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The Physical Content of Quantum Mechanics

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THE physical ideas involved in the system of thought and calculation known as "quantum mechanics" have developed to a considerable extent after the development of the mathematical methods. This is contrary to what might be thought to be the correct method of developing physical theory, namely to have a clear physical idea and then express it in mathematical language. On the other hand, it must be realized that the development of new concepts is much more difficult than the mathematical elaboration of old ones, and it need not be surprising that the development has been halting.

That there is at present no unanimity as to the physical content of quantum mechanics is evidenced by the number of papers which have recently appeared commenting on the treatment by Einstein, Rosen and Podolsky[1] of an apparent paradox in the quantum-mechanical treatment. These papers have presented a surprising number of different attitudes toward the subject. I shall attempt to give here, in very elementary form, an attitude toward quantum mechanics which is self consistent and which I believe approximates very closely to that of Bohr[2] and others.

Classical and quantum-mechanical prediction

The object of a physical theory is to predict from a certain set of experiments the results of other experiments or measurements to be performed in the future. In classical mechanics the behavior of a system is predicted by

(*a*) measuring the coordinates and velocities, or momenta, of the system at some time $t=t_0$; and

(*b*) using the laws of motion, that is, Newton's, Lagrange's, or Hamilton's equations, to predict the future values of these coordinates and momenta. The knowledge of the physical nature of the system is included in the form of these equations. This is really a prediction of the results of future experiments, since it is presumed that the future values will be measured in some way.

Both parts of this process are essential. Without some initial conditions the solutions of the equations of motion give no specific results; and without the use of the laws of motion, a single observation of the system is of little use. It may be mentioned here that in finding the initial conditions it is the values of the coordinates and momenta after the determination is made which are desired. In classical theory, it is assumed that the various quantities can, in principle, be measured without interfering with them; but in case such methods are used as do interfere, it is certainly the values after the measurement which are inserted as initial conditions in the solutions of the differential equations.

The procedure in quantum mechanics is very analogous to that of classical theory. The "state of the system" is

(*a*) determined at some definite time $t=t_0$; and then

(*b*) predicted for some future time by means of the equation of motion, namely, the Schroedinger equation.

[1] Phys. Rev. **47**, 777 (1935).
[2] Phys. Rev. **48**, 696 (1935).

The state of the system can be represented in various ways. A familiar way is by means of the Schroedinger wave function $\psi(q, t)$ which is a function of the coordinates and the time. The determination of the state of the system is made by making measurements of certain quantities in ways which will be discussed in more detail later. When the state is determined at some initial time, it can then be known at any later time—as long as the system is kept isolated—by means of the equation

$$-(h/2\pi i)(\partial\psi/\partial t) = H\psi, \tag{1}$$

where H is a differential operator which corresponds to the Hamiltonian function of the system in question. This equation governs the change of the state as long as it is isolated; but the development of the quantum mechanics has emphasized the fact that an isolated system cannot be observed, that an observation is an interference with the system, and hence that predictions can be made only as far ahead as the first observation after the one which fixes the initial state.

Thus in either classical or quantum mechanics, a significant experiment can be performed by making an observation at $t=t_0$, predicting the result of an experiment at $t=t_1$, and checking this prediction at t_1. In the classical theory the prediction of any physical quantity is exact, provided an adequate determination of the initial conditions is made. In the quantum theory also some predictions are exact, but the normal case is that only a prediction of probability can be made. The wave function can always be exactly predicted, but the connection of this with the results of observations is usually statistical.

Interpretation of the state of a system

In case the state of the system at the time t is known it is possible to determine from it the probability of any result of an experiment. Consider the state to be described by the wave function $\psi(q, t)$, and suppose the experiment is designed to measure the quantity R. The rule is then to take the operator $\mathbf{R}$ which "corresponds to R" and to evaluate the integrals

$$\overline{R^n}(t) = \int \psi^*(q, t)\mathbf{R}^n\psi(q, t)dq \tag{2}$$

for all different values of n. These give the average, or expectation, values of the corresponding powers of R. A knowledge of all of these is equivalent to a knowledge of the distribution of R, or of the probability of any value of R, in that the probability of any value of R can be determined from these average values. If, for example, the average of the nth power of R is equal to the nth power of the average for all values of n, there is no spread at all in the possible values and the quantity is specified exactly by the function. If the average of all of the odd powers is zero, the probability of a certain positive value is equal to the probability of the same negative value.

The question arises as to whether there is an operator "corresponding" to every physical quantity. A physical quantity can be defined by the means used to measure it, and so the operator corresponds perhaps more to the particular operations involved in the experiment than to anything having a more abstract existence. Apparently this question has not been given a complete answer, but in any case the determination of the proper operator is a matter more for experiment than for analysis.

Certain operators are well known. When the $\psi(q, t)$ functions are expressed in Cartesian coordinates, the operator which corresponds to an experiment which, interpreted classically, measures the momentum of a particle in the x direction, is $(h/2\pi i)(\partial/\partial x)$, the operator for angular momentum about the z axis is $(h/2\pi i)(x(\partial/\partial y) - y(\partial/\partial x))$, the operator for the x coordinate is simply multiplication by x, etc. For functions of these quantities, the operator is usually the corresponding function of the component operators, but because the operators do not always commute with each other, this function of the operators is not always uniquely defined. In case there are several possible functions which satisfy the mathematical conditions for physically significant operators, and which differ only because the operators do not commute, these may be considered to represent physical quantities that strictly are different, but that coincide in the limit in which classical mechanics is valid.

Another question is as to whether there corresponds a possible physical quantity, or experimental operation, to every mathematically suitable operator. This question has apparently not been given a general answer.

As an example of the interpretation of the state of a system consider a particle which can move in one dimension only, and let its coordinate be x. Then suppose that at the time t_0, the particle is known to be in the state described by the Schroedinger function

$$\psi(x, t_0) = Ae^{-\alpha x^2/2}. \quad (3)$$

How this knowledge can be acquired will be discussed later. This is a complete description of the system at the time t_0 and contains all that can be known about it. One may ask, for instance, for the value of the coordinate, that is, the position of the particle. The answer is that the coordinate is indeterminate, that the coordinate is not a well-defined characteristic of the particle in this state. Nevertheless an average value for the coordinate can be obtained from the integral

$$\bar{x} = |A|^2 \int_{-\infty}^{\infty} xe^{-\alpha x^2} dx = 0.$$

This means that if an experiment is performed to determine the value of the coordinate x, and is performed a great many times on different systems all of which are known to be in the state represented by $\psi(x, t_0)$, the average of the results obtained will be zero. Similarly the average of any odd power will be zero while the averages of the even powers will be different from zero. The knowledge of the averages of all of the powers is equivalent to the knowledge of the probabilities of all the values.

Furthermore, in the state represented by Eq. (3) the momentum of the particle is also indeterminate but the wave function gives some information about the result of an experiment to measure it. The average value of the momentum in the x direction, in the sense just described for the coordinate, is given by

$$\bar{p} = \frac{h}{2\pi i} |A|^2 \int_{-\infty}^{\infty} e^{-\alpha x^2/2} \frac{\partial}{\partial x} (e^{-\alpha x^2/2}) dx = 0.$$

This implies that the probability of finding the momentum in one direction is the same as that of finding it in the other. The averages of other powers of the momentum can be found from similar integrals. For this particular wave function it is possible to show that the probability of finding a momentum between p and $p+dp$ is just given by the function

$$\frac{2\pi}{\alpha h} |A|^2 e^{(-4\pi^2/\alpha h^2)p^2} dp.$$

It is possible to show quite generally that with this method of determining probabilities from a wave function there results the inequality

$$(\overline{x^2} - \bar{x}^2)(\overline{p^2} - \bar{p}^2) \geqslant h^2/16\pi^2. \quad (4)$$

For the particular wave function used here as an example, the equality sign holds; but for functions in general the inequality is applicable. This is Heisenberg's principle of indetermination. It can be regarded as a direct consequence of the use of a wave function and the methods used for interpreting it, so that whatever physical significance is ascribed to this principle must be regarded as equivalent to the physical significance of the wave function itself.

The principle of indetermination has been illustrated with the coordinate and the conjugate momentum. It applies, however, to every pair of canonically conjugate quantities. The derivation depends only upon the fact that the operators for the two quantities are connected by the commutation rule

$$\mathbf{PQ} - \mathbf{QP} = (h/2\pi i)\mathbf{1}. \quad (5)$$

In the case of the example given, the quantities x and p were indeterminate and the predictions made about experiments to measure them could be statistical only. However, the knowledge of the state is very definite and not all quantities are indeterminate. The quantity $[p^2 + (\alpha^2 h^2/\pi^2)x^2]$ has the definite value $\alpha h/2\pi^2$, and any function of this quantity has a precisely known value. The state is characteristic of this quantity $[p^2 + (\alpha^2 h^2/\pi^2)x^2]$ and the Schroedinger function $Ae^{-\alpha x^2/2}$ is a characteristic function or eigenfunction with the eigenvalue $\alpha h^2/2\pi^2$.

Determination of the state of a system

Without a knowledge of the state of the system at some time t_0 it is impossible to make any precise predictions about its state at a future time, and so some attention should be given to the kinds of experiments which can be used for determining the initial state. The way in which the question is put implies an attitude toward the answer. One might ask, "How can the state of the system at the time t_0 be discovered?" This would imply the existence of the system in a definite state and the use of an experiment to determine what the state is. On the other hand, one might ask, "How can the state of the system be prescribed?" The latter question would imply that only the state of the system after the experiment is of importance, and would ignore the state before the experiment. It implies that if the experiment changes the state of the system, this change is to be taken into account, and that the experiment is really a preparation of the system in a given state. Only this latter attitude can be consistently maintained in quantum mechanics. Let us then consider a few examples of the prescription of a state at the time t_0.

1. As a first example consider the famous "gamma-ray microscope." This is a classically customary method of locating a particle by looking at it or photographing it. The particle is illuminated by light of which some is scattered through the lens to form an image on the plate at P, Fig. 1. If only a single photon is scattered, the time at which it was scattered can be inferred from the time at which it strikes the plate. More than one scattered photon cannot be used, for in general the particle would move between the impacts of one photon and the next, so that only the last would give a suitable measure of its position. But since one photon will form only one spot on the plate it is necessary to use light of very short wave-length, so that one spot will effectively coincide with the center of the interference pattern which would have been formed had many photons been scattered from this same point. In the limit, when light of infinitely short wave-length is used, the position of the particle, relative to the microscope and the framework to which it is attached, can be determined with any desired precision.[3] This position measurement will then be an "initial condition" for the future

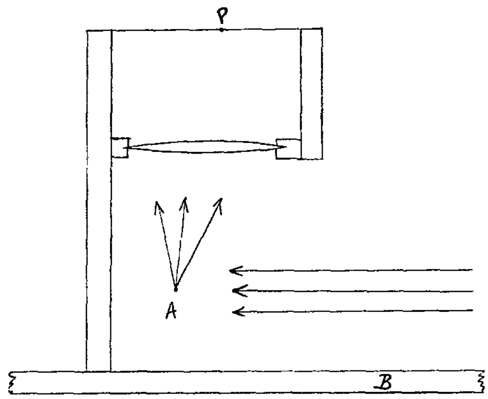

FIG. 1. The gamma-ray microscope. The lens and the plate are rigidly fastened to the base B which serves to define the coordinate system. A position P on the plate can therefore be interpreted as a coordinate measurement without any uncertainty.

motion of the particle, and the wave function which is ascribed to the particle must be such that it represents the particle at the observed point at the time t_0. This function can differ from zero only at the point r_0 and will be[4]

$$\psi(r, t) = \{\delta(r-r_0)\}^{\frac{1}{2}}. \tag{6}$$

This state will then change in accordance with the Schroedinger equation and will give the subsequent behavior of the particle.

2. Another type of experiment gives a state in which the momentum is defined and the wave function is that of a plane wave. Let a beam of particles be incident on a grating through a set of slits which will define the angle of incidence. If, then, a particle is scattered so as to pass through a set of slits defining a definite angle of diffraction, the wave function which represents the state will be that for a plane wave. The wave-length will be connected with the momentum of the particle by the de Broglie relationship $\lambda = h/p$, so that

$$\psi(x, t_0) = Ae^{2\pi i(p/h)x}, \tag{7}$$

where x is measured along the direction defined by the slits through which the particle is scattered. Of course this simple plane wave function is a limit which is approached as the grating and the slits are made larger so that the resolving power increases and the wave-length is more sharply defined. For this reason the position of the particle is entirely undetermined by the

[3] Since we are dealing with nonrelativistic quantum mechanics only, the finiteness of c is ignored and h/mc is treated as zero. Hence the uncertainty which is introduced because the wave-length after scattering is at least h/mc is neglected.

[4] This δ function is defined by the fact that it is zero when $r \neq r_0$, and $\int \delta(r-r_0)dv = 1$. It may be pictured as the limit of a set of functions $\lim_{\alpha\to\infty} (\alpha/\pi)^{3/4} e^{-\alpha\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}}$.

experiment. This type of experiment is the antithesis of the gamma-ray microscope since the very property which makes possible an accurate determination of λ and p entirely precludes any knowledge of the position of the particle under consideration.

These two types of experiment are commonly known in quantum mechanics since they correspond to the classical determination of position and momentum. There are, however, others which can be used.

3. The angular momentum of an atom can be prescribed by sending a beam through an inhomogeneous magnetic field. By means of slits the atoms having different components of angular momentum in the direction of the field can be separated and their future behavior observed. This corresponds to a determination of only one factor in the wave function since the wave function requires for its unique determination the specification of as many quantities as there are degrees of freedom.

4. If it is desired to produce a particle in the state described by the function $\psi(r, t_0) = Ae^{-r^2/2a^2}$, it can be connected to the origin by some elastic device such as a spring with a suitable force constant and a vanishingly small viscosity. In the case of a charged particle the viscosity is unnecessary since the necessary damping is produced by radiation. After a long time the particle will be known to be in the state of lowest energy because of the dissipation of energy in the viscous spring or in radiation. It will then be in the desired state, and if the connection with the origin is quickly broken the state will start to change according to the equation of motion for a free particle.

In all of these experiments there is an exact determination of some quantity, and in quantum mechanics as well as in classical mechanics this represents an idealization. A crude determination will not fix the state exactly. If the position of a particle is determined by using light of a finite wave-length in a microscope, or by allowing the particle to pass through a slit of finite width, the position will be determined within certain limits, and it will be known that the absolute value of the function differs from zero only within these limits. There is an infinity of functions, however, which satisfy this condition, since any originally chosen function can be multiplied by an arbitrary function of absolute value one without affecting its representation of this approximate knowledge of position. In particular there can be a δ function at every point within the allowed region, and these can be added together with various phase factors to give a suitable wave function. The state of the particle is thus not determined by this kind of an experiment. It is customary to describe this situation by saying that the system is in a "mixed state." This term is not very descriptive; it means that the state of the system is not precisely known, that it might be any one of a large number or an infinity of states which satisfy the conditions of the experiment, and that predictions as to the future state of the system can be made only in the sense of ordinary statistical mechanics.

Physical aspects of principle of indetermination

It has already been indicated that the principle of indetermination is a necessary concomitant of the description of the state of a system by means of a wave function and the prescribed methods of interpreting it. Nevertheless the question naturally arises as to whether this description is complete; as to whether the wave function gives everything about the system which has any physical significance. The question is sometimes put as to whether a particle does not *really have* both a definite coordinate and conjugate momentum which are, however, only inaccurately known. To some persons this latter form of question seems to have a definite meaning, but it is not easy to make it precise. One possible point of view is that it should be possible to make such experiments that the position and the momentum of a particle could both be determined accurately, and that the failure to find such experiments is due merely to lack of insight or ingenuity.

In contradiction to this attitude, Bohr has maintained that canonically conjugate quantities are, in their very concepts and definitions, complementary to each other and mutually exclusive. A position, for example, must be determined by reference to some coordinate system, which must in fact be a heavy material body. Momentum is defined essentially by its conservation law.[5] If now the position of a particle is to be determined, the particle must come in contact, perhaps indirectly, with the coordinate system. In this interaction momentum will be conserved, but it is necessarily impossible to know how much momentum has been transferred to the coordinate system. Any attempt to determine

[5] See E. Mach, *Science of Mechanics*, p. 238, for a discussion of the conceptual basis of Newtonian mechanics.

this quantity involves treating the coordinate system as a free body and referring it to another coordinate system, so that the usefulness of the first as an ultimate reference system is destroyed. On this account any measurement of position will destroy the value of a previous measurement of momentum.

Let us consider more in detail the gamma-ray microscope. Let us suppose that the momentum of the particle has been previously measured and is known to be p_0, and that the microscope is to be used to determine the position so that both quantities will then be known. The incident light may be taken to have a definite direction and wave-length, and hence each quantum will have a definite momentum which may be called p_1. In the interaction between the particle and the light quantum which is scattered there will be conservation of momentum, and the change of the momentum of the particle would be known if the final momentum of the quantum could be determined. However, in order that an observation can be made of the position of the particle the quantum must be allowed to pass undisturbed through the lens to impinge on the plate. The exact direction of the scattered quantum cannot be determined, for any reduction in the aperture of the lens would reduce the resolving power and hence the accuracy of the position determination. Thus the momentum of the scattered quantum cannot be determined before it is absorbed in the microscope. In addition the momentum given to the microscope by the light quantum cannot be determined, or even considered, for the microscope is necessarily rigidly fastened to the coordinate system and hence, by definition, cannot take on any motion. The very property of rigidity which makes it suitable for a position measurement entirely precludes the possibility of taking any account of the transfer of momentum from the particle to the microscope. A more detailed analysis of the operation of this particular instrument shows that the disturbance of the momentum of the particle can be reduced by the use of light of finite wave-length at the cost of precision in the determination of position, and that the relation between the maximum precision obtained in the position measurement and the minimum disturbance of the momentum is that given by the indetermination relation. Thus Bohr's contention is that position and momentum are, in their very definitions, quantities which cannot be precisely defined at the same time.

Another possible point of view might be that it is possible to imagine a classical description of the state of a particle, that is, a precise position and momentum, even though it is in principle impossible to determine them. It is, however, very difficult to give precision to such an idea. If it is admitted that a classical description of a state is really unattainable because of incompatibility in the definitions involved, the idea that such a state nevertheless exists is rather unsatisfactory.

A consistent point of view is that classical quantities exist only for limiting cases of quantum-mechanically defined states; and that canonically conjugate quantities are complementary in the sense that they cannot exist at the same time, as they characterize different limiting quantum-mechanical states which can be produced by suitable but mutually exclusive experiments.

Observation and causality in the quantum mechanics

The principle of indetermination has led to a good deal of discussion of the disappearance of causality from the world of science. Much of this discussion has been without any very adequate definition of what is meant by causality. It is true that classical mechanics permits a physicist to predict the behavior of an isolated system if its initial state is known, but quantum mechanics permits the same thing. In fact, without an equation describing the time behavior of the system there could be no mechanics at all. The difference lies in the idea of what constitutes a complete description of the state, and the emphasis placed on the fact that a really isolated system cannot be observed at all when quantities of the order of h are to be considered.

The quantum mechanics emphasizes the distinction between two different kinds of interactions. One is the ordinary interaction between the parts of the system under consideration, and the other is the interaction with a measuring instrument. It is only the latter kind of interaction to which the principle of indetermination applies. The ordinary interaction between parts of a system goes on in a perfectly causal (although not necessarily classical) way according to the Schroedinger equation. The interaction with a measuring instrument, however, must be described in classical terms, and can be predicted in a statistical way only.

In making an observation there is usually a chain of interactions involved. Any one of these may be taken as the dividing line between the system and the measuring instrument. In the example of the gamma-ray microscope one may say that the microscope measures the position of the particle. Then the state of the particle, just after the interaction, is known; its position is exactly known although its momentum is entirely indeterminate, and the action of the microscope is described classically. In particular, the subsequent determination of the position of the spot on the plate in terms of the coordinate system and the perception of this result by the physicist is supposed to be entirely classical. On the other hand, the microscope may be considered as part of the system and the observation can be made on the microscope itself. This is a different selection of the interaction which is to be taken as dividing the system from the observer. In this case the microscope could not be fastened to the coordinate system, but must be free to move in its interaction with the particle. The position of the spot on the plate must then be measured with reference to the fixed coordinate system and the principle of indetermination will apply to this measurement. This change in the interaction which is considered as the observation does not carry with it any loss of choice on the part of the observer as to what is to be measured. When the particle alone is considered as the system its momentum could be measured instead of its position. The same thing is true when the microscope is part of the system, for the momentum of the microscope could be measured instead of its position. This measurement would entirely preclude a measurement of position, but because of the conservation of momentum in the interaction between the particle and the microscope, the measurement of the momentum of the microscope would serve to measure the momentum of the particle. The interaction to which the principle of indetermination is applied can be arbitrarily chosen, but it must always be included at some point in the chain between the system of interest and the observing physicist.

The necessity of clearly distinguishing between the observer and the things observed is common to all science. In the classical mechanics it is often overlooked because all interactions are treated in the same way. The quantum mechanics has emphasized this distinction by treating the interaction with the observer in this peculiar fashion, and by bringing to light the paradoxes which appear when this distinction is not carefully made.

One may say, then, that according to quantum mechanics, processes in the outside world go on in a perfectly causal way in that the state of a system changes in accordance with the Schroedinger equation. This does not mean that the processes go on in a classical fashion or can be at all classically described. It merely means that the development of the state of the system can be described by the Schroedinger equation as long as the system is isolated. As soon, however, as the observer interferes with the system, the causal development of the state is broken off and the interaction with the observer can be described only in terms of probabilities.

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