is part of the surface of the conductor, such a surface can be excluded in the integration in Eq. (2). In addition, we can check by uniqueness that $f_j=\delta_{ij}$ in the volume of the cavity $V_{i,c}$ so that $\nabla f_j=0$ in such a volume, and hence it can be excluded from the volume integral (10). In conclusion neither $S_{i,c}$ nor $V_{i,c}$ contributes in this case.

The situation is different if there is another conductor in the cavity. In this case the surface of the cavity contributes in Eq. (2). Similarly the volume between the cavity and the embedded conductor contributes in the volume integral (10). The arguments can be extended for successive embedding of conductors in cavities as shown by Fig. 2 or for conductors with several cavities.

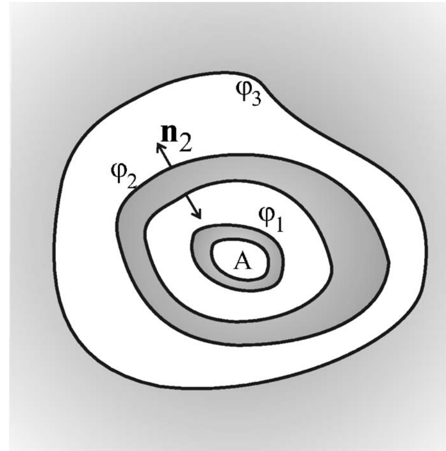

Fig. 2. Example of system in which there is a successive embedding of conductors. The volume V_{S_T} corresponds to the region in white. The regions corresponding to empty cavities (and their associated surfaces and volumes) can be excluded without affecting the calculations. In this picture cavity A is empty and its surface and volume need not be considered for calculations.

III. SOME ADDITIONAL PROPERTIES

We define a function F,

$$F\equiv\sum_{j=1}^{N+1}f_j, \tag{14}$$

and see from Eq. (6) that

$$\nabla^2 F=0,\quad F(S_i)=1\quad(i=1,\ldots,N+1). \tag{15}$$

Because $F=1$ on the surface S_T, we see by uniqueness that $F=1$ in the volume V_{S_T} from which we find that

$$\sum_{j=1}^{N+1}f_j=1. \tag{16}$$

In addition, by summing over j in Eq. (7b) and taking into account Eq. (16), we find that

$$\sum_{j=1}^{N+1}C_{ij}=0\quad(i=1,\ldots,N+1). \tag{17}$$

The symmetry of the C_{ij} elements leads also to

$$\sum_{i=1}^{N+1}C_{ij}=0\quad(j=1,\ldots,N+1). \tag{18}$$

Equations (17) and (18) imply that the sum of the elements over any row or column of the matrix is zero. Appendix A gives some proofs of consistency for these important properties. If we take into account the symmetrical nature of the C_{ij} matrix with dimension $(N+1)\times(N+1)$ and the $N+1$ constraints in Eq. (18), we see that for a system of N conductors surrounded by another conductor $N+1$, the number of independent capacitance coefficients is

$$(N+1)^2-\left[\frac{N(N+1)}{2}\right]-(N+1)=\frac{N(N+1)}{2}. \tag{19}$$

Other important properties are that

$$C_{ii}\geq 0, \tag{20a}$$

$$C_{ij} \leq 0 \quad (i \neq j). \tag{20b}$$

Equation (20a) follows straightforwardly from Eq. (10). To demonstrate Eq. (20b), we recall that the solutions of Laplace's equation cannot have local minima nor local maxima in the volume in which the equation is valid.[1,2] Consequently, the f_j functions must lie in the interval

$$0 \leq f_j \leq 1. \tag{21}$$

Because $f_j=0$ on any surface S_i for $i \neq j$, we see that f_j acquires its minimum value on such surfaces. Therefore the function ∇f_j should point outward with respect to the conductor i for $i \neq j$. Hence

$$\mathbf{n}_i \cdot \nabla f_j \geq 0 \quad \text{for } i \neq j. \tag{22}$$

We substitute Eq. (22) into Eq. (7) and obtain $C_{ij} \leq 0$ for $i \neq j$. An additional derivation of the fact that $C_{ii} \geq 0$ can be obtained by taking into account that f_j acquires its maximum value on the surface S_j.

Equation (18) can be rewritten as

$$\sum_{i=1}^{N} C_{ij} = -C_{N+1,j}. \tag{23}$$

From Eq. (20) we have that $C_{N+1,j} \leq 0$ for $j=1,\ldots,N$ and $C_{N+1,N+1} \geq 0$. Hence

$$\sum_{i=1}^{N} C_{ij} \geq 0 \quad (j=1,\ldots,N), \tag{24a}$$

$$\sum_{i=1}^{N} C_{i,N+1} \leq 0. \tag{24b}$$

The following properties follow from Eqs. (11), (18), (20), and (24):

$$|C_{ii}| \geq \sum_{i \neq j}^{N} |C_{ij}|, \tag{25a}$$

$$|C_{ii}| \geq |C_{ij}|, \tag{25b}$$

$$C_{ii}C_{jj} \geq C_{ij}^2, \tag{25c}$$

$$|C_{N+1,N+1}| = \sum_{i=1}^{N} |C_{i,N+1}|, \tag{25d}$$

$$|C_{N+1,N+1}| \geq |C_{i,N+1}|, \tag{25e}$$

where $i,j=1,\ldots,N$.

A particularly interesting case arises when the external conductor is at zero potential. In such a case, although the elements of the form $C_{N+1,j}$ do not necessarily vanish, they do not appear in the contributions to the charge on the internal conductors as can be seen from Eq. (7) by setting $\varphi_{N+1}=0$. For this reason, the capacitance matrix used to describe N free conductors (that is, not surrounded by another conductor) has dimensions $N \times N$.[11]

IV. TWO CONDUCTORS

We illustrate our method by deriving the basic properties of a system of two conductors. These examples will show the usefulness of Eq. (7) and some of the properties derived from our approach. We analyze a single internal conductor with an external conductor, that is, $N=1$. The internal conductor is labeled as conductor 1. From Eqs. (11) and (18) we have

$$C_{21} = C_{12} = -C_{11} = -C_{22}. \tag{26}$$

Therefore, there is only one independent coefficient, say C_{11} [in agreement with Eq. (19) with $N=1$]. The charges on the internal and external conductors can be calculated from Eq. (7):

$$Q_1 = C_{11}(\varphi_1 - \varphi_2), \tag{27a}$$

$$Q_2 = -C_{11}(\varphi_1 - \varphi_2) = -Q_1. \tag{27b}$$

Equation (27b) is consistent with Eq. (A4) and shows that the charge induced on the surface of the cavity of conductor 2 is opposite to the charge on conductor 1.

In Table I we display the results of three well known configurations of two conductors. The second column shows the f_i functions, which can be found by Laplace's equation (6) and used to calculate C_{11} with Eq. (7).

Table I. C_{11} and f_1 factors for three systems of two conductors with $a \leq r \leq b$ and $0 \leq x \leq d$. We neglect edge effects for the cylinders and planes.

System	f_1	C_{11}
Spherical shell with radius b and concentric solid sphere with radius a.	$\frac{ab}{b-a}\left(\frac{1}{r}-\frac{1}{b}\right)$	$\frac{4\pi\varepsilon_0 ab}{b-a}$
Cylindrical shell with radius b and concentric solid cylinder with radius a, both with length L.	$\frac{\ln(r/b)}{\ln(a/b)}$	$\frac{2\pi\varepsilon_0 L}{\ln(b/a)}$
Two parallel planes with area A at $x=0$ and $x=d$ (conductor 1).	x/d	$\varepsilon_0\frac{A}{d}$

V. EXAMPLES

We use our approach to study a system with embedded conductors. In addition, the case of two internal conductors is examined, and we show the limit in which the configuration of two conductors without an external conductor is obtained. These examples show how the properties we have derived can be used to calculate the capacitance coefficients.

Example 1. Consider two concentric spherical shells with radii b and c and a solid spherical conductor (concentric with the others) with radius a such that $c>b>a$. The potentials are denoted by φ_1, φ_2, and φ_3, respectively. The general solution of Laplace's equation for f_i can be written as

$$f_i = \frac{A_i}{r} + B_i. \tag{28}$$

From Eqs. (6) and (28) we obtain f_1 and f_3:

$$f_1 = \begin{cases} \dfrac{ab}{b-a}\left(\dfrac{1}{r} - \dfrac{1}{b}\right) & (a \leq r \leq b), \\ 0 & (b \leq r \leq c); \end{cases} \tag{29}$$

$$f_3 = \begin{cases} 0 & (a \le r \le b), \\ \dfrac{bc}{c-b}\left(\dfrac{1}{b} - \dfrac{1}{r}\right) & (b \le r \le c). \end{cases} \tag{30}$$

Although f_2 can be obtained the same way, it is easier to extract it from Eq. (16). The result is

$$f_2 = \begin{cases} \dfrac{ab}{b-a}\left(\dfrac{1}{a} - \dfrac{1}{r}\right) & (a \le r \le b), \\ \dfrac{bc}{c-b}\left(\dfrac{1}{r} - \dfrac{1}{c}\right) & (b \le r \le c). \end{cases} \tag{31}$$

The nine capacitance coefficients can be evaluated explicitly from Eq. (7), but it is easier to use Eqs. (11) and (18) and to take into account that $C_{31}=0$ [$\nabla f_1(r)=0$ for $r>b$]. We have

$$C_{13} = 0, \quad C_{12} = -C_{11}, \tag{32a}$$

$$C_{22} = C_{11} - C_{32}, \quad C_{33} = -C_{32}. \tag{32b}$$

From Eq. (7) the charge on each conductor is

$$Q_1 = C_{11}(\varphi_1 - \varphi_2), \tag{33a}$$

$$Q_2 = -Q_1 + C_{32}(\varphi_3 - \varphi_2), \tag{33b}$$

$$Q_3 = C_{32}(\varphi_2 - \varphi_3) = -(Q_1 + Q_2). \tag{33c}$$

Hence, we only have to calculate C_{11} and C_{32}.[12] The result gives

$$C_{11} = 4\pi\varepsilon_0 \frac{ab}{b-a}, \quad C_{32} = -4\pi\varepsilon_0 \frac{bc}{c-b}. \tag{34}$$

If $\varphi_2=\varphi_3$, we find that $Q_1=-Q_2$ and $Q_3=0$. It can be shown that Eqs. (32) and (33) are valid even if the conductors are neither spherical nor concentric, because those equations come from Eqs. (7a), (11), and (18), which reflect general properties independent of specific geometries.

Example 2. Consider two internal conductors and a grounded external conductor. As customary, we begin with $Q_1=Q_2=0$. By transferring charge from one internal conductor to the other, we keep $Q_1=-Q_2$. From Eq. (7a) and the definition $V \equiv \varphi_1-\varphi_2$ we find

$$Q_1 = (C_{11} + C_{12})\varphi_1 - C_{12}V, \tag{35}$$

$$Q_1 = -C_{13}\varphi_1 - C_{12}V, \tag{36}$$

where we have used Eq. (18). Similarly $Q_2=-C_{23}\varphi_1-C_{22}V$, and using again Eq. (18) we find

$$Q_1 + Q_2 = C_{33}\varphi_1 - C_{32}V. \tag{37}$$

Because the system is neutral, $Q_1+Q_2=0$ and hence

$$\varphi_1 = -\frac{C_{32}}{C_{33}}V; \tag{38}$$

substituting Eq. (38) into Eq. (36) we obtain

$$Q_1 = CV \quad C \equiv \frac{C_{13}C_{32} - C_{33}C_{12}}{C_{33}}. \tag{39}$$

Because $N=2$ only three of the coefficients in the definition of C are independent. From Eqs. (20) we see that this effective capacitance is non-negative. The procedure is not valid if $C_{33}=0$; in that case we see by using Eqs. (18) and (20) that $C_{i3}=C_{3i}=0$, and from Eq. (36) we find $C=-C_{12}=C_{22}$, which is also non-negative. The limit in which there is no external conductor is obtained by taking all the dimensions of the cavity to infinity while keeping the external conductor grounded as discussed in Ref. 11.

VI. CONCLUSIONS

We have used an approach based on Laplace's equation to demonstrate that the capacitance matrix depends only on purely geometrical factors. The explicit use of Laplace's equation permits us to demonstrate many properties of the capacitance coefficients. The geometrical relations and properties shown here permit us to simplify many calculations of the capacitance coefficients. Equation (6) shows that Laplace's equations required in our development are purely geometrical. Laplace's equation is usually easier than Green function formalism for both analytical or numerical calculations. Appendix A shows some proofs of consistency.

ACKNOWLEDGMENTS

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APPENDIX A: PROOFS OF CONSISTENCY

A proof of consistency for the identity (18) is achieved by using Eq. (7) to calculate the total charge on the N internal conductors:[13]

$$Q_{\text{int}} = \sum_{i=1}^{N} Q_i = \sum_{j=1}^{N+1} \left[\varphi_j \sum_{i=1}^{N} C_{ij} \right]. \tag{A1}$$

We use Eq. (18) to find

$$Q_{\text{int}} = -\sum_{j=1}^{N+1} C_{N+1,j}\varphi_j. \tag{A2}$$

Note that Eq. (A2) requires many fewer elements of the C_{ij} matrix than Eq. (A1). This difference becomes more significant as N increases. If we again use Eq. (7), we can find the charge on the cavity of the external conductor,

$$Q_{N+1} = \sum_{j=1}^{N+1} C_{N+1,j}\varphi_j, \tag{A3}$$

and therefore

$$Q_{N+1} = -Q_{\text{int}}, \tag{A4}$$

a property that can also be obtained from Gauss's law.[1,2]

Another proof of consistency for Eq. (18) is found by employing Eqs. (7) and (A2) to calculate Q_{int} [taking into account that Eq. (A2) comes directly from Eq. (18)]:

$$Q_{\text{int}} = -\sum_{j=1}^{N+1} C_{N+1,j}\varphi_j \tag{A5a}$$

$$= \varepsilon_0 \oint_{S_{N+1}} \nabla \left(\sum_{j=1}^{N+1} f_j \varphi_j \right) \cdot \mathbf{n}_{N+1}\, dS. \tag{A5b}$$

We use Eq. (5) to write Q_{int} as

$$Q_{\mathrm{int}} = \varepsilon_0 \oint_{S_{N+1}} \nabla \phi \cdot \mathbf{n}_{N+1}\, dS = \varepsilon_0 \oint_{S_{N+1}} \mathbf{E} \cdot (-\mathbf{n}_{N+1})\, dS. \tag{A6}$$

This relation is clearly correct because $\mathbf{n}_{N+1}$ points inward with respect to the volume V_{S_T}.

A proof of consistency for Eq. (10) that shows the symmetry of C_{ij} can be obtained by calculating the electrostatic internal energy, which in terms of the electric field is

$$U = \frac{\varepsilon_0}{2}\int_{V_{S_T}} E^2\, dV = \frac{\varepsilon_0}{2}\int_{V_{S_T}} \nabla \phi \cdot \nabla \phi\, dV \tag{A7a}$$

$$= \frac{1}{2}\sum_{i,j}^{N+1} \varphi_i \varphi_j \left[\varepsilon_0 \int_{V_{S_T}} \nabla f_i \cdot \nabla f_j\, dV \right], \tag{A7b}$$

where we have used Eq. (5). From Eq. (10) we find

$$U = \frac{1}{2}\sum_{i,j}^{N+1} C_{ij}\varphi_j\varphi_i = \frac{1}{2}\sum_{i}^{N+1} Q_i \varphi_i, \tag{A8}$$

consistent with standard results.[1,2]

APPENDIX B: SUGGESTED PROBLEMS

To enhance the understanding of this approach and its advantages, we suggest the following exercises for the reader.

(1) Implement a numerical method to solve Laplace's equation (6) for the f_i functions associated with a nontrivial geometry (for example, two non-concentric ellipsoids). Use Eqs. (16) and (21) to either simplify your calculations or to check the consistency of your results. Then use Eq. (7) to obtain the C_{ij} factors numerically. Use Eq. (11) and Eqs. (17)–(25) either to simplify your calculations or to check the consistency of your results.

(2) We have emphasized that to calculate the total charge on the internal conductors Eq. (A2) requires many fewer C_{ij} elements than Eq. (A1). How many fewer elements are required for an arbitrary value of N?

(3) For a successive embedding of concentric spherical shells, calculate the capacitance coefficients for an arbitrary number of spheres.

(4) Show that for the successive embedding of three conductors with arbitrary shapes, Eqs. (32) and (33) still hold. Generalize your results for an arbitrary number of conductors.

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[9]Q_{N+1} is not necessarily the total charge on the external conductor, but the charge accumulated on the surface of the cavity that encloses the other conductors. The value of the charge is calculated with the surface integral (2), which for the case of the internal conductors encompasses the whole surface, but for the external conductor is only the surface of the cavity that encloses the other conductors.
[10]Equation (10) is an integral over the volume for the C_{ij} factors. We might be tempted to use Gauss' theorem to obtain an integral of the volume directly from Eq. (7). However, f_j is not defined in the region inside the conductors. The gradient of f_j in Eq. (7) is evaluated in an external neighborhood of the conductor surface.
[11]By uniqueness, the solution for this problem is equivalent to the solution for a system consisting of the same N conductors contained in the cavity of a surrounding conductor, such that all the dimensions of the cavity tend to infinity, and the potential of the external conductor is set to zero.
[12]If we take into account that C_{13} is another degree of freedom (although zero), we have a total of three degrees of freedom, in agreement with Eq. (19) for $N=2$.
[13]For a derivation of some of these results based on the energy of the electrostatic field see L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd ed. (Elsevier Butterworth-Heinemann, New York, 1984), p. 3.