Ejercicio 1

Antes de comenzar, recordar que

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} \psi^*(x) f(x) \psi(x)\, dx = \int_{-\infty}^{\infty} f(x) |\psi(x)|^2 \, dx, \tag{1}$$

$$\langle f(p)\rangle = \int_{-\infty}^{\infty} \phi^*(p) f(p) \phi(p)\, dp = \int_{-\infty}^{\infty} f(p) |\phi(p)|^2 \, dp, \tag{2}$$

donde

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-\frac{\imath px}{\hbar}} dx, \ \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{\frac{\imath px}{\hbar}} dp. \tag{3}$$

Además,

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^*(x) \hat{p}\psi(x)\, dx = -\imath\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d\psi}{dx}(x)\, dx, \tag{4}$$

$$\left\langle p^2\right\rangle = \int_{-\infty}^{\infty} \psi^*(x) \hat{p}^2\psi(x)\, dx = -\hbar^2 \int_{-\infty}^{\infty} \psi^*(x) \frac{d^2\psi}{dx^2}(x)\, dx = \hbar^2 \int_{-\infty}^{\infty} \left|\frac{d\psi}{dx}(x)\right|^2 dx. \tag{5}$$

Si $\psi(x) = \psi(-x)$ como en muchos casos, $\langle x\rangle = \langle p\rangle = 0$.

Ítem a

Por un lado, como la función de onda es par, sabemos que $\langle x\rangle = \langle p\rangle = 0$.
Por el otro,

$$\left\langle x^2\right\rangle = \int_{-\infty}^{\infty} \psi^*(x) x^2 \psi(x)\, dx = \frac{\int_{-\infty}^{\infty} e^{-2a^2x^2} x^2 dx}{\int_{-\infty}^{\infty} e^{-2a^2x^2} dx} = -\frac{d}{d\alpha} \ln\left(\int_{-\infty}^{\infty} e^{-\alpha x^2}\right), \tag{6}$$

donde $\alpha = 2a^2$. Luego,

$$\left\langle x^2\right\rangle = -\frac{d}{d\alpha} \ln\left(\sqrt{\frac{\pi}{\alpha}}\right) = \frac{d}{d\alpha} \ln\left(\sqrt{\frac{\alpha}{\pi}}\right) = \frac{1}{2}\frac{d}{d\alpha} \ln\alpha = \frac{1}{2\alpha} = \frac{1}{4a^2} \tag{7}$$

y por lo tanto $\sigma_x = \frac{1}{2a}$.
Además,

$$\left\langle p^2\right\rangle = \hbar^2 \int_{-\infty}^{\infty} \left|\frac{d\psi}{dx}(x)\right|^2 dx = \hbar^2 a^2 \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx = 4\hbar^2 a^4 \left\langle x^2\right\rangle = \hbar^2 a^2 \tag{8}$$

y por ende $\sigma_p = \hbar a$.
Finalmente

$$\sigma_x\sigma_p = \frac{\hbar}{2}. \tag{9}$$

Ítem b

Por la misma razón de antes, $\langle x\rangle = \langle p\rangle = 0$.
Además,

$$\left\langle x^2\right\rangle = \int_{-\infty}^{\infty} \psi^*(x) x^2 \psi(x)\, dx = \frac{\int_0^{\infty} e^{-2ax} x^2 dx}{\int_0^{\infty} e^{-2ax} dx} = \frac{\frac{d^2}{d\alpha^2}\int_0^{\infty} e^{-\alpha x} dx}{\int_0^{\infty} e^{-\alpha x} dx} = \frac{\frac{d^2}{d\alpha^2}\frac{1}{\alpha}}{\frac{1}{\alpha}} = \frac{2}{\alpha^2} = \frac{1}{2a^2}, \tag{10}$$

$$\left\langle p^2\right\rangle = \hbar^2 \int_{-\infty}^{\infty} \left|\frac{d\psi}{dx}(x)\right|^2 dx = 2\hbar^2 \int_0^{\infty} \left|\frac{d\psi}{dx}(x)\right|^2 dx = 2\hbar^2 a^2 \frac{\int_0^{\infty} e^{-2ax} dx}{2\int_0^{\infty} e^{-2ax} dx} = \hbar^2 a^2. \tag{11}$$

Luego,

$$\sigma_x\sigma_p = \frac{\hbar}{\sqrt{2}} \geq \frac{\hbar}{2}. \tag{12}$$

Notar que

$$\left\langle p^2\right\rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^*(x) \frac{d^2\psi}{dx^2}(x)\, dx \neq -\hbar^2 \left[\int_{-\infty}^{0} \psi^*(x) \frac{d^2\psi}{dx^2}(x)\, dx + \int_0^{\infty} \psi^*(x) \frac{d^2\psi}{dx^2}(x)\, dx\right] = -\hbar^2 a^2 \tag{13}$$

ya que

$$\lim_{x\to 0} \frac{d^2\psi}{dx^2}(x) = \infty. \tag{14}$$

En efecto, para hacer la cuenta de esta forma hay que usar que el salto de la derivada primera en el origen es $-2a$ y por lo tanto

$$\frac{d^2\psi}{dx^2} = \left(a^2 - 2a\delta\left(x\right)\right)\psi\left(x\right). \tag{15}$$

Luego,

$$\left\langle p^2\right\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi^*\left(x\right)\frac{d^2\psi}{dx^2}\left(x\right)dx = -\hbar^2\int_{-\infty}^{\infty}\left(a^2 - 2a\delta\left(x\right)\right)\left|\psi\left(x\right)\right|^2 dx = -\hbar^2 a^2 + 2\hbar^2 a\left|\psi\left(0\right)\right|^2 = \hbar^2 a^2 \tag{16}$$

lo cual es consistente con lo anterior.

Haciéndolo con transformada de Fourier se tiene

$$\phi\left(p\right) \propto \int_{-\infty}^{\infty} e^{-\imath\frac{px}{\hbar}} e^{-a|x|}dx \propto \int_{-\infty}^{0} e^{-\imath\frac{px}{\hbar}} e^{ax}dx + \int_{0}^{\infty} e^{-\imath\frac{px}{\hbar}} e^{-ax}dx \propto \left.\frac{e^{\left(a-\imath\frac{p}{\hbar}\right)x}}{a-\imath\frac{p}{\hbar}}\right|_{-\infty}^{0} + \left.\frac{e^{-\left(a+\imath\frac{p}{\hbar}\right)x}}{-\left(a+\imath\frac{p}{\hbar}\right)}\right|_{0}^{\infty} \tag{17}$$

$$\propto \frac{1}{a-\imath\frac{p}{\hbar}} + \frac{1}{\left(a+\imath\frac{p}{\hbar}\right)} \propto \frac{1}{a^2+\frac{p^2}{\hbar^2}}. \tag{18}$$

Luego,

$$\left\langle p^2\right\rangle = \frac{\int_{-\infty}^{\infty} p^2\left(a^2+\frac{p^2}{\hbar^2}\right)^{-2}dp}{\int_{-\infty}^{\infty}\left(a^2+\frac{p^2}{\hbar^2}\right)^{-2}dp} = \frac{\int_{-\infty}^{\infty} p^2\left(1+\frac{p^2}{\hbar^2 a^2}\right)^{-2}dp}{\int_{-\infty}^{\infty}\left(1+\frac{p^2}{\hbar^2 a^2}\right)^{-2}dp} = \hbar^2 a^2\frac{\int_0^{\infty} u^2\left(1+u^2\right)^{-2}du}{\int_0^{\infty}\left(1+u^2\right)^{-2}du} = \hbar^2 a^2. \tag{19}$$

Ítem c

En primera lugar,

$$\left\langle x\right\rangle = A^2\int_{-\frac{a}{2}}^{\frac{a}{2}} x = 0, \tag{20}$$

$$\left\langle x^2\right\rangle = \frac{\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 dx}{\int_{-\frac{a}{2}}^{\frac{a}{2}} dx} = \frac{\int_0^{\frac{a}{2}} x^2 dx}{\int_0^{\frac{a}{2}} dx} = \frac{a^2}{12}. \tag{21}$$

En cuanto a los valores medios del momento, aquí la función de onda es discontinua y además es compleja por lo que no hay otra opción mas que hacerlos con transformada de Fourier. Se tiene que

$$\phi\left(p\right) \propto \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-\imath\frac{px}{\hbar}} e^{\imath\frac{p_0 x}{\hbar}}dx \propto \left.\frac{e^{\imath\frac{(p_0-p)x}{\hbar}}}{\imath\frac{(p_0-p)}{\hbar}}\right|_{-\frac{a}{2}}^{\frac{a}{2}} \propto \frac{\sin\left[\frac{(p-p_0)a}{2\hbar}\right]}{p-p_0}. \tag{22}$$

Luego,

$$\left\langle p\right\rangle = \frac{\int_{-\infty}^{\infty} p\frac{\sin^2\left[\frac{(p-p_0)a}{2\hbar}\right]}{(p-p_0)^2}dp}{\int_{-\infty}^{\infty}\frac{\sin^2\left[\frac{(p-p_0)a}{2\hbar}\right]}{(p-p_0)^2}dp} = p_0\frac{\int_{-\infty}^{\infty}\frac{\sin^2\left[\frac{ap'}{2\hbar}\right]}{p'^2}dp'}{\int_{-\infty}^{\infty}\frac{\sin^2\left[\frac{ap'}{2\hbar}\right]}{p'^2}dp'} = p_0, \tag{23}$$

$$\left\langle p^2\right\rangle = \frac{\int_{-\infty}^{\infty} p^2\frac{\sin^2\left[\frac{(p-p_0)a}{2\hbar}\right]}{(p-p_0)^2}dp}{\int_{-\infty}^{\infty}\frac{\sin^2\left[\frac{(p-p_0)a}{2\hbar}\right]}{(p-p_0)^2}dp} = \frac{\int_{-\infty}^{\infty}\left(p_0+p'\right)^2\frac{\sin^2\left[\frac{ap'}{2\hbar}\right]}{p'^2}dp'}{\int_{-\infty}^{\infty}\frac{\sin^2\left[\frac{ap'}{2\hbar}\right]}{p'^2}dp'} = p_0^2 + \frac{\int_{-\infty}^{\infty}\sin^2\left[\frac{ap'}{2\hbar}\right]dp'}{\int_{-\infty}^{\infty}\frac{\sin^2\left[\frac{ap'}{2\hbar}\right]}{p'^2}dp'} \to \infty. \tag{24}$$

Esto implica

$$\sigma_x\sigma_p \to \infty. \tag{25}$$