Electric fields and charge distributions associated with steady currents

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By means of the example of a very long straight wire, we investigate the question of what determines the electric field outside current-carrying conductors.

Merzbacher[1] raises the question of the conditions which determine the electrical field *outside* of a straight current-carrying wire. The answer is related to the answer to another question, viz., what causes the electric field *inside* a current-carrying wire. We are not aware of any literature where these questions are answered and would like to offer the following solution.

For a steady current in a homogeneous conductor, the charge density ρ is zero inside the conductor. This familiar result follows when the continuity equation for steady currents,[2]

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = \nabla \cdot \mathbf{J}, \tag{1}$$

is combined with Gauss's law

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 \tag{2}$$

and the relation between the electric field **E** and the current density **J**,

$$\mathbf{J} = \sigma \mathbf{E}, \tag{3}$$

since the conductivity σ is constant.

Let us now consider an infinitely long wire of constant circular cross section. Let us also imagine a Cartesian coordinate system whose origin is somewhere on the axis of the wire and whose z axis coincides with the axis of the wire. It follows from Eq. (1) combined with symmetry considerations that only the z component of **J** is nonzero and that it can only be a function of x and y. It follows from Maxwell's equation

$$\nabla \times \mathbf{E} = 0, \tag{4}$$

that such a $J_z(x,y)$ cannot be a function of x and y either. Thus **J** is a constant vector everywhere inside the wire pointing in the z direction and, because of Eq. (3), the same is true for **E**.

If there exists a static electric field somewhere in space, we require as a boundary condition that there must be charges somewhere from which this field "originates." As we have just seen, in the present case the charges cannot reside inside the wire. Neither are there any charges outside the wire. The only place where charges can be is on the surface of the wire because the preceding argument does not exclude the possibility of a surface charge density. We will now guess a distribution of surface charge and then verify that it leads to the correct field inside the wire. Let us introduce cylindrical polar coordinates r, θ, z related to the Cartesian coordinates x, y, z in standard fashion. We will assume there to be an amount of charge $cz\, d\theta\, dz$ within a wire surface element defined by $d\theta$ and dz, where c is independent of θ and z. Thus the assumed charge density varies linearly with z and has cylindrical symmetry.

At any point (r, θ, z) either inside or outside a wire extending from $-L$ to L, the electric potential $\phi(r, z)$ due to such a charge distribution is given by

$$\phi(r,z) = \frac{c}{4\pi\epsilon_0} \int_0^{2\pi} d\theta \int_{-L}^{L} dz' \times \frac{z'}{\sqrt{(z'-z)^2 + R^2[\sin^2\theta + (\cos\theta - r/R)^2]}}, \tag{5}$$

where R is the radius of the wire. By an elementary but somewhat tedious calculation, done in the Appendix, one can show that for $L \gg R$, expression (5) is equal to

$$\phi(r,z) = (c/\epsilon_0)(\ln L/R)z \quad L \gg R,\ L \gg z,r. \tag{6}$$

Expression (6) is valid for points that are within a distance much smaller than L away from the origin. As $L \to \infty$, formula (6) becomes valid for all points both inside and outside the wire. We see that formula (6) does indeed give the correct electric field inside the wire,

$$\mathbf{E} = -\nabla\phi = -(c/\epsilon_0)(\ln L/R)\hat{z}. \tag{7}$$

For a given field whose z component equals E, the necessary surface charge density equals

$$cz/R = [-E\epsilon_0/R \ln(L/R)]z. \tag{8}$$

As $L \to \infty$, this surface charge density goes to zero at any given point z.

Thus we obtain to the questions asked at the beginning the surprising answer that an infinitely long wire in which a steady current is flowing has a vanishing surface charge density (8) and a uniform electric field (7) both inside and outside the wire. It is particularly noteworthy that as $L \to \infty$ the electric field outside the wire has a vanishing component normal to the wire. This is so because of cancellations of contributions to this component from the positive and negative charges on the wire. One should realize that charge distribution (8) assumes that the net charge on the wire is zero. Other solutions exist, which superimpose on (8) a uniform charge density. For such solutions there would be a nonvanishing normal electric field component outside of the wire.

Let us consider the Poynting vector **S** outside the wire. The magnetic field outside a straight wire carrying a current I has the magnitude

$$H = I/2\pi r \tag{9}$$

and points clockwise when viewed in the current direction. The uniform electric field outside the wire has magnitude E and is directed in the current direction. Thus the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{10}$$

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everywhere points directly towards the axis of the wire and is of magnitude

$$S = IE/2\pi r. \tag{11}$$

When S is integrated over a cylindrical ring of unit length and concentric with the wire, we obtain the power crossing that ring towards the wire. The result, which equals IE, is independent of r and is equal to the Joule heating that takes place inside the wire.

Summing up, to determine the electric field outside a system of conductors carrying steady currents one has to solve the following problem. First, one has to determine a solution to Laplace's equation for the electric potential inside the conductors,

$$\nabla^2\phi = 0, \tag{12}$$

such that the currents have the given values. We are assuming that the conductivities are given. Next, one determines a surface charge density which produces the electric potential inside the conductors. The answer should be unique if the entire system is assumed to have zero net charge. At places where the conductivity is not constant, nonzero charge densities will also exist inside a conductor. The charge distribution determined in this manner will allow calculation of the electric potential outside of the conductors as well. It may be an interesting exercise to determine in this manner the surface charges on two parallel current carrying wires and the corresponding fields. It is clear that a linearly varying charge density on either wire as in Eq. (8) cannot be the answer if the two wires have different internal electric fields because density (8) would imply that both wires are producing uniform electric fields throughout space whose superposition would also be uniform everywhere in contradiction to the assumption.

APPENDIX

We now would like to evaluate integral (5). With the transformation of variables $\omega = z' - z$ and the abbreviation

$$f(r,\theta) = \sin^2\theta + (\cos\theta - r/R)^2 \tag{A1}$$

we obtain

$$\frac{4\pi\epsilon_0}{c}\phi(r,z) = \int_0^{2\pi} d\theta \int_{-L-z}^{L-z} \frac{\omega\, d\omega}{\sqrt{\omega^2 + R^2 f}} + z\int_0^{2\pi}\int_{-L-z}^{L-z} \frac{d\omega}{\sqrt{\omega^2 + R^2 f}}. \tag{A2}$$

The first and second terms on the right-hand side of Eq. (A2), denoted A and B, can be evaluated as follows:

$$A = \int_0^{2\pi} d\theta \sqrt{\omega^2 + R^2 f}\Big|_{\omega=-L-z}^{L-z} = \int_0^{2\pi} d\theta\, [\sqrt{(L-z)^2 + R^2 f} - \sqrt{(L+z)^2 + R^2 f}]. \tag{A3}$$

Assuming the conditions

$$L \gg z, \quad L \gg r, \quad L \gg R \tag{A4}$$

and neglecting everything that is of order $(z/L)^2$ compared to the leading term, Eq. (A3) reduces to

$$A = \int_0^{2\pi} d\theta \left[(L-z)\left(1 + \frac{R^2 f}{2(L-z)^2}\right) - (L+z)\left(1 + \frac{R^2 f}{2(L+z)^2}\right)\right] = -4\pi z. \tag{A5}$$

Similarly,

$$\begin{aligned} B &= z\int_0^{2\pi} d\theta \ln(\omega + \sqrt{\omega^2 + R^2 f})\Big|_{\omega=-L-z}^{L-z} \\ &= z\int_0^{2\pi} d\theta \ln\left(\frac{L - z + \sqrt{(L-z)^2 + R^2 f}}{-L - z + \sqrt{(L+z)^2 + R^2 f}}\right) \\ &= z\int_0^{2\pi} d\theta \ln\left(\frac{2(L-z) + R^2 f/2(L-z)}{R^2 f/2(L+z)}\right) \\ &= z\int_0^{2\pi} d\theta \ln\left(\frac{4L^2}{R^2 f}\right) \\ &= \left(4\pi \ln\frac{L}{R} + \int_0^{2\pi} d\theta \ln\frac{4}{f(r,\theta)}\right) z. \end{aligned} \tag{A6}$$

Combining results (A5) and (A6) with Eq. (A2) we obtain for the electric potential under conditions (A4) the expression

$$\phi(r,z) = \frac{c}{\epsilon_0}\left(\ln\frac{L}{R} + \frac{1}{4\pi}\int_0^{2\pi} d\theta \ln\frac{4}{f(r,\theta)} - 1\right). \tag{A7}$$

For sufficiently large L, Eq. (A7) is dominated by the first term on the right-hand side, and we obtain formula (6).

[1]E. Merzbacher, Am. J. Phys. **48**, 178 (1980).
[2]All formulas and calculations will be in SI units.

LETTERS TO THE EDITOR

Letters express personal opinions and may critically examine any aspect of physics or physics instruction. They need not conform to our regular editorial policy and ordinarily are not reviewed. From the large number submitted, published letters are selected for their expected interest for our readers. They must be brief and are subject to editing, with the author's approval of significant changes. Comments on regular articles and notes are reviewed according to a special procedure and appear in the Notes and Discussions section (see the "Statement of Editorial Policy" in the January issue). Running controversies among letter writers will not be published.

LETTER TO THE EDITOR

The concept of the atom is usually ascribed to Democritus, and its development from a philosophical abstraction attributed to Dalton (1808) and Avogadro (1811). Their formulation was the culmination of more than a century of work on the quantitative investigation of chemical reactions. This is thoroughly explored in the first two of the four articles making up the symposium "History of the Atom" [Am. J. Phys. **49** (3) (1981)].

There is, however, a parallel path that has its origins in the study of crystals, and that began in the 17th century. These two lines of development were finally united in 1912, with the discovery of x-ray diffraction.

As early as 1633 Kepler had published diagrams in *A New Year's Gift of Hexagonal Snow*, in which the hexagonal symmetry of the snowflake was related to the hexagonal patterns that could be made by arranging spherical balls in a plane. In 1665, Robert Hooke published very similar diagrams in *Micrographia* and described experiments in which he made models of various solid objects from "bullets" (presumably lead shot). A review of this period would not be complete without mention of the most striking drawing of Iceland spa, published in 1690 by Huygens, in which the shape of the rhomb and the "easy cleavage" planes were explained in terms of the close packing of some ellipsoidal elementary units.

What all three writers show is the insight that the external shape, or morphology, of a crystal is controlled by the packing of what Hooke described as "globular particles" and Huygens "small invisible equal particles."

There is in all this a confusion between the concept of the atom or molecule and that of the unit cell, but, by 1801, Haüy had replaced the "particles" by a parallelopiped, or unit cell, which he called the "molécule intégrante," this, in turn, being composed of "molécules élémentaires" or atoms, and in so doing had laid the foundations of modern crystallography.

The exact relationship between the unit cell and the constituent atoms was a matter of considerable speculation throughout the 19th century, spurred on by the development of symmetry theory, but the problem was not solved until the experiment of von Laue on the x-ray diffraction patterns of zinc blende in 1912 and its subsequent interpretation by W. L. Bragg in terms of what we would now call a face-centered-cubic lattice. Perhaps the final vindication of the views of Kepler, Hooke, and Huygens came in the late 1920s with the development of Fourier methods, which have proved decisive in our understanding of the solid state in physics, chemistry, and biology.

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LETTER TO THE EDITOR

The solution to the problem by G. P. Sastry[1] is incorrect in part. If one makes the reasonable assumption that specular reflection is intended, then Snell's law immediately allows only $\Theta = 0$, except when $l = 0$, without use of Fermat's principle. The quantity l is a distance and, by definition, $l > 0$. The required change in sign of the second derivative results from the fact that, for point A outside the hemisphere, the angle Θ becomes the external angle of the triangle and the form of the law of cosines must be changed accordingly. Thus for this case,

$$\left(\frac{d^2D}{d\Theta^2}\right)_{\Theta=0} = \frac{-lR}{R+l},$$

which also eliminates the apparent problem at $l = R$ and the artificial limitation $|l| < R$.

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[1]G. P. Sastry, Am. J. Phys. **49**, 345 (1981).

LETTER TO THE EDITOR

The recent article "Electric fields and charge distributions associated with steady currents," which appeared in Am. J. Phys. **49**, 450 (1981), gives rise to questions and doubts concerning the efficiency of the editorial process at the Journal.

As many readers have probably discovered for themselves, the results obtained in this paper can all be found explained with much greater mathematical clarity and physical insight in A. Sommerfeld's *Electrodynamics* (Academic, New York, 1952). The authors of the paper have to assume (actually, guess) that the surface charge density on the wire is proportional to z, whereas, in Sommerfeld's elegant and very thorough treatment, this result is derived clearly and unambiguously from the basic equations.

I find it rather surprising that the authors of the paper are not aware of the above reference. Sommerfeld's *Electrodynamics* is the 3rd volume of the famous series "Lectures on Theoretical Physics" and is one of the classic treatises of the literature on electromagnetism. It is even more astonishing that these facts have totally escaped the scrutiny of the referees.

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