ATTOSECOND PHYSICS

UNIT III
REVIEW OF ELECTROMAGNETISM

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Electromagnetic fields

System of Units: We use MKS despite the most used is the Gaussian

Maxwell Equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

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Electromagnetic fields (cont.)

$$\vec{\nabla}\cdot\vec{B}=0 \quad\Rightarrow\quad \boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$$

$\vec{A}$ is the vector potential

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{\nabla}\times\vec{A}}{\partial t}=0$$

$$\vec{\nabla}\times\underbrace{\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)}_{-\vec{\nabla}\phi}=0 \quad\Rightarrow\quad \boxed{\vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}}$$

ϕ is the scalar potential

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Gauge Invariance

Electric and magnetic fields *E* and *B* are unaltered if one changes:

$$\vec{A}\rightarrow\vec{A}'=\vec{A}+\vec{\nabla}\chi$$

$$\phi\rightarrow\phi'=\phi-\frac{\partial\chi}{\partial t}$$

where $\chi(\vec{r},t)$ is an arbitrary function

Proof:

$$\vec{B}'=\vec{\nabla}\times\vec{A}'=\vec{\nabla}\times\left(\vec{A}+\vec{\nabla}\chi\right)=\vec{\nabla}\times\vec{A}+\underbrace{\vec{\nabla}\times\vec{\nabla}\chi}_{=0}=\vec{\nabla}\times\vec{A}=\vec{B}$$

$$\vec{E}'=-\vec{\nabla}\phi'-\frac{\partial\vec{A}'}{\partial t}=-\vec{\nabla}\left(\phi-\frac{\partial\chi}{\partial t}\right)-\frac{\partial}{\partial t}\left(\vec{A}+\vec{\nabla}\chi\right)=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}=\vec{E}$$

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Gauge Invariance (cont.)

$$\vec{\nabla}\cdot\vec{A}=0$$

Example: Coulomb gauge

$$\vec{\nabla}\cdot\vec{A}'=0=\vec{\nabla}\cdot\left(\vec{A}+\vec{\nabla}\chi\right)=\underbrace{\vec{\nabla}\cdot\vec{A}}_{=0}+\nabla^2\chi$$

$$\Rightarrow \nabla^2\chi=0$$

Still freedom to choose χ

We fix

$$\chi=-\vec{A}\cdot\vec{r};\quad \text{supposing that } \vec{A}\neq\vec{A}(\vec{r})$$

$$\Rightarrow \vec{\nabla}\chi=-\vec{\nabla}\left(\vec{A}\cdot\vec{r}\right)=-\vec{A}$$

$$\Rightarrow \begin{cases} \vec{A}'=\vec{A}+\vec{\nabla}\chi=\vec{A}-\vec{A}=0 \\ \phi'=\phi-\dfrac{\partial\chi}{\partial t}=\phi+\dfrac{\partial\vec{A}\cdot\vec{r}}{\partial t}=\phi+\underbrace{\dfrac{\partial\vec{A}}{\partial t}}_{-\left(\vec{E}+\vec{\nabla}\phi\right)}\cdot\vec{r}=\phi-\vec{r}\cdot\vec{\nabla}\phi-\vec{E}\cdot\vec{r}=-\vec{E}\cdot\vec{r} \end{cases}$$

Supposing that $\phi\neq\phi(\vec{r})$ and doing $\phi=\text{const}=0$

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Classical Electromagnetic Radiation Field

Lorentz gauge: $\vec{\nabla}\cdot\vec{A}+\dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t}=0$

From the Maxwell equation:

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}\quad\text{and}\quad \vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}$$

$$\Rightarrow -\vec{\nabla}\cdot\vec{\nabla}\phi-\frac{\partial}{\partial t}\underbrace{\vec{\nabla}\cdot\vec{A}}_{\frac{-1}{c^2}\frac{\partial\phi}{\partial t}}=\frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}=0 \quad \text{in absence of charges}$$

Wave equation

Exercise 3: From the Maxwell equation $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t}$

Derive a wave equation for the vector potential

$$\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J}=0 \quad \text{in absence of currents}$$

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Classical Electromagnetic Radiation Field (cont.)

Solution to the wave equation:

$$\vec{A}_{\lambda\vec{k}} = A_N e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta_\omega)}\hat{\varepsilon}_{\lambda\vec{k}} \;\; +\text{c.c.}$$

$$= A_N \hat{\varepsilon}_{\lambda\vec{k}}\, 2\cos\left(\vec{k}\cdot\vec{r}-\omega t+\delta_\omega\right)$$

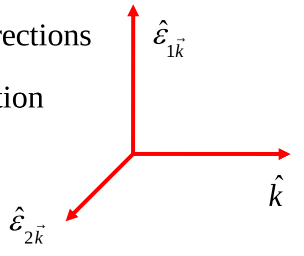
We still have the freedom to resctrict :

$$\phi = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} = 0 \quad \text{(Coulomb gauge)}$$

$\vec{\nabla}\cdot\vec{A} \propto \vec{k}\cdot\hat{\varepsilon}_{\lambda\vec{k}} = 0 \quad \Rightarrow \quad \vec{k}\perp\hat{\varepsilon}_{\lambda\vec{k}}$: two directions

When we introduce $\vec{A}$ into the wave equation

$\Rightarrow \omega = ck \quad$ dispersion relation



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Classical electromagnetic radiation field (cont.)

Electric field: $$\vec{E}_{\lambda\vec{k}} = -\frac{\partial\vec{A}_{\lambda\vec{k}}}{\partial t} = A_N\hat{\varepsilon}_{\lambda\vec{k}}\, 2\omega\sin\left(\vec{k}\cdot\vec{r}-\omega t+\delta_\omega\right)$$

Magnetic field: $$\vec{B}_{\lambda\vec{k}} = \vec{\nabla}\times\vec{A}_{\lambda\vec{k}} = A_N\left(\vec{k}\times\hat{\varepsilon}_{\lambda\vec{k}}\right)2\sin\left(\vec{k}\cdot\vec{r}-\omega t+\delta_\omega\right)$$

$$\Rightarrow \vec{B}_{\lambda\vec{k}} \perp \vec{E}_{\lambda\vec{k}} \text{ and in phase.}$$

$$\left|\vec{E}_{\lambda\vec{k}}\right| \propto \omega \quad \text{and} \quad \left|\vec{B}_{\lambda\vec{k}}\right| \propto k \quad \Rightarrow \frac{\left|\vec{B}_{\lambda\vec{k}}\right|}{\left|\vec{E}_{\lambda\vec{k}}\right|} = \frac{k}{\omega} = \frac{1}{c}$$

In the non-relativistic regime, as c is big, we can neglect the magnetic field (next slide).

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Classical Electromagnetic Radiation Field (cont.)

The volumetric density of energy

$$\rho(\omega) = \frac{\varepsilon_0}{2}\left|\vec{E}_{\lambda\vec{k}}\right|^2 + \frac{1}{2\mu_0}\left|\vec{B}_{\lambda\vec{k}}\right|^2$$

$$= \frac{\varepsilon_0}{2} A_N^2\, 4\omega^2 \sin^2\left(\vec{k}\cdot\vec{r} - \omega t + \delta_\omega\right) + \frac{1}{2\mu_0} A_N^2\, 4 \underbrace{k^2}_{\frac{\omega^2}{c^2} = \frac{\omega^2}{1/\mu_0\varepsilon_0}} \sin^2\left(\vec{k}\cdot\vec{r} - \omega t + \delta_\omega\right)$$

$$= \varepsilon_0 A_N^2\, 4\omega^2 \sin^2\left(\vec{k}\cdot\vec{r} - \omega t + \delta_\omega\right), \quad \text{which is time dependent}$$

$$\Rightarrow \langle \rho(\omega) \rangle = 2\varepsilon_0 A_N^2 \omega^2 \quad \text{(time average)}$$

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Classical electromagnetic radiation field (cont.)

The Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\underbrace{\vec{B}}_{E_0/c} \propto \vec{k}$$

$$= \frac{E_0\,B_0}{\mu_0}\hat{k}\sin^2\left(\vec{k}\cdot\vec{r} - \omega t + \delta_\omega\right)$$

$$= \frac{E_0^2}{\mu_0 c}\hat{k}\sin^2\left(\vec{k}\cdot\vec{r} - \omega t + \delta_\omega\right) = \varepsilon_0 c E_0^2 \hat{k}\sin^2\left(\vec{k}\cdot\vec{r} - \omega t + \delta_\omega\right)$$

Intensity or irradiance: $I = \left\langle \left|\vec{S}\right| \right\rangle = \frac{\varepsilon_0 c E_0^2}{2}$ $\quad [I] = \frac{\mathrm{W}}{\mathrm{m}^2}$

Classical electromagnetic radiation field (cont.)

Example 1: What is the intensity of the solar radiation on the Earth?

$$I = 1.4\frac{\text{kW}}{\text{m}^2} = 0.14\frac{\text{W}}{\text{cm}^2};\quad E_0 \sim 750\frac{\text{V}}{\text{m}};\quad B_0 \sim 2.4\mu\text{Tesla}$$

Example 2: What is the intensity of the radiation of a He-Ne laser?

$$\text{Power} \sim 3.2\text{mW};\quad \text{diameter beam} \sim 2.5\text{mm}$$

$$I \simeq \frac{3.2\text{mW}}{\pi\left(2.5\text{mm}/2\right)^2} = 0.065\frac{\text{W}}{\text{cm}^2}$$

Example 3: What is the intensity of the radiation of a Ti-sapphire laser?

$$I \sim 10^{10} - 10^{16}\frac{\text{W}}{\text{cm}^2}$$

Classical electromagnetic radiation field (cont.)

Dipole approximation:

Spatial dependence of the vector potential: $\vec{A}_{\lambda\vec{k}} \propto e^{\pm i\vec{k}\cdot\vec{r}}$ where $k = \frac{\omega}{c} = \frac{2\pi\nu}{c}$

As c is big, we can neglect the wave number k of the photon and, therefore, the spatial dependence of the vector potential

$$\text{(foton momentum)}\ \hbar k = \frac{\hbar\omega}{c} = \frac{\varepsilon}{c} \ll 1$$

Example: Ti-sapphire laser

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{8\times10^2\times10^{-9}\text{m}} = 7.8\times10^6\text{m}^{-1}$$

The dimension of the atom is about some Bohr radii

$$a_0 = 5.3\times10^{-11}\text{m}$$

$$ka_0 = \frac{2\pi a_0}{\lambda} = 7.8\times10^6\text{m}^{-1}\times5.3\times10^{-11}\text{m} \simeq 4\times10^{-4} \ll 1$$

The dipole approximation fails when:

$$kr \sim 1 \;\Rightarrow\; \nu > \frac{c}{2\pi r} \quad \text{with} \quad \frac{c}{2\pi a_0} = 9\times10^{17}\text{Hz} \quad \text{(hard X and } \gamma \text{ rays)}$$

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Radiation-matter coupling

Lorentz Force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Inserting B and E into the Lorentz law:

$$\vec{F} = q\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \vec{\nabla} \times \vec{A}\right) \qquad (1)$$

Exercise 4: Prove that (Goldstein pgs. 27-28)

$$\vec{v} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\left(\vec{v} \cdot \vec{A}\right) - \frac{d\vec{A}}{dt} + \frac{\partial \vec{A}}{\partial t} \qquad (2)$$

$$\text{where } \frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \underbrace{\left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z}\right)}_{\vec{v} \cdot \vec{\nabla} A_x}$$

Replacing (2) in (1)

$$\vec{F} = q\left(-\vec{\nabla}\left(\phi - \vec{v} \cdot \vec{A}\right) - \frac{d\vec{A}}{dt}\right)$$

And rewriting

$$\vec{A} = \vec{\nabla}_{\vec{v}}\left(\vec{v} \cdot \vec{A}\right) = -\vec{\nabla}_{\vec{v}}\left(\phi - \vec{v} \cdot \vec{A}\right)$$

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Lagrangian Formalism

$$\vec{F} = -\vec{\nabla}\left[\underbrace{q\left(\phi - \vec{v} \cdot \vec{A}\right)}_{U}\right] + \frac{d}{dt}\left\{\vec{\nabla}_{\vec{v}}\left[\underbrace{q\left(\phi - \vec{v} \cdot \vec{A}\right)}_{U}\right]\right\}$$

$$\vec{F} = -\vec{\nabla}U + \frac{d}{dt}\vec{\nabla}_{\vec{v}}U$$ Definition of generalized force

$$U = q\left(\phi - \vec{v} \cdot \vec{A}\right)$$ *U* is the generalized potential

The Lagrangian of a charged particle in an electromagnetic field is given by

$$L(\vec{r}, \dot{\vec{r}}, t) = T_{\text{kin}} - U = \tfrac{1}{2} m v^2 - q\left(\phi - \vec{v} \cdot \vec{A}\right); \quad \text{where } \dot{\vec{r}} = \vec{v}$$

Generalized momentum:

$$\vec{p} = \frac{\partial L}{\partial \vec{v}}$$

$$\vec{p} = m\vec{v} + q\vec{A}; \quad \vec{p} \neq m\vec{v}$$

Watch out!

Hamiltonian Formalism

By means of a Legendre transform the Hamiltonian is defined by:

$$H(\vec{p},\vec{r},t) = \dot{\vec{r}}\cdot\vec{p} - L(\vec{r},\dot{\vec{r}},t)$$

$$= \vec{v}\cdot\vec{p} - \tfrac{1}{2}mv^2 + q\left(\phi - \vec{v}\cdot\vec{A}\right)$$

$$= \left(\frac{\vec{p}-q\vec{A}}{m}\right)\cdot\vec{p} - \tfrac{1}{2}m\left(\frac{\vec{p}-q\vec{A}}{m}\right)^2 + q\phi - q\left(\frac{\vec{p}-q\vec{A}}{m}\right)\cdot\vec{A}$$

$$H(\vec{p},\vec{r},t) = \frac{\left(\vec{p}-q\vec{A}\right)^2}{2m} + q\phi$$

Exercise 5: Derive the Lorentz force from the Hamilton equations:

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}; \quad -\dot{\vec{p}} = \frac{\partial H}{\partial \vec{r}}$$

Hamiltonian formalism (cont.)

This is called the length gauge:

$$H'(\vec{p},\vec{r},t) = \frac{\left(\vec{p}-q\vec{A}'\right)^2}{2m} + q\phi' = \frac{\vec{p}^2}{2m} - q\vec{E}\cdot\vec{r}$$

where $\vec{p} = m\vec{v} + q\underbrace{\vec{A}'}_{=0} = m\vec{v}$

In the velocity gauge:

$$H(\vec{p},\vec{r},t) = \frac{\left(\vec{p}-q\vec{A}\right)^2}{2m} + q\underbrace{\phi}_{=0} = \frac{\left(\vec{p}-q\vec{A}\right)^2}{2m}$$

where $\vec{p} = m\vec{v} + q\vec{A} \neq m\vec{v}$

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Classical electron in a sinusoidal electric field

Linearly polarized plane wave (never started and never finished):

$$\vec{E} = \hat{z}E_0 \sin\omega t; \quad \hat{z}: \text{ polarization direction}$$

$$q = -e; \quad e = 1.6\times10^{-19}C$$

$$\text{Lorentz Force:} \quad m_e\ddot{z} = -eE_0 \sin\omega t$$

$$\dot{z} = \frac{eE_0}{m_e\omega}\cos\omega t \qquad \text{supposing adiabatic switch on and off}$$

$$\text{kinetic energy: } T_{\text{kin}} = \tfrac{1}{2}m_e\dot{z}^2 = \frac{e^2E_0^{\ 2}}{2m_e\omega^2}\cos^2\omega t$$

$$\text{time average: } \langle T_{\text{kin}}\rangle = \frac{1}{T}\int_t^{t+T} T_{\text{kin}}\,dt = \frac{e^2E_0^{\ 2}}{4m_e\omega^2}$$

$$U_p = \frac{e^2E_0^{\ 2}}{4m_e\omega^2} \qquad \text{ponderomotive energy}$$

Energy an electron has because it is oscillating driven by E(t)

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Classical electron in a sinusoidal electric field (cont.)

Electron position:

$$z(t) = \alpha_0 \sin\omega t; \quad \alpha_0 = \frac{eE_0}{m_e\omega^2} \quad \text{quiver (oscillation) amplitude}$$

$$\beta = \frac{v}{c} = \frac{eE_0}{m_e\omega c} \Rightarrow \begin{cases} \beta \ll 1 \Rightarrow \text{ non-relativistic case} \\ \beta \simeq 1 \Rightarrow \text{ relativistic case} \end{cases}$$

Exercise 6: How much is β for an electron excited with a Ti-Sapphire laser?
λ = 800nm, E_0 = 5x10^{10} V/m, m_e = 9.1x10^{-31} kg

Only this century it has been possible to build lasers with β ~ 1

Quantum Electron in Electromagnetic Field

First quantization: $\vec{r} \rightarrow \hat{\vec{r}}$

$$\vec{p} \rightarrow \hat{\vec{p}} = \frac{\hbar}{i}\vec{\nabla}$$

$$\varepsilon \rightarrow \hat{H} = i\hbar\frac{\partial}{\partial t}$$

Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \left[\frac{1}{2m_e}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2 + q\phi\right]\Psi(\vec{r},t)$

Exercise 7: How does gauge invariance work for the Schrödinger equation?

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi$$

$$\phi \rightarrow \phi' = \phi - \frac{\partial\chi}{\partial t}$$

Solution: $\Psi'(\vec{r},t) = \Psi(\vec{r},t)e^{i\frac{q}{\hbar}\chi}$

In the Coulomb gauge: $\nabla^2\chi = 0$

Quantum Electron in Electromagnetic Field (cont.)

As we have already seen $\chi(\vec{r},t) = -\vec{A}\cdot\vec{r}$

$$\Rightarrow \vec{A}' = 0, \quad \text{and} \quad \phi' = -\vec{E}(t)\cdot\vec{r}$$

Length gauge: $i\hbar\frac{\partial}{\partial t}\Psi'(\vec{r},t) = \left[\frac{-\hbar^2}{2m_e}\vec{\nabla}^2 - q\vec{E}(t)\cdot\vec{r}\right]\Psi'(\vec{r},t)$

Velocity gauge: $i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \frac{1}{2m_e}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2\Psi(\vec{r},t)$

Let's solve the Schrödinger equation in the velocity gauge.
In the trivial case that there are not electromagnetic fields (A = 0) we have the free particle:

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \frac{-\hbar^2}{2m_e}\vec{\nabla}^2\Psi(\vec{r},t)$$

We use the method of separation of variables: $\Psi(\vec{r},t) = \psi(\vec{r})e^{-\frac{i}{\hbar}\varepsilon_k t}$

$$\frac{-\hbar^2}{2m_e}\vec{\nabla}^2\psi(\vec{r}) = \varepsilon_k\psi(\vec{r}) \;\Rightarrow\; \psi(\vec{r}) = \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} \quad \text{with} \quad \varepsilon_k = \frac{\hbar^2k^2}{2m_e}$$ (kinetic energy)

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Quantum Electron in Electromagnetic Field (cont.)

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Exercise 8: Prove that if $\vec{A}=\vec{A}(t)\neq 0 \Rightarrow \Psi(\vec{r},t)=\psi(\vec{r})e^{-\frac{i}{\hbar}S(t)}$

where $S(t)=\frac{1}{2m_e}\int_{t_0}^{t} dt'\left[\hbar\vec{k}-q\vec{A}(t')\right]^2$ classical action

Let's consider the case of a linearly polarized sinusoidal electric field with adiabatic switch on and off

$$\vec{E}(t)=\hat{z}E_0\sin\omega t$$

$$\vec{E}(t)=-\frac{\partial\vec{A}}{\partial t} \quad\Rightarrow\quad \vec{A}(t)=-\int_{-\infty}^{t}dt'\vec{E}(t')=-\hat{z}E_0\int_{-\infty}^{t}dt'\sin\omega t'=\hat{z}\frac{E_0}{\omega}\cos\omega t$$

For the sinusoidal case we can derive an analytical action:

$$S(t)=\frac{1}{2m_e}\int_{t_0}^{t}dt'\left[\hbar\vec{k}-q\vec{A}(t')\right]^2=\varepsilon_k(t-t_0)+\hbar\vec{k}\cdot\vec{\alpha}(t)+\beta(t)$$

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Quantum Electron in Electromagnetic Field (cont.)

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$$\vec{\alpha}(t)=\frac{-q}{m_e}\int_{t_0}^{t}dt'\vec{A}(t')=\frac{-q}{m_e}\frac{E_0}{\omega}\hat{z}\int_{t_0}^{t}dt'\cos\omega t'=\frac{-q}{m_e}\frac{E_0}{\omega^2}\hat{z}\sin\omega t \quad \text{(adiabatic switch on)}$$

is the integral of the integral of the force $\Rightarrow$ displacement of the classical electron

$$\langle\vec{\alpha}(t)\rangle=\frac{1}{T}\int_0^T dt\vec{\alpha}(t)=0$$

$$\beta(t)=\frac{q^2}{2m_e}\int_{t_0}^{t}dt'A^2(t')=\frac{q^2}{2m_e}\frac{E_0^2}{\omega^2}\int_{t_0}^{t}dt'\cos^2\omega t'=\frac{q^2}{4m_e}\frac{E_0^2}{\omega^2}\left[(t-t_0)+\frac{\sin 2\omega t}{2\omega}\right]$$

where I used adiabatic switch on

$$\langle\beta(t)\rangle=\frac{1}{T}\int_0^T dt\beta(t)=\frac{q^2}{4m_e}\frac{E_0^2}{\omega^2}=U_p \quad \text{(ponderomotive energy)}$$

$q=-e$ for the electron

If E_0 = 0 everything reduces to the free particle