$$p = \frac{1}{3} u(T)$$

$$E = V u(T)$$

- Planteo primer principio:

$$TdS = dE + pdV \quad \Rightarrow dS = \frac{1}{T} dE + \frac{p}{T} dV \qquad (1)$$

- Escribo $dS$ como función de $T$ y $V$:

$$dS = \left.\frac{\partial S}{\partial T}\right|_V dT + \left.\frac{\partial S}{\partial V}\right|_T dV \qquad (2)$$

- Escribo $dE$ como función de $T$ y $V$:

$$dE = \left.\frac{\partial E}{\partial T}\right|_V dT + \left.\frac{\partial E}{\partial V}\right|_T dV = V \frac{du}{dT} dT + u dV \qquad (3)$$

- Meto (3) en (1) e igualo (1) y (2):

$$\left.\frac{\partial S}{\partial T}\right|_V dT + \left.\frac{\partial S}{\partial V}\right|_T dV = \frac{1}{T}\left( V \frac{du}{dT} dT + u dV \right) + \frac{p}{T} dV$$

$$\left.\frac{\partial S}{\partial T}\right|_V dT + \left.\frac{\partial S}{\partial V}\right|_T dV = \frac{V}{T} \frac{du}{dT} dT + \frac{1}{T}(u + p) dV \qquad (4)$$

- De (4), se ve que:

$$\frac{\partial S}{\partial T} = \frac{V}{T} \frac{du}{dT}$$

$$\frac{\partial S}{\partial V} = \frac{1}{T}(u + p) = \frac{4}{3} \frac{1}{T} u \qquad \text{(usando que } p = \frac{1}{3} u \text{)}$$

- Igualando las derivadas cruzadas:

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V} \quad \Rightarrow \quad \frac{1}{T} \frac{du}{dT} = -\frac{4}{3} \frac{u}{T^2} + \frac{4}{3} \frac{1}{T} \frac{du}{dT}$$

- Reordenando:

$$\frac{4}{3} \frac{u}{T^2} = \frac{4}{3} \frac{1}{T} \frac{du}{dT} - \frac{1}{T} \frac{du}{dT}$$

$$4 \frac{u}{T} = \frac{du}{dT} \quad \Rightarrow \quad \frac{du}{u} = 4 \frac{dT}{T}$$

- Integrando:

$$\ln u = 4 \ln T + cte \quad \Rightarrow \quad \boxed{u(T) = aT^4} \quad \text{con } a = cte$$