Electromechanical implications of Faraday's law: A problem collection

W. M. Saslow

Citation: American Journal of Physics **55**, 986 (1987); doi: 10.1119/1.15281
View online: http://dx.doi.org/10.1119/1.15281
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/55/11?ver=pdfcov
Published by the American Association of Physics Teachers

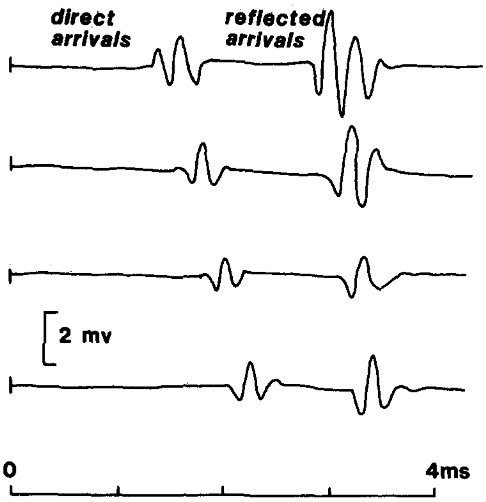

Fig. 8. Typical waveforms obtained for $z = 0.4$ m and x values of 0.4, 0.5, 0.6, and 0.7 m. The earlier arrivals are the direct wave and the later arrivals are the reflected wave.

val times. Also be sure to measure the height of the bar at each end, and its length so you can calculate the dip.

Using your plot of the direct wave for the four microphone experiment, extrapolate the travel time back to zero distance. You will probably find some residual time for zero distance. This value must be subtracted from all the travel time values before proceeding. What does this time correspond to? (Answer: This time probably corresponds to the time elapsed from when the pulse was generated to the time the sound pulse passed the zero distance mark.)

B. Data display and computation

Make a table of the arrival times versus x all the configurations measured and plot on the same graph. Include on the plot a solid line indicating the theoretical arrival time for the direct wave (sound in air). For the reflections from the flat surface, compute x^2 and t^2 values and plot them on a separate plot. For best results you will want to make a plot with a scale for t^2 that is appropriate for the values of t^2 obtained from a single reflection. That is, do not set your origin at zero. Set it a little less than the value you expect for t_0^2. Extrapolate these plots to $x = 0$ to find t_0. Compute the velocity of sound in air from the slope of the x^2-t^2 plot. Compare t_0 with the value determined from measuring the depth to the reflector and compare V to its theoretical value.

For the dipping reflector, use the difference in up-dip and down-dip arrivals to compute the dip according to the approximation given earlier.

C. Evaluation

Summarize in words what you have accomplished. Comment on sources of error including: (a) your estimation of the error in "picking" arrival times and (b) the inaccuracies of the dip computation above.

V. CONCLUSION

Apparatus has been described here which models a reflection seismic experiment. It has proved to be a popular experiment, and, with care, students can obtain values of the speed of sound in air, as well as the distance to and inclination of a reflector that are accurate to within a few percent.

[1]W. M. Telford, L. P. Gedart, R. E. Sheriff, and D. A. Keys, *Applied Geophysics* (Cambridge U. P., New York, 1976).
[2]M. B. Dobrin, *Introduction to Geophysical Prospecting* (McGraw-Hill, New York, 1976), 3rd ed.
[3]D. H. Griffiths and R. F. King, *Applied Geophysics for Geologists and Engineers* (Pergamon, New York, 1981), 2nd ed.

Electromechanical implications of Faraday's law: A problem collection

W. M. Saslow
Department of Physics, Texas A&M University, College Station, Texas 77843

(Received 7 July 1986; accepted for publication 28 October 1986)

A collection of problems illustrating the electromechanical implications of Faraday's law is presented. They are appropriate for well-prepared freshmen and all undergraduate physics majors. A number of interesting examples are worked out analytically, including Thomson's jumping ring demonstration. They are of interest in part because they include the effects of inductance and capacitance more fully than in the usual textbook treatments.

I. INTRODUCTION

The collections of problems given to undergraduate physics majors typically do not treat a number of aspects of Faraday's law that are crucial for understanding many of its practical implications. The present collection is designed to develop insights not normally obtained from the usual Faraday's law problems. They nearly all involve forces and torques on current loops.

Typical textbook treatments neglect self-inductance when first introducing Faraday's law. When self-inductance is later introduced, however, the same current loop is not reanalyzed, so that the implications of self-inductance for the electrical and (especially) the mechanical behavior of the loop are not fully developed. By discussing the behavior of the loop with resistance and inductance (and capacitance), it has been possible to generate new problems.

The fascinating Elihu Thomson jumping ring demonstration has been a major impetus for the creation of this problem set. Recently, in order to pose a test question with the same basic physics as the jumping ring, I came across a very simple geometry that clearly illustrates the relevant physics. (See Sec. VI.) When capacitance was added to the circuit, even more complex possibilities developed: At my suggestion an enterprising undergraduate put a capacitor into an inductive loop, thereby making what we facetiously call the "sucking" ring demonstration. I hope that the next generation of physics students will be able to put the insights from these problems to good use.

Sections II and III deal with linear motion, Secs. IV–VII deal with ac response, and Secs. VIII and IX deal with rotational motion.

Before presenting the problems, I remind the reader to apprise his (or her) students of the following essential (but often neglected) sign conventions. (i) If a complete circuit C has associated with it an open surface S, the direction of circulation determines the orientation of the surface by the right-hand rule; curl the fingers of the right hand along the direction of circulation $d\mathbf{l}$ and the thumb gives the direction of the normal to the surface $d\mathbf{S}$. (ii) The current is taken to be positive along the direction of circulation. (iii) The direction of the magnetic moment of a circuit is, by definition, along the normal to the plane of the circuit using the right-hand rule with fingers curling along the direction of positive current.

II. A BASIC PROBLEM—THROWING A CURRENT LOOP INTO A B FIELD

A. An inductanceless loop

This situation is depicted in Fig. 1, where the loop has resistance R, inductance L, mass M, rightward velocity v, vertical dimension l, and horizontal dimension h. The magnetic field has the value B, pointing into the paper, for $x>0$, and is zero for $x<0$. The usual treatment sets $L=0$. Therefore, the (counterclockwise) total emf is

$$\epsilon = Blv, \tag{1}$$

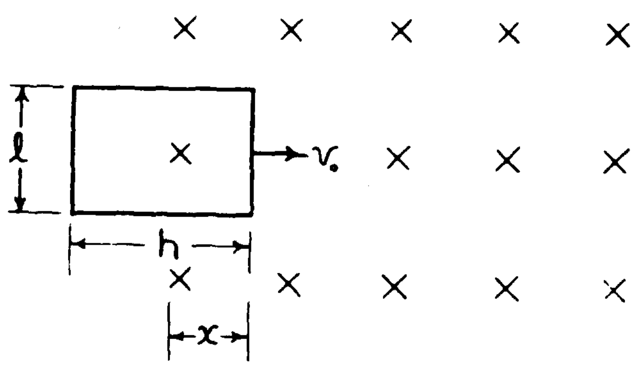

Fig. 1. A current loop of resistance R, inductance L, and mass M moving into a fixed magnetic field **B**.

obtained either from the motional emf

$$\epsilon = \oint \mathbf{E}_{\rm mot}\cdot d\mathbf{l}, \quad \mathbf{E}_{\rm mot} \equiv \mathbf{v}\times\mathbf{B}, \tag{2}$$

or the standard form of Faraday's law

$$\epsilon = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S}. \tag{3}$$

Here, the usual rule, that $d\mathbf{S}$ pointing out of the paper implies that $d\mathbf{l}$ circulates counterclockwise, is satisfied. Using the circuit equation $\epsilon = iR$ gives

$$i = Blv/R. \tag{4}$$

The force on the right arm, found from

$$\mathbf{F} = i\mathbf{l}\times\mathbf{B}, \tag{5}$$

yields, with

$$\mathbf{F} = M\frac{d\mathbf{v}}{dt}, \tag{6}$$

$$M\frac{dv}{dt} = -ilB = -\frac{B^2l^2}{R}v, \tag{7}$$

whose solution is

$$v = v_0\exp(-t/\tau), \quad x = v_0\tau[1-\exp(-t/\tau)], \tag{8}$$

where v_0 is the initial velocity and

$$\tau = MR/B^2l^2. \tag{9}$$

Equation (8) applies as long as the current loop lies only partly in the **B** field. This is a very standard problem. We now perform a variation on the theme.

B. A resistanceless loop

Consider that the loop is made of a superconducting material so that $R\approx 0$. In this case we cannot neglect the self-inductance, which produces an emf $-L(di/dt)$ that must be added to Blv of Eq. (1). Their sum gives $iR\approx 0$. Hence

$$Blv - L\frac{di}{dt} = 0 \tag{10}$$

so that, on integrating,

$$i = (Bl/L)x, \tag{11}$$

which satisfies the initial condition $i=0$ when $x=0$. The equation of motion yields

$$M\frac{d^2x}{dt^2} = -ilB = -\frac{B^2l^2}{L}x. \tag{12}$$

This is, of course, the equation of motion for a harmonic oscillator subject to $x=0$ and $dx/dt=v_0$ at $t=0$. Its solution is

$$x = (v_0/\omega_0)\sin\omega_0 t, \quad v = v_0\cos\omega t, \tag{13}$$

where

$$\omega_0 = Bl/(ML)^{1/2}. \tag{14}$$

Therefore, as long as v_0/ω_0 does not exceed the width h, so that the loop never completely enters the region of **B** field, the loop will eventually be repelled from the **B** field, leaving with velocity v_0 in the opposite direction.

For $B=0.1\tau$, $l=0.1$ m, $M=0.1$ kg, and $L=10^{-7}$ H, Eq. (14) gives $\omega_0=10^2\ {\rm s}^{-1}$, so that $v_0 = 1\ {\rm ms}^{-1}$ gives $v_0/\omega_0 = 10^{-2}$ m. These are not implausible numbers. Note that B, l, and M, with $R=10^{-3}\ \Omega$, yield $\tau = 1$ s, another easily observable time scale.

C. A loop with both resistance and inductance

We now have the circuit equation

$$Blv - L\frac{di}{dt} = iR \tag{15}$$

and the force equation

$$M\frac{dv}{dt} = -ilB. \tag{16}$$

Differentiating Eq. (15) and substituting in Eq. (16) gives

$$\frac{1}{\omega_0^2}\frac{d^2 i}{dt^2} + \tau\frac{di}{dt} + i = 0, \tag{17}$$

where τ is given by (9) and ω_0 is given by (14). The same equation holds for v, with $i \to v$. Clearly, we obtain underdamping, critically damped, or overdamped motion according to

$$\omega_0\tau > \sqrt{2}, \quad \text{(underdamping)},$$
$$\omega_0\tau = \sqrt{2}, \quad \text{(critical damping)},$$
$$\omega_0\tau < \sqrt{2}, \quad \text{(overdamping)}. \tag{18}$$

The initial conditions are $i = 0$ and $di/dt = Blv_0/L$. The solution of Eq. (17) is standard, so we will not dwell on it other than to note that in the overdamped and the critically damped case, and even for the underdamped case (if there is a sufficient amount of damping), the loop will come to rest in the **B** field without being expelled.

D. Variation on the problem—pulling a B field past a current loop

Here we have a current loop initially at rest, to be drawn into motion by the **B** field moving to the left. This is the principle of the induction motor, simplified here to the case of linear motion. The solutions can be obtained from a Galilean boost by $-v_0$ of the solutions given above. Note that the currents do not change.

III. A BASIC PROBLEM—PULLING A CURRENT LOOP INTO A B FIELD

This resembles the previous basis problem. Now a constant force **F** is applied to the loop as it is drawn into the **B** field. Let $v = v_0$ initially.

A. An inductanceless loop

We again have

$$i = Blv/R, \tag{19}$$

but now the force equation reads

$$M\frac{dv}{dt} = F - ilB = F - \frac{B^2 l^2}{R}v. \tag{20}$$

The solution is

$$v = v_\infty + (v_0 - v_\infty)\exp(-t/\tau),$$
$$x = v_\infty t + [(v_\infty - v_0)/\tau][\exp(-t/\tau) - 1], \tag{21}$$

where the terminal velocity v_∞ is

$$v_\infty = FR/B^2 l^2 \tag{22}$$

and τ is given by Eq. (9). Note that as $t \to \infty$, $i_\infty \to i = F/lB$.

B. A resistanceless loop

We again have

$$i = (Bl/L)x, \tag{23}$$

but now the force equation reads

$$M\frac{d^2x}{dt^2} = F - ilB = F - \frac{B^2 l^2}{L}x. \tag{24}$$

This is a harmonic oscillator equation, subject to $x = 0$ and $dx/dt = v_0$ at $t = 0$, with the solution

$$x = (v_0/\omega_0)\sin\omega_0 t + (FL/B^2 l^2)(1 - \cos\omega_0 t),$$
$$v = v_0\cos\omega_0 t + (FL\omega_0/B^2 l^2)\sin\omega_0 t. \tag{25}$$

If the loop is wide enough ($h \to \infty$), no matter how large F is, the loop will eventually be pushed back out of the **B** field (to be returned by the force F).

C. A loop with both resistance and inductance

We now have Eq. (15) for the emf and

$$M\frac{dv}{dt} = F - ilB \tag{26}$$

for the force. Differentiating Eq. (15) and substituting in Eq. (26) gives

$$\frac{1}{\omega_0^2}\frac{d^2 i}{dt^2} + \tau\frac{di}{dt} + (i - i_\infty) = 0, \tag{27}$$

where ω_0 is given by (14) and $i_\infty = F/lB$, as in Sec. III C. The analogous equation for v is

$$\frac{1}{\omega_0^2}\frac{d^2 v}{dt^2} + \tau\frac{dv}{dt} + (v - v_\infty) = 0. \tag{28}$$

where v_∞ is given by Eq. (22). Except for the shift in v and i, the solutions will be like those of Eq. (17).

D. Variation on the problem—pulling a B field past a current loop acted on by a constant force

The solutions in this case can be obtained from the above solutions by making a Galilean boost by $-v_0$.

IV. A CURRENT LOOP PARTLY IN AN OSCILLATING B FIELD

Here we consider a rectangular loop initially at rest, partly in a **B** field oscillating at a frequency ω. The loop extends a distance x into the **B** field, and has vertical di-

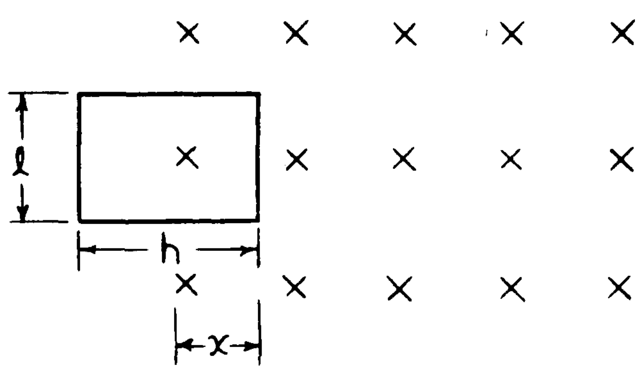

Fig. 2. A current loop partially in an oscillating magnetic field **B**.

mension l, resistance R, and inductance L. See Fig. 2. If we take $B = B_0 \sin \omega t$, where $B > 0$ means that **B** points into the paper, then the induced emf is given by $-B_0(lx)(d/dt)\sin \omega t = -B_0 lx\omega \cos \omega t$, where a negative emf tends to cause current flow counterclockwise through the loop. Hence, if $i > 0$ is taken to be counterclockwise, we have

$$B_0 lx\omega \cos \omega t = iR + L\frac{di}{dt}. \tag{29}$$

The force on the right arm, following Eqs. (5)–(7), gives the net force

$$F = M\frac{d^2x}{dt^2} = -ilB_0 \sin \omega t. \tag{30}$$

A. An inductanceless loop

We now take $L(di/dt) \ll R$, in which case (if $\omega x \gg dx/dt$, so that the motional emf may be neglected)

$$i \approx (B_0 lx\omega/R)\cos \omega t, \tag{31}$$

so that

$$F = -(B_0{}^2 l^2 \omega/R)(\sin \omega t \cos \omega t)x. \tag{32}$$

If we hold the loop at fixed x, we see that there is an oscillating force at a frequency of 2ω. Further, if the loop is entirely inside the **B** field, so that $x = h$, there is a force on the left arm which is equal and opposite to Eq. (32). In addition, there are forces on the upper and lower arms. All four of these forces have the effect of causing an alternate compression and expansion of the loop at a frequency 2ω. This causes a vibration of the surrounding air at a frequency 2ω, which in many cases is quite audible. It is possible to feel a loop vibrating in such circumstances, as this author has done on many occasions, when moving a conducting ring about the fringes of a solenoid used for the jumping rings demonstration. (See Sec. VI.) Indeed, this fringing phenomenon suggested the present problem.

B. A resistanceless loop

We now take $L(di/dt) \gg iR$, so that Eq. (29) implies

$$i \approx (B_0 lx/L)\sin \omega t \tag{33}$$

if $i = 0$ at $t = 0$. The force on the right arm becomes

$$F = -(B_0^2 l^2/L)(\sin^2 \omega t)x. \tag{34}$$

Thus there is a constant force pushing the loop out of the **B** field and an oscillating force at frequency 2ω, since $\sin^2 \omega t = \frac{1}{2} - \frac{1}{2}\cos 2\omega t$. If ω is so high that we may set $\sin^2 \omega t$ to its time average, then

$$\overline{F} = M\frac{d^2x}{dt^2} \approx -\frac{B_0^2 l^2}{2L}x \tag{35}$$

and the loop is expelled from the **B** field according to

$$x = x_0 \cos \Omega t, \tag{36}$$

where

$$\Omega = B_0 l/(2ML)^{1/2}. \tag{37}$$

Clearly, Eq. (36) holds only if $\Omega \ll \omega$, so that the motional emf may be neglected. In a time $\pi/2\Omega$, the loop (which was initially at rest at x_0) is expelled from the region of the **B** field. This is easily understood in terms of Lenz's law; the system moves out of the region to minimize the rate of change of the magnetic flux.

C. A loop with both resistance and inductance

We assume that ω is so high that, for purposes of solving Eq. (29), we may treat x as constant. Then the solution for i is

$$i = [B_0 lx\omega/(R^2 + \omega^2 L^2)][R \cos \omega t + \omega L \sin \omega t - R \exp(-t/\tau_L)], \tag{38}$$

where $\tau_L = L/R$ and we include the transient which enables us to satisfy $i = 0$ at $t = 0$. If we now drop this transient, we find that Eq. (30) becomes

$$F = M\frac{d^2x}{dt^2} = -\frac{B_0^2 l^2 \omega}{R^2 + \omega^2 L^2}(R \sin \omega t \cos \omega t + \omega L \sin^2 \omega t)x. \tag{39}$$

If we replace $\sin \omega t \cos \omega t$ and $\sin^2 \omega t$ by their time averages, then Eq. (39) becomes

$$\overline{F} = M\frac{d^2x}{dt^2} = -\frac{B_0^2 l^2 \omega^2 L}{2(R^2 + \omega^2 L^2)}x, \tag{40}$$

so that

$$x = x_0 \cos \overline{\Omega} t, \tag{41}$$

where

$$\overline{\Omega} = \Omega[\omega L/(R^2 + \omega^2 L^2)^{1/2}]. \tag{42}$$

Hence, the presence of resistance increases the time it takes to expel the loop from the **B** field, since $\overline{\Omega} < \Omega$.

We repeat that we have neglected the motional emf that would oppose the outward motion of the current loop. This would give a term $-B_0 l \sin \omega t(dx/dt)$ which would be added to the left-hand side of Eq. (29). If the fractional rate of change of the **B** field is much greater than the fractional rate of change of position, this term should be relatively negligible.

Finally, note that the introduction of capacitance to the circuit can cause significant changes in the response of the system. Mathematically, this can be done by letting $\omega L \to \omega L - (\omega C)^{-1}$. In particular, note that if the capacitance dominates the phase, the circuit will feel a net force pushing it into, or a net torque orienting it along, the **B** field. This seems to violate Lenz's law, but can be understood in terms of the capacitor dominating the inductance (of course, at very short times, when the transients are included, the inductance dominates), whose emf does satisfy Lenz's law. In fact, this suggests a method to find the direction of an ac **B** field if it has a fixed, but unknown, direction. Take a coil with a large capacitance in its circuit, so that the capacitance dominates the phase. Mount the coil so that it can rotate freely about some axis. The coil, because of the dominant capacitive coupling, will orient as much as possible along **B**. Hence, by changing the direction of the rotation axis it should be possible to locate the direction of **B**. Actually, use of an inductively dominated coil, free to rotate about some axis, would more easily locate the direction of **B**. In two orientations of the axis, two normals $\hat{n}_1$ and $\hat{n}_2$ will be located. Their cross product will give the direction of **B**.

Comparison of the inductively and capacitatively dominated cases provides an excellent example of the significance of the seemingly abstract concept of "phase": For the inductively dominated case the ring is pushed out of the field and for the capacitatively dominated case it is pushed into the field.

V. A RING IN AN OSCILLATING B FIELD

We consider here a ring of radius a in a **B** field with $B = B_0 \sin \omega t$, as in Sec. IV. The emf equation is

$$B_0 \pi a^2 \omega \cos \omega t = iR + L \frac{di}{dt}, \quad (43)$$

where $B > 0$ implies that i is counterclockwise. The net force on the ring will be zero, by symmetry, but there is a radial force per unit length obtained from

$$d\mathbf{F} = i d\mathbf{l} \times \mathbf{B}. \quad (44)$$

With $d\mathbf{l}$ counterwise and $B > 0$, we find that dF points radially inward, with

$$\frac{dF}{dl} = -iB. \quad (45)$$

The solution of Eq. (43), for fixed R, is analogous to that of Eq. (39). If we neglect the initial transient, then

$$i = [B_0 \pi a^2 \omega / (R^2 + \omega^2 L^2)](R \cos \omega t + \omega L \sin \omega t), \quad (46)$$

so that

$$\frac{dF}{dl} = -\frac{B_0^2 \pi a^2 \omega}{R^2 + \omega^2 L^2}(R \sin \omega t \cos \omega t + \omega L \sin^2 \omega t). \quad (47)$$

In other words, there is a net compressional force

$$\left.\frac{dF}{dl}\right|_{\text{net}} = -\frac{B_0^2 \pi a^2 \omega^2 L}{2(R^2 + \omega^2 L^2)} \quad (48)$$

and a net oscillatory force

$$\left.\frac{dF}{dl}\right|_{\text{osc}} = -\frac{B_0^2 \pi a^2 \omega^2 L}{2(R^2 + \omega^2 L^2)}(R \sin 2\omega t - \omega L \cos 2\omega t). \quad (49)$$

As for the case in Sec. IV, it is possible to feel the vibration and to hear the noise associated with it. For $\omega / 2\pi = 60$ Hz, this is a lovely example of 120-cycle hum. As in the previous problem, a capacitor may be included in the circuit by letting $\omega L \to \omega L - (\omega C)^{-1}$, leading to the possibility that the compressional force can become negative—a force of expansion.

VI. THOMSON'S JUMPING RING

A diagram of Thomson's jumping ring system is given in Fig. 3. Currents I_r and I_s flow in the ring and in the solenoid, and are considered to be positive as viewed from below. The mutual inductance M near the center of the solenoid is given by[1]

$$M \approx \mu_0 n N \pi a^2, \quad (50)$$

where n is the number of turns per unit length of the solenoid, N is the number of turns in the ring (we will take $N = 1$), and a is the radius of the ring. The magnetic field due to the solenoid has a radial component

$$B_\rho = \mu_0 n I_s K, \quad (51)$$

where K is a geometrical constant and is a measure of the flaring out of the magnetic field due to the solenoid. Using Eq. (44), now applied to the vertical component of the

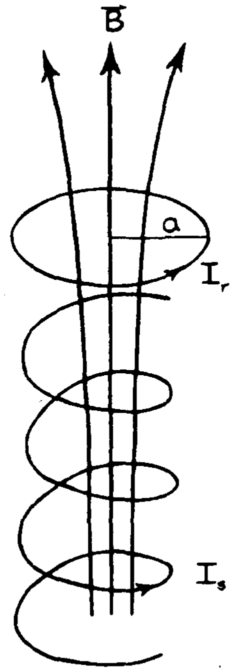

Fig. 3. Schematic of Thomson's jumping ring, placed somewhat above a concentric solenoid.

force on the ring, one finds that

$$dF_z = -I_r dl\, B_\rho, \quad (52)$$

so that

$$F_z = -I_r (2\pi a) B_\rho = -(\mu_0 n K 2\pi a) I_r I_s. \quad (53)$$

The emf equation for the ring is given by

$$-M \frac{dI_s}{dt} = I_r R + L \frac{dI_r}{dt}. \quad (54)$$

We will assume that an external power source produces an I_s of the form

$$I_s = I_{s0} \sin \omega t. \quad (55)$$

Neglecting transients, the solution of Eqs. (54) and (55) is

$$I_r = -[M I_{s0} \omega / (R^2 + \omega^2 L^2)](R \cos \omega t + \omega L \sin \omega t), \quad (56)$$

which gives

$$F_z = (\mu_0 n K 2\pi a)[M \omega I_{s0}^2 / (R^2 + \omega^2 L^2)] \times (R \sin \omega t \cos \omega t + \omega L \sin^2 \omega t). \quad (57)$$

There is a net vertical force and an oscillatory force at frequency 2ω.

It should be remarked that the value of I_{s0} very much depends upon the presence of the ring. We have blown fuses on our jumping ring apparatus at Texas A&M by holding our ring in place, thus forcing the solenoid to provide a large amoung of power which goes into Joule heating. The work done by F_z, which causes the ring to jump, is usually negligible compared to the Joule heating. However, cooling the ring (by dipping it in liquid N_2) gives spectacular results: On one occasion, a ceiling tile was knocked off by the force of impact (in which case the work done by F_z must have been fairly significant). Note that, for oscillations rapid enough that the vertical displacement may be considered constant, we have the time average

$$\bar{F}_z = (\mu_0 nKM 2\pi a)\{\omega^2 L I_{s0}^2/[2(R^2+\omega^2 L^2)]\}. \quad (58)$$

Because K varies with height (as does M), this force is not constant. At the center of a symmetrical solenoid, $K=0$, so KM is at a minimum. As one moves above the center, KM increases to a maximum somewhere near the top of the solenoid, after which KM decreases. In the high-frequency limit (where $I_r R \ll L dI_r/dt$),

$$\bar{F}_z = (\mu_0 nKM\pi a/L) I_{s0}^2, \quad (59)$$

a value which is independent of frequency.

It should be noted that if the ring moves, one must also include a term $-I_s(dM/dt)$ on the left-hand side of Eq. (54) to account for the emf induced by the motion of the ring. It is expected, however, that this term is relatively small, because the magnetic (or inductive) inertia of the current is typically much less than the mechanical inertia of the ring.

If the ring is replaced by a circuit which is dominated by a capacitor [again, let $\omega L \to \omega L - (\omega C)^{-1}$], then the system will be drawn into the region of the **B** field, as in Sec. V. As mentioned in the Introduction, a resourceful undergraduate has built a working model of this "sucking" ring. Note that $\omega_0 = (LC)^{-1/2}$ typically exceeds $2\pi(60\,\text{s}^{-1})$ by a large factor, so that almost any coil in series with a capacitor will be dominated by the capacitor.

The reader should note that the jumping ring was an invention of the American scientist Elihu Thomson, who employed its principles to make the first ac motor. An account of his discovery can be found in Ref. 2.

VII. INDUCTION HEATING AND INTERACTION OF NEIGHBORING CURRENT LOOPS

Consider two rectangular current loops in a plane, with sides parallel. Let one of the loops be much larger than the other, and let the smaller loop be outside of, but so close to the larger one that the larger loop may be treated as if only one arm mattered, so it is like an infinite wire. Let the nearest arm of the smaller loop be a distance x away from the nearest (effectively infinite) arm of the larger loop. Let its parallel arm be a distance b further away and its height be h. It is a standard calculation to show that the mutual inductance is given by (with $B=\mu_0 I/2\pi r$)

$$M = \int \mathbf{B}\cdot d\mathbf{S}/I_l = \frac{\mu_0 h}{2\pi}\ln\left(\frac{b+x}{x}\right). \quad (60)$$

where I_l is the current through the larger loop.

The emf equation for the smaller loop is [neglecting the $-I_s(dM/dt)$ term from the motion of the smaller loop]

$$-M\frac{dI_l}{dt} = I_s R + L\frac{dI_s}{dt}, \quad (61)$$

where I is the current through the smaller loop.

The net force on the smaller loop, of mass m, is given by

$$F = m\frac{d^2x}{dt^2} = I_s h\frac{\mu_0 I_l}{2\pi}\left(\frac{1}{x}-\frac{1}{x+b}\right), \quad (62)$$

where Eq. (5) was employed. Here I_s (I_l) is considered to be positive if it is upward (downward) along the nearest arms of the two loops. If we take $I_l = I_{l0}\sin\omega t$, then the solution of Eq. (61) is

$$I_s = -[M\omega I_{l0}/(R^2+\omega^2L^2)](R\cos\omega t + \omega L\sin\omega t), \quad (63)$$

so that

$$F = m\frac{d^2x}{dt^2} = -\frac{M\omega\mu_0 I_{l0}^2}{2\pi(R^2+\omega^2L^2)}\left(\frac{1}{x}-\frac{1}{x+b}\right) \times (R\sin\omega t\cos\omega t + \omega L\sin^2\omega t). \quad (64)$$

Taking the time average (this assumes a more rapid variation of I than of x) we have

$$\bar{F} = m\frac{d^2x}{dt^2} \approx -\frac{M\omega^2\mu_0 L I_{l0}^2}{4\pi(R^2+\omega^2L^2)}\left(\frac{1}{x}-\frac{1}{x+b}\right). \quad (65)$$

Since M depends upon x, it is not trivial to obtain a solution to Eq. (66). Certainly it gives the appearance of motion in a potential field, which causes the smaller loop to be repelled by the larger one. However, it differs from true potential motion in that there is a constant Joule heating of the smaller loop, is given by

$$\bar{P} = \bar{I}^2 R = M^2\omega^2 I_{l0}^2 R/[2(R^2+\omega^2L^2)]. \quad (66)$$

If the circuits are held fixed, then Eq. (66) gives a constant rate of so-called *induction heating*. Usually, industrial applications of induction heating are made at such high frequencies that the skin effect is important; indeed, to produce heating localized near the surface, the skin effect is essential. Note that such induction heating occurs for all of the cases where the magnetic field is oscillating (Secs. IV–VII). Indeed, the jumping ring phenomenon is employed to levitate small samples of metal in a high vacuum, which are simultaneously subjected to induction heating.[3] When the metal has become molten the field is turned off and the sample falls under gravity, to be splat cooled by a piston and anvil.

VIII. ROTATIONAL EDDY CURRENT BRAKE AND INDUCTION MOTOR

A. Eddy current brake

This is a well-known problem,[1] but we include it because typically its mechanical significance is glossed over and because it is a useful preliminary to Sec. XI. A diagram of the situation is given in Fig. 4 for the case of a rotational eddy current brake. The disk has a radius a, thickness h,

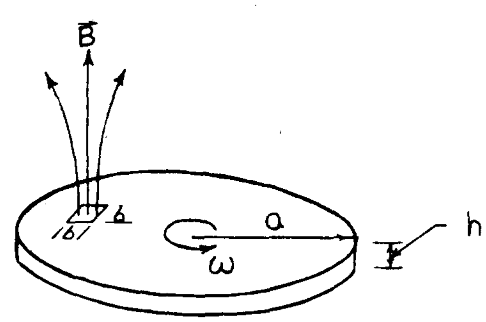

Fig. 4. Schematic of an eddy current brake, with a disk of radius a spinning about a vertical, with the axis in a fixed vertical **B** field confined to a square region of side b.

conductivity σ, and rotates with instantaneous angular velocity ω about an axis along which there is a field **B**. This field is localized about a small square of side b at a distance r from the center of the disk.

If the current flow pattern is uniform through the square and the resistance is important only within this square, we may employ the familiar equation $R = \sigma^{-1} l / A$, where the wire length l is b and the flow has a cross-sectional area $A = bh$. Thus $R \approx (\sigma h)^{-1}$. (This is a severe approximation.) The magnetic torque is approximately given by

$$\mathbf{\Gamma}_{\rm mag} = \mathbf{r} \times \mathbf{F}, \qquad (67)$$

where

$$\mathbf{F} = i\mathbf{l} \times \mathbf{B}. \qquad (68)$$

Here, the force is opposite to the disk's velocity with respect to the square, with $F = -ibB$, where $i > 0$ for radially outward current. Thus the torque is opposite to the disk's angular velocity, with

$$\Gamma_{\rm mag} = -ibrB. \qquad (69)$$

The induced emf, from $\epsilon = \int \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$, with $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, is ωBbr. Thus the emf equation is

$$\omega Bbr = iR + L \frac{di}{dt}. \qquad (70)$$

Here, L is of the order of $\mu_0 b$, so we have $L \approx (\mu_0 b) K'$, where K' is a geometrical constant. If we include friction in the bearing, with relaxation time τ, then with (69) the torque equation is

$$I\dot{\omega} = -I\omega/\tau - ibrB. \qquad (71)$$

Solving Eq. (71) for i, and employing this in Eq. (70), yields

$$\ddot{\omega} + \left(\frac{R}{L} + \frac{1}{\tau}\right)\dot{\omega} + \left(\frac{R}{L\tau} + \frac{B^2 b^2 r^2}{LI}\right)\omega = 0. \qquad (72)$$

The same equation holds for i, with $\dot{\omega}$ replaced by i.

In the limit of large $R\tau/L$, the $\ddot{\omega}$ term in (72) can be dropped (it yields a small amplitude and very rapid transient), giving

$$\dot{\omega} + (1/\tau_1)\omega = 0 \quad (R\tau/L \gg 1), \qquad (73)$$

where

$$1/\tau_1 = 1/\tau + \sigma B^2 b^2 r^2 h / I \qquad (74)$$

and we have used $R \approx (\sigma h)^{-1}$. This has a solution

$$\dot{\omega} = \dot{\omega}_0 \exp(-t/\tau_1), \qquad (75)$$

so that

$$\dot{\theta} = \omega = \omega_0 \exp(-t/\tau_1), \qquad (76)$$

where ω_0 is the initial angular velocity.

In the limit of small R/L and small $1/\tau$, ω oscillates according to

$$\ddot{\omega} + (B^2 b^2 r^2 / LI)\omega = 0, \qquad (77)$$

so that

$$\omega = \omega_0 \cos \Omega t, \qquad (78)$$

where

$$\Omega = Bbr/\sqrt{LI}. \qquad (79)$$

We will not discuss the more general case, whose possibilities may be considered by the reader. It should be clear that, when the resistance dominates, it provides a significant source of damping. The reader can compute the total $i^2 R$ Joule heating loss and verify that $\int_0^\infty i^2 R \, dt = \frac{1}{2} I\omega_0^2$, the initial kinetic energy (before **B** was turned on).

B. Induction motor

Now let **B** rotate about the axis at the fixed angular velocity ω_0. This induces currents in the disk which bring the latter into rotation, eventually at the angular velocity ω_0. If there is no friction in the bearing the torque equation is still given by (71), but now the induced emf is $(\omega - \omega_0) Bbr$, so that the emf equation (70) is replaced by

$$(\omega - \omega_0) Bbr = iR + L \frac{di}{dt}. \qquad (80)$$

Elimination of i from Eqs. (71) and (80) yields

$$\ddot{\omega} + \left(\frac{R}{L} + \frac{1}{\tau}\right)\dot{\omega} + \left(\frac{R}{L\tau} + \frac{B^2 b^2 r^2}{LI}\right)\omega = \frac{B^2 b^2 r^2}{LI}\omega_0. \qquad (81)$$

The steady-state solution $\overline{\omega}$ of Eq. (82) is, with

$$\tau_B = I/(\sigma B^2 b^2 r^2 h), \qquad (82)$$

$$\overline{\omega} = \omega_0 (1 + I/\sigma B^2 b^2 r^2 h\tau)^{-1} = \omega_0 (1 + \tau_B/\tau)^{-1}. \qquad (83)$$

If the friction in the bearing is negligible, so that $\tau \to \infty$, then $\overline{\omega} \approx \omega_0$; otherwise, bearing friction is a factor in determining the operating speed of the induction motor. Clearly, we have chosen to ask only a few questions about this system. Note that from (80) the steady-state current is given by

$$\overline{i} = (Bbr/R)(\overline{\omega} - \omega_0) = -(I\omega_0/Bbr\tau)(1 + \tau_B/\tau)^{-1} \qquad (84)$$

and the steady-state rate of dissipation $\overline{P}$ is given by

$$\overline{P} = \overline{i}^2 R + I\overline{\omega}^2/\tau = (I\omega_0^2/\tau)(1 + \tau_B/\tau)^{-1}. \qquad (85)$$

Thus as $\tau/\tau_B \to \infty$,

$$\overline{\omega} \to \omega_0, \quad \overline{i} \to 0, \quad \overline{P} \to 0; \qquad (86)$$

whereas, as $\tau/\tau_B \to 0$,

$$\overline{\omega} \to \omega_0 (\tau/\tau_B), \quad \overline{i} \to -I\omega_0 (Bbr\tau_B)^{-1}, \quad \overline{P} \to I\omega_0^2/\tau_B. \qquad (87)$$

In practice, it is of interest to consider the case where the motor must support a constant load torque $-\Gamma_L$ due to an object which the motor must turn (e.g., hoisting an elephant). In that case Eq. (71) becomes

$$I\dot{\omega} = -I\omega/\tau - ibrB - \Gamma_L. \qquad (88)$$

The emf equation is still given by Eq. (80), but the equation for ω is now given by

$$\ddot{\omega} + \left(\frac{R}{L} + \frac{1}{\tau}\right)\dot{\omega} + \left(\frac{R}{L\tau} + \frac{B^2 b^2 r^2}{LI}\right)\omega = \frac{B^2 b^2 r^2}{LI}\omega_0 - \frac{R\Gamma_L}{LI}. \qquad (89)$$

The steady-state solution $\overline{\omega}_L$ is

$$\overline{\omega}_L = (1 + \tau_B/\tau)^{-1}(\omega_0 - \tau_B \Gamma_L / I). \qquad (90)$$

Addition of a constant load torque decreases the steady-state angular velocity. Equation (90) is not valid if $\overline{\omega}_L < 0$.

IX. DAMPING OF A PERMANENT MAGNET OSCILLATING IN AN EXTERNAL B FIELD

In 1825, Arago noticed that a magnetic needle, when placed in a magnetic field, would oscillate for a much shorter time if placed near a large piece of copper than if placed far away from the copper.[4] This phenomenon was later explained by Faraday as a consequence of the action of induced currents. Without the copper present, three sources of damping come to mind: bearing friction, air friction, and currents induced in the magnetic needle itself. For small oscillations, the latter become negligible because the emf varies as $\dot{\theta}\sin\theta$, where the angle θ between the magnetic moment $\mathbf{m}$ of the needle and the external $\mathbf{B}$ field is small. The other effects may be incorporated into a rotational drag term, as done previously.

In the presence of a sheet of copper, we model the damping as the torque on the magnetic needle, which we assume to have a length $2r$, and to produce a field B_0 localized over an area b^2 of a sheet of copper having thickness h and conductivity σ. Following Sec. VIII, but including the $\mathbf{m}\times\mathbf{B}$ torque on the needle, the torque on the magnetic needle is

$$I\ddot{\theta} = -mB\sin\theta - I\dot{\theta}/\tau - ibrB_0, \tag{91}$$

where the last term is the torque due to currents induced in the copper sheet. It is obtained by the product of the moment arm r of the magnetic needle with the reactive force on the magnet ibB_0. The emf in the copper sheet satisfies [by analogy to (70)]

$$\dot{\theta}B_0br = iR + L\frac{di}{dt}. \tag{92}$$

Now consider θ to be small, so $\sin\theta\approx\theta$. Then (91) becomes

$$I\ddot{\theta}\approx -mB\theta - I\dot{\theta}/\tau - ibrB_0. \tag{93}$$

If $R \gg L(di/dt)$, as was probably the case in Arago's experiments, then (92) yields $i\approx B_0br\dot{\theta}/R$ and Eq. (93) becomes

$$\ddot{\theta} + \dot{\theta}\left(\frac{1}{\tau} + \frac{b^2r^2B_0^2}{IR}\right) + \frac{mB}{I}\theta = 0. \tag{94}$$

If this corresponds to underdamping, then with

$$\frac{1}{\tau_B} = \frac{b^2r^2B_0^2}{IR} = \frac{\sigma B_0^2b^2r^2h}{I} \tag{95}$$

[which is the same as Eq. (82)], the frequency of oscillation is

$$\Omega = \Omega_0[(1 - 1/4\Omega_0^2)(1/\tau + 1/\tau_B)^2]^{1/2}, \tag{96}$$

where

$$\Omega_0 = (mB/I)^{1/2}. \tag{97}$$

The characteristic decay rate is

$$\alpha = 2(1/\tau + 1/\tau_B). \tag{98}$$

It is thus clear how the presence of the copper sheet could have increased the rate of decay. Further, since $1/\tau_B$ is proportional to h, the decay rate increases as the thickness of the copper sheet increases, in qualitative agreement with another of Arago's findings.

One experiment Arago could not have performed involves using a superconducting aluminum sheet, with $R = 0$. In that case, by (92) one has $i = B_0br\theta/L$ (up to a constant, which we set to zero), so that Eq. (93) becomes

$$\ddot{\theta} + (1/\tau)\dot{\theta} + (mB/I + B_0^2b^2r^2/LI)\theta = 0. \tag{99}$$

For $\tau\to\infty$ this gives an oscillation frequency

$$\overline{\Omega}_0 = [(mB + B_0^2b^2r^2/L)/I]^{1/2}, \tag{100}$$

which is higher than Ω_0. In other words, the presence of a superconductor causes the needle to oscillate more rapidly. Physically, a repulsive image magnet has been induced in the superconductor; this provides an additional restoring force on the magnetic needle.

X. CONCLUSION

It is hoped that this collection of problems has added to the reader's insight and perhaps has suggested new ways to employ Faraday's law. Would it not be interesting to move an object about simply by beaming microwaves at a metal plate on the object and thereby pushing it in the opposite direction? No motors would be needed: only wheels, metal plates, and a directable source of microwaves. Moreover, by using capacitors and a second (and lower) microwave frequency, it might be possible to move things about by *pulling* them. That is, the lower frequency would be below the object's resonant frequency, so that the capacitor would dominate, leading to an attractive force; whereas the higher frequency would be above the object's resonant frequency, so that the inductor would dominate, leading to a repulsive force.

Note added in proof: Since submitting this paper, I have become aware of the following work that treats the "levitating" ring: S. Y. Mak and K. Young, Am. J. Phys. **54**, 808 (1986).

ACKNOWLEDGMENTS

I would like to acknowledge D. G. Naugle, S. Ramirez, and A. Leitner for useful conversations and correspondence.

[1]D. Halliday and R. Resnick, *Physics* (Wiley, New York, 1978).
[2]D. O. Woodbury, *Elihu Thomson, Beloved Scientist* (Museum of Science, Boston, 1960), p. 178.
[3]E. C. Okrass, D. M. Wroughton, G. Comenetc, T. H. Brace, and J. C. R. Kelly, J. Appl. Phys. **23**, 545 (1952).
[4]E. T. Whittaker, *A History of the Theories of Aether and Electricity* (Philosophical Library, New York, 1951).