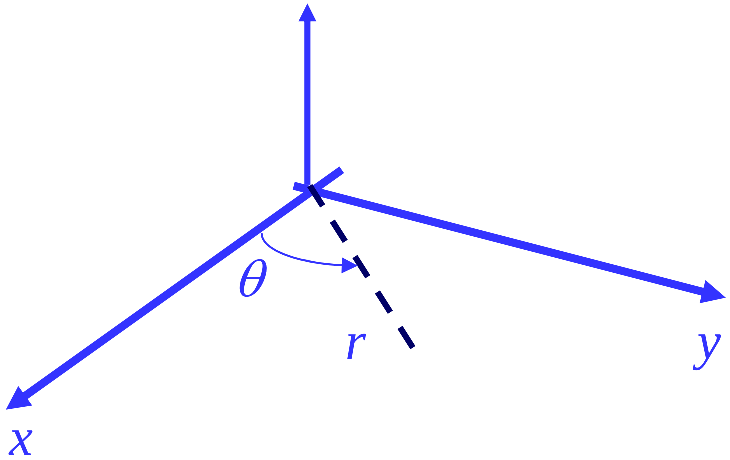
2D Laplace's Equation in Polar Coordinates



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ where $$x = x(r,\theta),\ \ y = y(r,\theta)$$

$$u(x,y) = u(r,\theta)$$

So, Laplace's Equation is $\nabla^2 u(r,\theta) = 0$

We next derive the ***explicit*** polar form of Laplace's Equation in 2D

Recall the chain rule: $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Use the product rule to differentiate again

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} \qquad (*)$$

and the chain rule again to get these derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial \theta}{\partial x} = \frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} = \frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}$$

The required partial derivatives

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r^2 = x^2 + y^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}, \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$

in like manner ….

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{r^4}, \frac{\partial^2 \theta}{\partial y^2} = \frac{2xy}{r^4}$$

Back to Laplace's Equation in polar coordinates

Plugging in the formula for the partials on the previous page to the formulae on the one before that we get:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\frac{x^2}{r^2} + \frac{\partial u}{\partial r}\frac{y^2}{r^3} + \frac{\partial^2 u}{\partial r \partial \theta}\frac{-2xy}{r^3} + \frac{\partial u}{\partial \theta}\frac{2xy}{r^4} + \frac{\partial^2 u}{\partial \theta^2}\frac{y^2}{r^4}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\frac{y^2}{r^2} + \frac{\partial u}{\partial r}\frac{x^2}{r^3} + \frac{\partial^2 u}{\partial r \partial \theta}\frac{2xy}{r^3} - \frac{\partial u}{\partial \theta}\frac{2xy}{r^4} + \frac{\partial^2 u}{\partial \theta^2}\frac{x^2}{r^4}$$

So Laplace's Equation in polars is

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is equivalent to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$