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Phase Waves of Louis deBroglie

This article is a translation of the first chapter of Louis deBroglie's PhD thesis, "Recherches sur la théorie des Quanta" (Univ. of Paris, 1924). The translation is by Jared W. Haslett of the Physics Department of the University of Chicago and is published with the kind permission of Louis deBroglie. The help in translation given by Mr. J. A. Kotzman and Mr. M. A. Lipson is gratefully acknowledged.

Two factors suggest the desirability of considering Louis deBroglie's dissertation. Both recent renewed interest (F. Kubli, "Louis deBroglie und die Entdeckung der Materiewelle," doctoral dissertation ETH, Zurich 1970) and the fact that it has never appeared in the English language suggests that there are very likely readers who will note with interest not only the results of deBroglie's work but also the method by which he arrived at his conclusions. It is curious that virtually all physics texts in English refer to the results of deBroglie's investigations but do not touch upon the technique by which he arrived at this widely accepted conclusion ($\lambda = h/mv$). The great computational success of Schrödinger and Born in connection with wave mechanics and especially the emphasis on probability considerations tended to divert the attention of physicists from the fundamental hypothesis of deBroglie. Recently deBroglie has diplomatically recalled attention to his original emphasis by way of what he calls his "double solution." That is, normalization may follow along the usual lines of probability or along the lines of matter–energy density. It is hoped that publication of this classic of physics, or rather one chapter of it, will fill in a long standing gap.

JARED W. HASLETT

I. THE RELATIONSHIP BETWEEN THE QUANTUM AND RELATIVITY

One of the most important new concepts introduced by relativity is that of the inertia of energy. According to Einstein, energy ought to be considered as having mass, and all mass exhibits some energy. Mass and energy are always connected with each other by the general relationship

$$\text{energy} = \text{mass} \times c^2,$$

c being the constant called "the speed of light," but we prefer to call it "the limiting speed of energy" for reasons brought out further on. Since there is always a fixed ratio between mass and energy, matter and energy should be considered as two synonymous terms designating the same physical reality. At first the atomic theory, and later the electron theory, taught us to consider matter as essentially discontinuous, and this leads us to affirm that all forms of energy (contrary to old ideas about light) if not entirely concentrated in small regions of space, are at the very least condensed around certain singular points.

The principle of inertia of energy attributes to a body whose rest mass is m_0 (i.e., as measured by an observer who is at rest with respect to it) a proper energy, m_0c^2. If the body is in uniform motion with a velocity $v = \beta c$ in relation to an observer whom we will call for simplicity the fixed observer, its mass will have for him the value $m_0/(1-\beta^2)^{1/2}$ according to a well known result of relativistic dynamics, and consequently, its energy will be $m_0c^2/(1-\beta^2)^{1/2}$. As kinetic energy can be defined as the augmentation which the energy of a body undergoes for the fixed observer, when it passes from a state of rest to the velocity $v = \beta c$, its value is found by the following expression:

$$E_{\text{kinetic}} = m_0c^2/(1-\beta^2)^{1/2} - m_0c^2$$
$$= m_0c^2[(1-\beta^2)^{-1/2} - 1],$$

which, in the event of small values of β, naturally leads to the classical formula

$$E_{\text{kinetic}} = \tfrac{1}{2} m_0 v^2.$$

Having reviewed this, let us find a form in which we can introduce the quantum into the dynamics of relativity. It seems to us that the fundamental idea of the theory of the quantum is the impossibility of depicting an isolated quantity of energy that is not accompanied by a certain frequency. This connection is expressed by what we shall call the relation of the quantum,

$$\text{energy} = h \times \text{frequency},$$

h being Planck's constant.

The progressive development of the theory of the quantum has focused attention on mechanical action several times, and there have been many attempts to express the relationship of the quantum where action is substituted for energy. Certainly the constant h has the dimensions of an action known as ML^2T^{-1}, and this is not due to chance, since the theory of relativity teaches us to classify action with the chief constants of physics. But action is a quantity of a very abstract character, and after numerous speculations on the quanta of light and the photoelectric effect, we have re-established as our base the energy expression, thereafter freed from the search as to why action plays such a great role in a number of questions.

The relationship of the quantum undoubtedly would not make as much sense if energy could be distributed in a continuous fashion in space, but we have come to see that it is certainly not so. Hence, it can be conceived, according to an important law of nature, that to each bit of energy of the proper mass m_0 there is connected a periodic phenomenon of the frequency ν_0, stated thus,

$$h\nu_0 = m_0 c^2,$$

ν_0 being measured, of course, in a system which is at rest with respect to a certain amount of energy. This hypothesis is the base of our system, and it is worth, as are all hypotheses, what the conclusions which can be drawn from it are worth.

Should we assume that the periodic phenomenon is localized in the *interior* of a particle of energy? Not necessarily. It will be seen in Sec. III that undoubtedly it is distributed throughout an extended region of space. Besides, what do we mean when we speak of the interior of a particle of energy? For us the electron is the archetype of the isolated particle of energy which we believe, perhaps wrongly, that we know best. But according to our customary ideas, the energy of the electron spreads throughout all space with a very high concentration in a region of very small dimensions whose properties, moreover, are not at all well known. What characterizes the electron as a particle of energy is not the small place it occupies in space, and I repeat that it occupies it entirely, but the fact that it is insecable, indivisible, that it forms *one unity*.

Having admitted the existence of a frequency that is connected with a particle of energy, let us try to discover how this frequency is manifested to the fixed observer mentioned above. The Lorentz–Einstein transformation of time teaches us that a periodic phenomenon connected with a body in motion appears to the fixed observer to be slowed down by the factor of 1 divided by $1-\beta^2$, which is the famous slowing down of clocks. Then the frequency observed by the fixed observer will be

$$\nu_1 = \nu_0 (1-\beta^2)^{1/2} = (m_0 c^2/h)(1-\beta^2)^{1/2}.$$

On the other hand, since the energy of the moving body is equal to $m_0c^2/(1-\beta^2)^{1/2}$ for the same observer, the corresponding frequency according to the relationship of the quantum is

$$\nu = h^{-1}[m_0 c^2/(1-\beta^2)^{1/2}].$$

The two frequencies ν_1 and ν are essentially different since the factor $(1-\beta^2)^{1/2}$ is not involved in the same way. This is a difficulty that has intrigued me for a long time; I have succeeded in eliminating it by demonstrating the following theorem that I shall call the theorem of the harmony of phases: The periodic phenomenon connected with a moving body whose frequency is for the fixed observer equal to

$$\nu_1 = h^{-1} m_0 c^2 (1-\beta^2)^{1/2}$$

appears to him to be constantly in phase with a wave of frequency

$$\nu = h^{-1} m_0 c^2 (1-\beta^2)^{-1/2}$$

emitted in the same direction as the moving body with the velocity $V = c/\beta$.

The demonstration is very simple. Let us suppose that at the time $t=0$, the periodic phenomenon connected with the moving body and the wave defined above are in phase with each other. At the time t, the moving body after the first instant travels a distance equal to $x=\beta ct$ and the phases of the periodic phenomenon has varied by

$$\nu_1 t = (m_0c^2/h)(1-\beta^2)^{1/2}(x/\beta c).$$

The phase of the portion of the wave that overlays the moving body has varied by

$$\nu[t-(\beta x/c)] = (m_0c^2/h)(1-\beta^2)^{-1/2} \times [(x/\beta c)-(\beta x/c)] = (m_0c^2/h)(1-\beta^2)^{1/2}(x/\beta c).$$

As we have stated, the agreement of phases persists.

It is possible to give another demonstration of this theorem, basically identical to the above, but perhaps more impressive. If t_0 represents time for an observer at rest with respect to the moving body (the proper time for the moving body), Lorentz's transformation gives

$$t_0 = (1-\beta^2)^{-1/2}[t-(\beta x/c)].$$

The periodic phenomenon that we are picturing is represented for the same observer by a sine-shaped function $\nu_0 t_0$. For the fixed observer, it is represented by the same sine shaped function $\nu_0(1-\beta^2)^{-1/2}[t-(\beta x/c)]$, which function represents a wave of frequency $\nu_0/(1-\beta^2)^{1/2}$ traveling with the speed c/β in the same direction as the moving body.

It is quite necessary now to reflect on the nature of the wave whose existence we have been led to conceive. The fact that its speed $V=c/\beta$ is necessarily greater than c(β always being less than 1, otherwise its mass would be infinite or imaginary) shows us that it cannot be a question of a wave carrying energy. Our theorem reveals to us, moreover, that it represents the distribution in space of the *phases* of a phenomenon; it is a "phase wave."

To be more precise on this last point, we shall employ a mechanical comparison that is a bit crude, but one that will catch the imagination. Let us picture a horizontal circular disk with a very great radius; from this disk are suspended identical systems, each made up of a spiral spring to which a weight is attached. The number of the systems thus suspended per unit area of the disk, or their density, diminishes very rapidly as we get farther from the center of the disk so that there is a condensation of the systems around the center. All these spring-weight systems, being identical, having the same period; let us have them oscillate with the same amplitude and the same phase. The surface passing through the centers of gravity of all the weights will be a plane that rises and falls in alternating movement. The total effect thus obtained forms a very rough analogy to the isolated particle of energy such as we conceive it.

The description that we have been giving concerns an observer at rest with respect to a disk. If another observer sees the disk move in straight line motion with the speed $v=\beta c$, each weight would appear to him like a little clock undergoing the slowing-down of Einstein; moreover, the disk and its distribution of oscillating systems would no longer be as isotropes around the center by reason of Lorentz contraction. But for us the fundamental fact (Sec. III will make this better understood) is the dephasing of the movements of the various weights. If at a given moment of his time, our fixed observer considers the geoposition of the centers of gravity of the various weights, he observes a cylindrical surface in the horizontal direction whose vertical sections, parallel to the velocity of the disk are sine-shaped. It corresponds, in the particular case under consideration, to our phase wave; according to the general theorem, the surface has a speed c/β parallel to that of the disk, and the frequency of vibration of a point fixed on the abscissa and constantly at rest on it is equal to the proper frequency of the oscillation multiplied by $1/(1-\beta^2)^{1/2}$. With this example we see clearly (and this is our excuse for such protracted insistence on it) how the phase wave corresponds to the transport of the phase and not at all to that of the energy.

The preceding results seem to us to be of extreme importance, because, with the aid of a hypothesis strongly suggested by the very notion of quantum, they establish a connection between the movement of a moving body and the propagation of a wave, and thus let us glimpse the possibility of a synthesis of the conflicting theories concerning the nature of radiations. Already we can note that the rectilinear propagation of the phase wave is connected with the rectilinear movement of the moving body; Fermat's principle applied to the phase wave determines the form of those rays which are vertical, whereas Maupertuis's principle applied to the moving body determines its rectilinear trajectory which is one of the rays of the wave. In Chap. II we will try generalizing on this coincidence.

II. PHASE VELOCITY AND GROUP VELOCITY

It is now necessary for us to demonstrate an important relationship existing between the speed of the moving particle and that of the phase wave. If the waves of nearly the same frequencies are propagated in the same direction Ox with velocities V which we call velocities of phase propagation, these waves will give by their superposition, heterodyning phenomena, if the velocity V varies with the frequency ν. These phenomena have been studied notably by Lord Rayleigh for the case of dispersive media.

Let us consider two waves of nearly the same frequencies ν and $\nu'=\nu+\delta\nu$ and the velocities V and $V'=V+(dV/d\nu)\delta\nu$; their superposition is expressed analytically by the following equation obtained by neglecting the second order term in ν which is $\delta\nu$:

$$\begin{aligned}\sin 2\pi[\nu t-(\nu x/V)+\phi]&\\ +\sin 2\pi[\nu' t-(\nu' x/V')+\phi']&\\ =2\sin 2\pi[\nu t-(\nu x/V)+\psi]&\\ \times\cos 2\pi\left(\tfrac{1}{2}(\delta\nu)t-x\frac{d(\nu/V)}{d\nu}\,\tfrac{1}{2}(\delta\nu)+\psi'\right).&\end{aligned}$$

We have therefore a resultant sinusoidal wave whose amplitude is modulated at the frequency $\delta\nu$ since the sign of the cosine is of little importance. This is a well known result. If one designates by U the velocity of propagation of the heterodyne or the group velocity of the waves, one finds

$$U^{-1}=\frac{d(\nu/V)}{d\nu}.$$

Let us return to the phase waves. If one assigns to the moving body a speed $v=\beta c$ not giving to β a completely determined value, but only that it be bounded by the limits of β and $\beta+\delta\beta$; the frequencies of the corresponding waves fill a small interval between ν and $\nu+\delta\nu$.

We are going to establish the following theorem which we will ultimately use:

The group velocity of the phase waves is equal to the velocity of the moving body.

In fact, this group velocity is determined by the formula given above in which the same V and ν can be considered as functions of β since one has

$$V=c/\beta,\qquad \nu=h^{-1}[m_0c^2/(1-\beta^2)^{1/2}].$$

One can write

$$U=\frac{d\nu/d\beta}{d(\nu/V)/d\beta}$$

Since

$$d\nu/d\beta=(m_0c^2/h)[\beta/(1-\beta^2)^{3/2}],$$

$$\frac{d(\nu/V)}{d\beta}=\frac{m_0c}{h}\frac{d[\beta/(1-\beta^2)^{1/2}]}{d\beta}=(m_0c^2/h)(1-\beta^2)^{-3/2}.$$

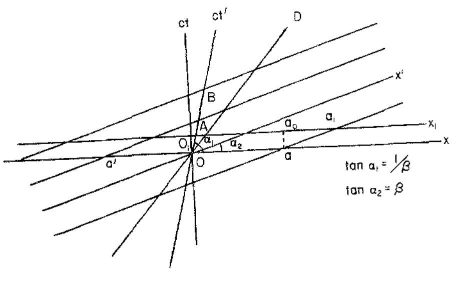
Therefore,

$$U = \beta c = v.$$

The group velocity of the phase waves is definitely equal to the velocity of the moving body. This result requires a remark: In the dispersion theory of waves, if one excludes the regions of absorption, the energy propagation velocity is equal to the group velocity.[1] Even though we examine it from a very different point of view, we find an analogous result since the velocity of the moving body is nothing but the energy propagation velocity.

III. PHASE WAVES IN SPACE–TIME

Minkowski was the first to show that one obtains a simple geometric representation of the relations of space and of time introduced by Einstein by considering a Euclidean multiplicity of four dimensions—called "universe" or "space–time." To do this he took three rectangular spacelike axes and a fourth axis normal to the first three on which was put time multiplied by $c(-1)^{1/2}$. Today one more usually puts the real quantity ct on the fourth axis, but then the mutually normal planes passing through this axis and normal to it have a pseudo-Euclidean hyperbolic geometry in which the fundamental invariant is $c^2dt^2 - dx^2 - dy^2 - dz^2$.



Let us consider now the space and time variables associated with the four axes of the fixed observer. We take the x axis for the straight path of the moving particle and we show in our paper [above] the plane Otx containing the time axis and the above mentioned trajectory. Under these conditions, the world line of the particle is represented by a straight line forming an angle of less than 45° with the time axis; this line is, moreover the time axis observed from the (reference frame) of the moving particle. We show in our figure two time axes intersecting at the origin, which does not violate our general principle.

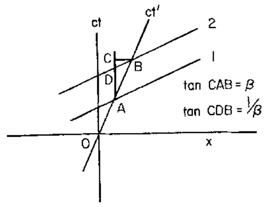
If the speed of the particle according to a fixed observer is βc, the slope of Ot' has a value of $1/\beta$. The line Ox' is in the plane tOx in the frame of the moving observer at time O, is symmetric to Ot' and the bisector OD; it is easy to demonstrate analytically by means of the Lorentz transformation; however, this results immediately in the fact that the limiting speed of energy c has the same value for all reference systems. The slope of Ox' is then β. If the space surrounding the particle is the site of a periodic phenomenon, the state of the space will become again the same for the moving observer each time at the elapsed (interval) $1/c(OA) - 1/c(AB)$ is equal to the proper period $T_0 = 1/\nu_0 = h/m_0c^2$ of the phenomenon.

The straight lines parallel to Ox' are then the "profiles" of the "equiphase" spaces of the moving observer in the xOt plane. The projection of the points $\cdots a'$, O, $a \cdots$ have their intersection in the frame of the fixed observer, at the instant O; these intersections of two surfaces in three dimensions are some surfaces of two dimensions and even planes because all the surfaces under consideration are Euclidean. When the time for the fixed observer passes, that part of the space–time continuum which is for him spacelike is represented by a straight line parallel to Ox moving with a uniform motion in the positive t sense. It is easily seen that equiphase planes $\cdots a'$, O, $a \cdots$ are displaced in the frame of the fixed observer with a speed c/β. In effect, if the line Ox_1 of the figure represents the frame of the fixed observer, at time $t = 1$, one has $aa_0 = c$. The phase which at time $t = 0$ was at a is now at a_1; for the fixed observer, it is then displaced in space by an amount a_0a_1 in the Ox direction, during a unit of time. One can then say that the speed is:

$$V = a_0a_1 = aa_0 \cot(xOx') = c/\beta.$$

The ensemble of equiphase planes constitutes that which we have called phase waves.

It remains to examine the question of frequencies. Let us once again draw a simple figure [below].



The straight lines 1 and 2 represent AB two successive equiphase planes for a fixed observer. AB is, as we have said, equal to c times the proper period $T_0 = h/m_0c^2$.

AC, the projection of AB on the Ot axis, is equal to

$$cT_1 = cT_0/(1-\beta^2)^{1/2}.$$

This results from a simple application of trigonometric relations; however, we may say that upon applying trigonometry to the figures of the xOt plane, it is always necessary to keep in mind the particular anisotropy of the plane. The triangle ABC gives us:

$$(AB)^2 = (AC)^2 - (CB)^2 = (AC)^2(1 - \tan^2 \angle CAB)$$
$$= (AC)^2(1-\beta^2),$$
$$(AC) = (AB)/(1-\beta^2)^{1/2}.$$

The frequency $1/T_1$ is that which the periodic phenomenon appears to have for the fixed observer as he follows it with his eyes as it is in motion. This is

$$\nu_1 = \nu_0(1-\beta^2)^{1/2} = (m_0c^2/h)(1-\beta^2)^{1/2}.$$

The period of the waves at a point in the frame of the fixed observer is not given by $1/c(AC)$, but by $1/c(AD)$. Let us calculate it.

In the small triangle BCD, one finds the relation $CB/DC = \beta^{-1}$ from which $DC = \beta CB = \beta^2 AC$. Moreover, $AD = AC - DC = AC(1-\beta^2)$. The new period T is then equal to

$$T = c^{-1}AC(1-\beta^2) = T_0(1-\beta^2)^{1/2}$$

and the frequency ν of the waves is expressed by

$$\nu = T^{-1} = \nu_0/(1-\beta^2)^{1/2} = m_0c^2/h(1-\beta^2)^{1/2}.$$

We have found once again all the results obtained analytically in Sec. I, but now we see better how they relate to the general conception of space–time and why the phase shift of periodic motion taking place at different points in space depends on the manner in which simultaneity is defined by the theory of relativity.

[1] See, e.g., L. Brillouin, *La théorie des quanta et l'atome de Bohr,* Journal de Physique (Paris, 1922), Chap. 1, pp. 181. (Recueil des conférences-rapports de documentation sur la physique, Vol. 2, 1re serie conferences, 4, 5, 6.)