LIST OF EXERCISES

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Exercise 1: **Detailed balance.** (a) Consider a close container with atoms and radiation in equilibrium at temperature *T*. The atomic states are not degenerate with energy E_d and E_u, so that $E_u > E_d$. Derive Einstein's absorption $B_{d\rightarrow u}$, spontaneous emission $A_{u\rightarrow d}$, and stimulated emission $B_{u\rightarrow d}$ coefficients from the master equation:

$$\begin{cases} \dot{n}_{d\rightarrow u} = B_{d\rightarrow u} n_d \rho(\omega_{ud}) \\ \dot{n}_{u\rightarrow d} = A_{u\rightarrow d} n_u + B_{u\rightarrow d} n_u \rho(\omega_{ud}) \end{cases}$$

where $\rho(\omega_{ud})$ is the energy density of the radiation, $\omega_{ji} = \left(E_j - E_i\right)/\hbar$, $\dot{n}_{i\rightarrow j}$ is the number of atoms doing the transition $i \rightarrow j$ per unit time due to the absorption or emission of radiation and n_i is the total number of atoms in state *i*. Hint: Consider the Boltzman and Planck distributions for the atomic levels and the radiation, respectively.

(b) Generalized the result in (a) for the case that E_d and/or E_u are degenerated.

Exercise 2: Consider a 10 fs Gaussian pulse for which $\Delta\nu\tau = 0.44$.

(a) How much is the bandwith $\Delta\nu$? Answer: 4.4×10^{13} Hz

(b) How much is the banwidth $\Delta\lambda$ for a central wavelength of 800 nm? Answer: 94 nm.

Exercise 3: From the Maxwell equation

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Derive a wave equation for the vector potential

$$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} = 0 \quad \text{in absence of currents}$$

Exercise 4: Prove that (Goldstein pgs. 27-28)

$$\vec{v} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\left(\vec{v} \cdot \vec{A}\right) - \frac{d\vec{A}}{dt} + \frac{\partial \vec{A}}{\partial t}$$

$$\text{where } \frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} \right)$$

Exercise 5: Derive the Lorentz force from the Hamilton equations:

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}; \quad -\dot{\vec{p}} = \frac{\partial H}{\partial \vec{r}}$$

where $H(\vec{p},\vec{r},t)=\frac{\left(\vec{p}-q\vec{A}\right)^2}{2m}+q\phi$

Exercise 6: How much is β=v/c for an electron excited with a Ti-Sapphire laser?
λ = 800nm, E_0 = 5x10^{10} V/m, m = 9.1x10^{-31} kg

Exercise 7: How does gauge invariance work for the Schrödinger equation?

$$\vec{A}\to\vec{A}'=\vec{A}+\vec{\nabla}\chi$$
$$\phi\to\phi'=\phi-\frac{\partial\chi}{\partial t}$$

Solution:

$$\psi'(\vec{r},t)=\psi(\vec{r},t)e^{i\frac{q}{\hbar}\chi}$$

in the Coulomb gauge

$$\nabla^2\chi=0$$

Exercise 8: Prove that if

$$\vec{A}=\vec{A}(t)\neq 0\Rightarrow\Psi(\vec{r},t)=\psi(\vec{r})e^{-\frac{i}{\hbar}S(t)}$$
$$\text{where } S(t)=\frac{1}{2m}\int_{t_0}^{t}dt'\left[\hbar\vec{k}-q\vec{A}(t)\right]^2 \quad \text{classical action}$$

Exercise 9: (a) How much is the electric field and potential felt by a classical electron in the first Bohr's orbit? What is the intensity due to 1 a.u. of electric field?

(b) How much is the electric potential felt by a classical electron in the first Bohr's orbit?

(c) What is the intensity due to 1 a.u. of electric field?

Exercise 10: How much is the intensity of a Ti-Sapphire laser (λ = 800 nm) impinging over H so that the Keldysh parameter is γ = 1?

Exercise 11: Calculate the classical action for a sinusoidal linearly polarized field $\vec{E}(t)=E_0\sin(\omega t)\hat{z}$. Solution:

$$S(t,\tau)=-\left[\frac{k^2\tau}{2}+\frac{F_0^2}{2\omega^2}\left(\frac{\tau}{2}+\frac{\sin 2\omega\tau}{4\omega}\right)+\frac{kF_0}{\omega}\sin\omega\tau+I_p\tau\right]+$$
$$+\left[\frac{k^2t}{2}+\frac{F_0^2}{2\omega^2}\left(\frac{t}{2}+\frac{\sin 2\omega t}{4\omega}\right)+\frac{kF_0}{\omega}\sin\omega t+I_p t\right]$$

Exercise 12: Prove that within the SFA the momentum distribution is symmetrical for a cosine-like pulse: $\vec{F}(t) = F_0 \cos(\omega t + \phi_{CE})\hat{z}$ with $\phi_{CE} = 0$

$$\frac{dP(k_z, k_\rho)}{d\vec{k}} = \frac{dP(-k_z, k_\rho)}{d\vec{k}} \quad \text{iff} \quad F(t) = F(\tau - t)$$

Exercise 13: Prove that in the interaction picture:

$$i\hbar \frac{d}{dt}|\psi_I(t)\rangle = H_{1,I}(t)|\psi_I(t)\rangle,$$
$$i\hbar \frac{d}{dt} A_I(t) = [A_I(t), H_{0,S}].$$

Exercise 14: Prove the return condition and the ionization condition from the saddle equation:

$$\nabla_{\vec{k}} S(\vec{k}, t, t-\tau) = 0 \quad \Rightarrow \quad \vec{r}(t) - \vec{r}(t-\tau) = 0$$
$$\frac{\partial}{\partial \tau} S(\vec{k}, t, t-\tau) = 0 \quad \Rightarrow \quad \tfrac{1}{2}\left(\vec{k} + \vec{A}(t-\tau)\right)^2 + I_p = 0$$

Exercise 15: Prove the energy conservation for emission of a harmonic photon from the saddle equation:

$$\frac{\partial}{\partial t} S(\vec{k}, t, t-\tau) = 0 \quad \Rightarrow \quad \tfrac{1}{2}\left(\vec{k} + \vec{A}(t)\right)^2 + I_p = q\omega$$

Exercise 16: Considering the relation between the measured angle θ and the ejection angle without IR, θ_i in LAPE:

$$\tan\theta = \frac{v(t)\sin\theta_i}{v(t)\cos\theta_i - A(t)}$$

prove that

$$v(t)\cos\theta_i = A(t)\sin^2\theta \pm \sqrt{A^2(t)\sin^4\theta - A^2(t)\sin^2\theta + v^2(t)\cos^2\theta}$$

Exercise 17: (Scattering 1D) Consider a rectangular potential barrier of height V_0 and width *a*, centered at the origin. Calculate the transmission group delay

$$\Delta t_g = \hbar \frac{d\varphi}{dE} \approx \frac{2m}{\hbar k \kappa} = \frac{2}{v\kappa}$$

where $1/\kappa$ is the penetration depth. Use the approximation of a very opaque barrier ($\kappa a >> 1$).