Lesson 4

Radiation

So far these notes have treated phenomena associated with electric charges that are either moving at constant velocity or flowing steadily through wires. The next level of complication involves sudden *changes* in the motion of charge. This is a very rich area for study, which could lead us into alternating currents, time-dependent magnetic fields, and the rest of Maxwell's equations. In the interest of brevity, however, I will go straight to the situation that I find most interesting of all: a point charge that undergoes a sudden change in motion (acceleration), and the electric field produced by this accelerated charge.

4.1 The Field of an Accelerated Charge

Recall from Section 1.2 that when a point charge moves at *constant* velocity, its electric field always points directly away from it, as shown in figure 1.5. (I'll assume for convenience that the point charge is positive.) In light of special relativity theory, this may seem strange, since no information can travel faster than the speed of light. Why then does the field at some faraway place point directly away from where the charge is *now*, rather than from where it was some time ago? Does this imply that information about the motion of the charge travels instantaneously throughout the whole universe? Well, not necessarily. You see, the particle has been traveling at constant velocity, along a predictable course, for some time. So if you're at a faraway place, you could have arranged for the particle to send you information about its position and velocity some time ago, so that when you receive this information you can extrapolate its motion from the past into the present and figure out where it must be by now.

Your scheme for predicting the position of the particle would be ruined, however, if the particle undergoes some acceleration between when it sends you the information and the present. You would think that the particle had continued to travel at constant velocity, and the field at your location would point away from where the particle *would* be now if it had done so, but in fact the particle is not there. For instance, suppose the particle is initially traveling to the right at 1/4 the speed of light, then suddenly bounces off a wall and recoils back to the left at the same speed (see figure 4.1). After one second, the news of the bounce can't have traveled farther than one light-second (300,000 km). If you're closer than one light-second to the location of the bounce then you've already received the news, and the field at your location points away from where the particle is now. But if you're farther than one light-second from the location of the bounce then the news hasn't

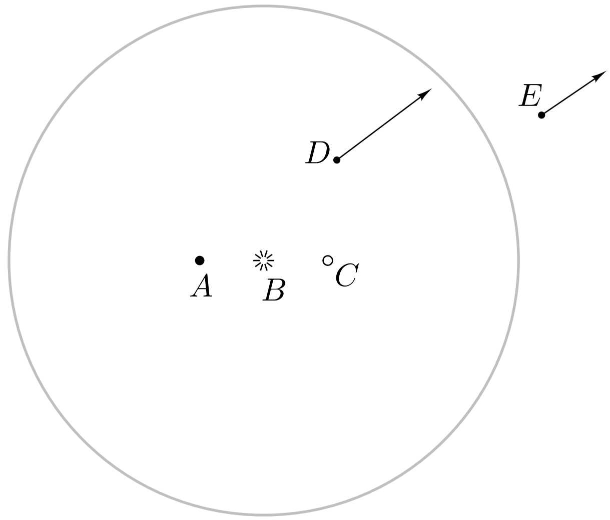

Figure 4.1. A positively charged particle, initially traveling to the right at 1/4 the speed of light, bounces off a wall at point B. The particle is now at point A, but if there had been no bounce it would now be at C. The circle (actually a cross section of a sphere) encloses the region of space where news of the bounce has already arrived; inside this circle (as at D) the electric field points directly away from A. Outside the circle (as at E) the news has not yet arrived, so the field points directly away from C. As time passes the circle expands outward at the speed of light, and points A and C move away from B at 1/4 the speed of light.

reached you yet, and the field at your location points away from where the particle would be now if there had been no bounce.

In this section I will assume that some mechanism like this, or at least equivalent to this, actually operates. We know from special relativity that no information can travel faster than the speed of light. I'll assume the best possible case: that the information travels at precisely the speed of light, but no faster. This assumption, together with Gauss's law, is enough to determine the electric field everywhere around the accelerated charge, and that is the goal of this section.

The complete map of the electric field of an accelerated charge turns out to be fairly complicated. Rather than representing the field as a bunch of arrows (like the two shown in figure 4.1), it is much more convenient to use a more abstract representation in terms of *field lines*. Field lines are continuous lines through space that run parallel to the direction of the electric field. A drawing of the field lines in a region therefore tells us immediately the direction of the electric field, although determining its magnitude is not so easy. A map of the field lines for the situation of figure 4.1 is shown in figure 4.2.

I have not drawn any field lines through the gray spherical shell in figure 4.2, since this is the region that is just in the midst of receiving the news of the particle's acceleration. To determine the direction of the field here, imagine a curved Gaussian "pillbox", indicated by the dashed line in the figure, which straddles the gray shell. (This surface is meant to be symmetrical about the line along which the particle is moving; viewed from along this line, it would be circular.) The Gaussian surface encloses no electric charge, so Gauss's law tells

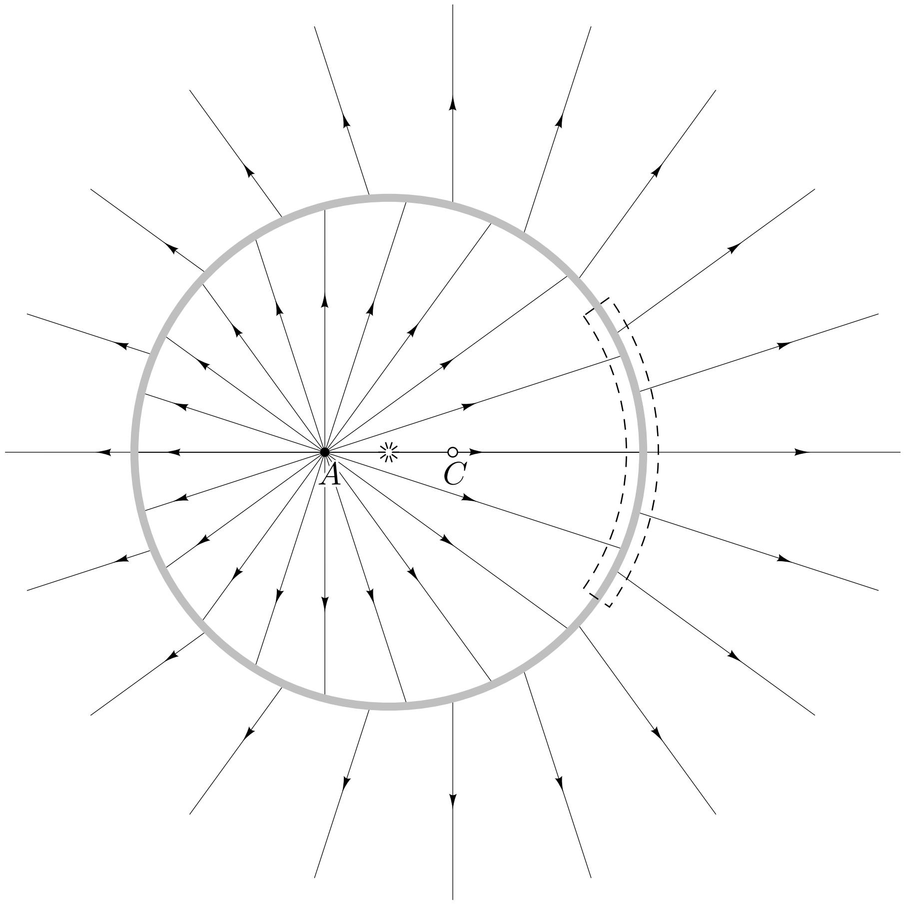

Figure 4.2. A map of the electric field lines for the same situation as in figure 4.1. The direction of the field within the gray spherical shell can be found be considering the flux through the curved Gaussian "pillbox" indicated by the dashed line.

us that the total flux of $\vec{E}$ through it must be zero. Now consider the flux through various parts of the surface. On the outside (right-hand) portion there is a positive flux, while on the inside (left-hand) portion there is a negative flux. But these two contributions to the flux do not cancel each other, since the field is significantly stronger on the outside than on the inside. This is because the field on the outside is that of a point charge located at C, while the field on the inside is that of a point charge located at A, and C is significantly closer than A. The net flux through the inside and outside portions of the surface is therefore positive. To cancel this positive flux, the remaining edges of the pillbox must contribute a negative flux. Thus the electric field within the gray shell must have a nonzero component along the shell, in toward the center of the Gaussian surface. I will refer to this component as the *transverse* field, since it points transverse (i.e., perpendicular) to the purely radial direction of the field on either side.

Exercise 4.1. Use a similar argument to determine the direction of the electric field within the portion of the gray shell on the left side of figure 4.2.

To be more precise about the direction of the field within the gray shell, consider the modified Gaussian surface shown in figure 4.3. Here I have shrunk the outer surface ef

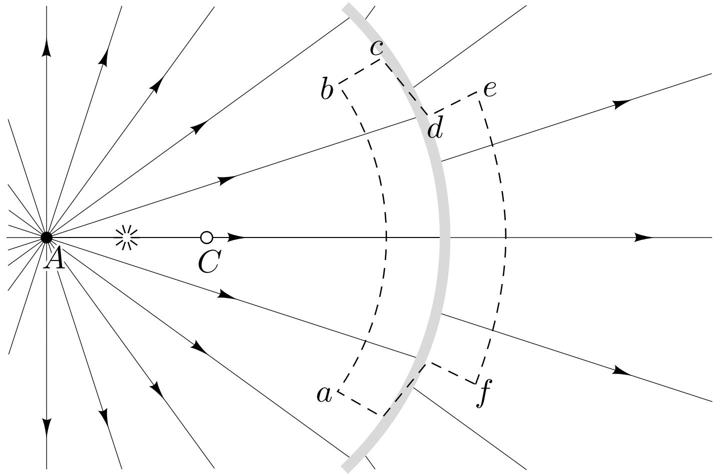

Figure 4.3. Another Gaussian surface applied to the same electric field as in figure 4.2. Since the flux along segment cd must be zero, the electric field within the gray shell must be parallel to this segment.

until it subtends the same angle, as viewed from C, that the inner surface ab subtends as viewed from A. Now the fluxes through ab and ef do indeed cancel. Segments bc and de are chosen to be precisely parallel to the field lines in their locations, so there is no flux through these portions of the surface. In order for the total flux to be zero, therefore, the flux must be zero through segment cd as well. This implies that the electric field within the gray shell must be parallel to cd. If you start at A and follow any field line outward, you will turn a sharp corner at the gray shell's inner edge, then make your way along the shell and slowly outward, turning another sharp corner at the outer edge. (The thickness of the gray shell is determined by the duration of the acceleration of the charge.) A complete drawing of the field lines for this particular situation is shown in figure 4.4.

Exercise 4.2. Sketch the field lines for a point charge that undergoes each of the following types of motion. (a) The charge moves to the right at 1/4 the speed of light, then suddenly stops. (b) The charge is initially at rest, then suddenly begins moving to the right at 1/4 the speed of light. (c) The charge is initially moving to the right at 1/2 the speed of light, then suddenly slows down to 1/4 the speed of light without changing direction. (d) The charge is bouncing back and forth, at 1/4 the speed of light, between two walls. (e) The charge is initially moving to the right at 1/4 the speed of light, then makes a sharp 90 degree turn without changing speed.

The transverse portion of the electric field of an accelerated charge is also called the *radiation field*, because as time passes it "radiates" outward in a sphere expanding at the speed of light. If the acceleration of the charged particle is sufficiently great, the radiation field can be quite strong, affecting faraway charges much more than the ordinary radial field of a charge moving at constant velocity. The radiation field can also store a relatively large amount of energy, which is carried away from the charge that created it. In the next section I will justify these claims by deriving a formula for the strength of the radiation field.

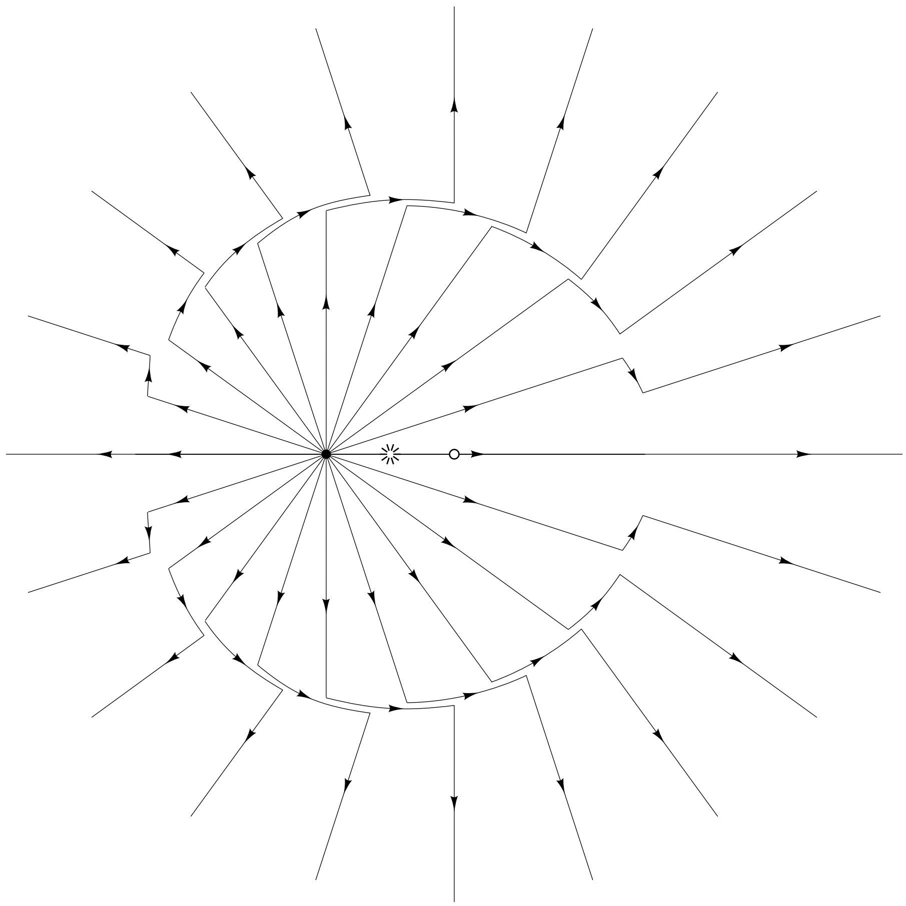

Figure 4.4. A complete sketch of the electric field lines for the situation shown in the preceding figures, including the transverse radiation field created by the acceleration of the charge.

4.2 The Strength of the Radiation Field

To turn the qualitative ideas of the previous section into quantitative formulas, let us consider a somewhat simpler situation, in which a positively charged particle, initially moving to the right, suddenly stops and then remains at rest. Let v_0 be the initial speed of the particle, and let the deceleration begin at time $t = 0$ and end at time $t = t_0$. I'll assume that the acceleration is constant during this time interval; the magnitude of the acceleration is then

$$a = |\vec{a}| = \frac{v_0}{t_0}. \tag{4.1}$$

I'll also assume that v_0 is much less than the speed of light, so that the relativistic compression and stretching of the electric field discussed in Section 1.2 is negligible.

Figure 4.5 shows the situation at some time T, much later than t_0. The "pulse" of radiation is contained in a spherical shell of thickness ct_0 and radius cT. Outside of this shell, the electric field points away from where the particle would have been if it had kept going; that point is a distance v_0T to the right of its actual location. (The distance that

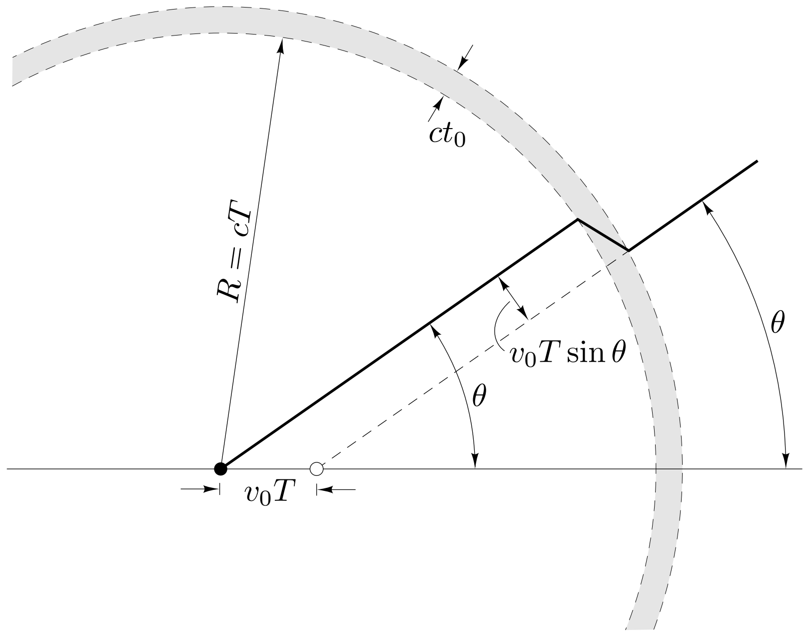

Figure 4.5. Figure for determining the strength of the electric field within the pulse of radiation. For clarity, only a single field line is shown here.

it traveled during the deceleration is negligible on this scale.) A single field line is shown in the figure, coming out at an angle θ from the direction of the particle's motion. There is a sharp kink in this line where it passes through the shell, as discussed in the previous section. We would like to know how strong the electric field is within the shell.

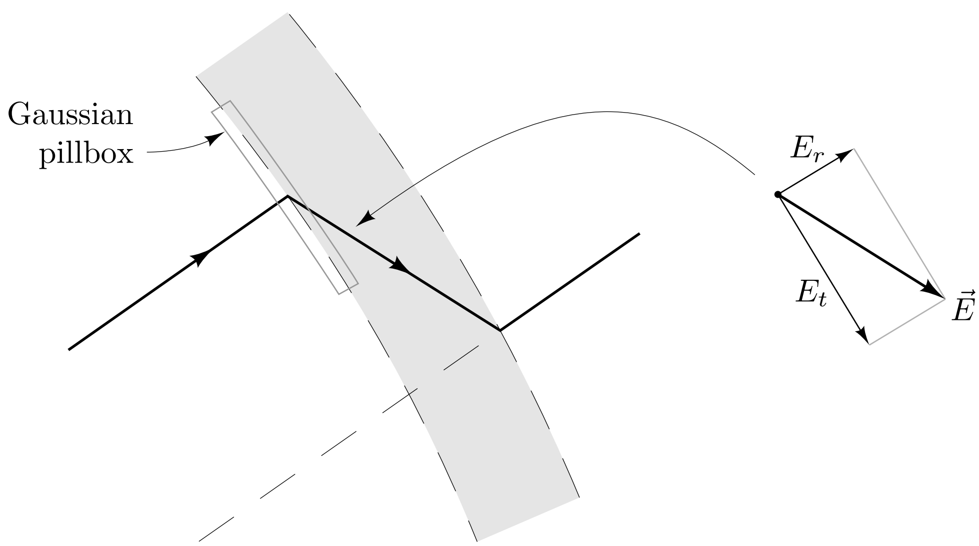

Figure 4.6. Close-up of the kink in the field in figure 4.5. The radial component E_r of the kinked field can be found by applying Gauss's law to the pillbox shown.

Let's break the kinked field up into two components: a radial component E_r that points away from the location of the particle, and a transverse component E_t that points in the perpendicular direction (see figure 4.6). The ratio of these components is determined by the direction of the kink; from figure 4.5 you can see that

$$\frac{E_t}{E_r} = \frac{v_0 T \sin\theta}{c t_0} = \frac{a T \sin\theta}{c}. \tag{4.2}$$

We can find the radial component E_r by applying Gauss's law to a tiny pillbox that straddles the inner surface of the shell (see figure 4.6). Let the sides of the pillbox be infinitesimally short so that the flux through them is negligible. Then since the net flux through the pillbox is zero, the radial component of $\vec{E}$ (that is, the component perpendicular to the top and bottom of the pillbox) must be the same on each side of the shell's inner surface. But inside the sphere of radiation the electric field is given by Coulomb's law. Thus the radial component of the kinked field is

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}, \tag{4.3}$$

where q is the charge of the particle. Combining equations (4.2) and (4.3) and using the fact that $R = cT$, you should now be able to show that

$$E_t = \frac{q a \sin\theta}{4\pi\epsilon_0 c^2 R}. \tag{4.4}$$

Exercise 4.3. Although I've derived formula (4.4) for the special case where the particle's final velocity is zero, it is true much more generally. To convince yourself of this, consider the case where the particle is *initially* at rest, then receives a sudden kick to the right. Draw a picture analogous to figure 4.5, and follow the same reasoning to arrive at equation (4.4). (Depending on your choice of coordinates, you may find an additional minus sign.)

Equation (4.4) tells us all we need to know about the strength of the pulse of radiation. First, note that the transverse field is proportional to $1/R$, not $1/R^2$. This means that as time goes on and R increases, the transverse field becomes much stronger than the radial field; at very large distances the radial field can be completely neglected and the field is purely transverse. Second, consider the dependence of E_t on the angle θ: It is weakest along the direction of motion ($\theta = 0$ or $180°$) and strongest at right angles to the motion ($\theta = 90°$). Looking back at figure 4.4, we see that the size of the kink in the field is a qualitative indication of the field strength. Finally, notice that the strength of the transverse field is proportional to a, the magnitude of the particle's acceleration. The greater the acceleration, the stronger the pulse of radiation.

This pulse of radiation carries energy. Recall from electrostatics that the energy per unit volume stored in any electric field is proportional to the square of the field strength. In our case, this implies

$$\text{Energy per unit volume} \propto \frac{a^2}{R^2}. \tag{4.5}$$

Since the volume of the spherical shell (the shell itself, not the region it encloses) is proportional to R^2, the total energy it contains does not change as time passes and R increases. Thus when a charged particle accelerates, it loses energy to its surroundings, in an amount proportional to the square of its acceleration. This process is the basic mechanism behind all electromagnetic radiation: visible light and its invisible cousins, from radio waves to gamma rays. Lesson 5 discusses a few applications of this all-important result.

4.3 The Larmor Formula

In this section I will derive a precise formula for the energy radiated by an accelerated charged particle. You've already read the most important part of the derivation, which ended with equation (4.4). The rest is mostly math.

The energy per unit volume stored in any electric field is

$$\text{Energy per unit volume} = \frac{\epsilon_0}{2}|\vec{E}|^2. \tag{4.6}$$

Once the pulse becomes large enough we can neglect the radial component of the field and simply plug in E_t for $|\vec{E}|$. The result is

$$\text{Energy per unit volume} = \frac{q^2 a^2 \sin^2\theta}{32\pi^2\epsilon_0 c^4 R^2}. \tag{4.7}$$

Notice again that this formula is largest when $\theta = 90°$.

If we don't care about the direction in which the energy goes, it is convenient to average equation (4.7) over all directions. I'll do this using a mathematical trick. Introduce a coordinate system with the origin at the center of the sphere and the x axis along the particle's original direction of motion. Then for any point (x, y, z) on the spherical shell, $\cos\theta = x/R$. Using angle brackets $\langle\,\rangle$ to denote an average over all points on the shell, I claim that

$$\langle\sin^2\theta\rangle = \langle 1 - \cos^2\theta\rangle = 1 - \frac{\langle x^2\rangle}{R^2}. \tag{4.8}$$

Now since the origin is at the center of the sphere, you must certainly agree that the average value of x^2 is the same as the average value of y^2 or z^2:

$$\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle. \tag{4.9}$$

But this implies that

$$\langle x^2\rangle = \frac{1}{3}\langle x^2 + y^2 + z^2\rangle = \frac{1}{3}\langle R^2\rangle = \frac{R^2}{3}, \tag{4.10}$$

since $x^2 + y^2 + z^2 = R^2$ and R is constant over the whole shell. Combining equations (4.8) and (4.10) gives

$$\langle\sin^2\theta\rangle = 1 - \frac{R^2}{3R^2} = \frac{2}{3}. \tag{4.11}$$

So the average energy per unit volume stored in the transverse electric field is

$$\text{Average energy per unit volume} = \frac{q^2a^2}{48\pi^2\epsilon_0 c^4 R^2}. \tag{4.12}$$

To obtain the total energy stored in the transverse electric field, we must multiply equation (4.12) by the volume of the spherical shell. The surface area of the shell is $4\pi R^2$ and its thickness is ct_0, so its volume is the product of these factors. Therefore the total energy is

$$\text{Total energy in electric field} = \frac{q^2a^2t_0}{12\pi\epsilon_0 c^3}. \tag{4.13}$$

Notice that the total energy is independent of R; that is, the shell carries away a fixed amount of energy that is not diminished as it expands.

Now, in order to be completely precise, I have to cheat. So far I've discussed only the *electric* field of the accelerated charge. But it turns out that there is also a *magnetic* field, which carries away an equal amount of energy. Since I've omitted so many details about magnetic fields from these notes, I have no way of justifying this claim. An error of a factor of 2 would hardly matter for the applications we'll be considering anyway, but I think it's better to go ahead and put it in for the record. Thus the *total* energy carried away by the pulse of radiation is twice that of equation (4.13), or

$$\text{Total energy in pulse} = \frac{q^2a^2t_0}{6\pi\epsilon_0 c^3}. \tag{4.14}$$

It is usually more convenient to divide both sides of this equation by t_0, the duration of the particle's acceleration. The left-hand side then becomes the energy radiated by the particle per unit time, or the *power* given off during the acceleration:

$$\text{Power radiated} = \frac{q^2a^2}{6\pi\epsilon_0 c^3}. \tag{4.15}$$

This result is called the *Larmor formula*, since it was first derived (using a more difficult method) by J. J. Larmor in 1897. The derivation given here was first published by J. J. Thomson (discoverer of the electron) in 1907. Although I have derived it for the special case where the final velocity of the particle is zero, the Larmor formula is true for any sort of accelerated motion provided that the speed of the particle is always much less than the speed of light.