Clase 19

Martes 10/11/2020

La clase pasada vimos:

i) Partículas, huecos, vacío

ii) Orden normal de operadores

iii) Pairing de operadores

iv) Teorema de Wick para productos de operadores

Chapter 19

Particle and hole operators and Wick's theorem

Gian Carlo Wick

Turín 1909-1992

Hoy vemos :

i) Contracciones

ii) Teorema de Wick para productos ordenados temporalmente

REPASO

Partículas: $\left.\begin{aligned} a_i^\dagger &\equiv c_i^\dagger \\ a_i &\equiv c_i \end{aligned}\right\}$ for $\epsilon_i > \epsilon_F$

Huecos: $\left.\begin{aligned} b_j^\dagger &\equiv c_j \\ b_j &\equiv c_j^\dagger \end{aligned}\right\}$ for $\epsilon \leq \epsilon_F$

$\{c_i^\dagger, c_k\} = \delta_{ik}$ ⟶ $\{a_i^\dagger, a_k\} = \delta_{ik}$ and $\{b_j^\dagger, b_\ell\} = \delta_{j\ell}$

$a_i \mid \Phi_0\rangle = 0$ and $b_j \mid \Phi_0\rangle = 0$ ⟶ Un nuevo vacío, sin partículas ni huecos

Los operador de campo se dividen en dos partes:

$$\begin{aligned} \psi(x) &= \sum_{\epsilon_i \leq \epsilon_F} \varphi_i(x) c_i + \sum_{\epsilon_i > \epsilon_F} \varphi_i(x) c_i \\ &= \sum_{\epsilon_i \leq \epsilon_F} \varphi_i(x) b_i^\dagger + \sum_{\epsilon_i > \epsilon_F} \varphi_i(x) a_i \equiv \psi_+(x) + \psi_-(x) \end{aligned}$$

REPASO

Orden normal en un producto de operadores: llevar operadores de **destrucción** (de partículas o huecos) a la **derecha** (y un signo)

Ejemplo:

$$
\begin{aligned}
& N[c_i(t_1)c_j(t_2)c_k^\dagger(t_3)] \\
= \quad & N[a_i(t_1)b_j^\dagger(t_2)a_k^\dagger(t_3)] \\
= \quad & (-1)^2 b_j^\dagger(t_2)a_k^\dagger(t_3)a_i(t_1) \\
= \quad & +c_j(t_2)c_k^\dagger(t_3)c_i(t_1),
\end{aligned}
$$

Supongamos:

$$\epsilon_i, \epsilon_k > \epsilon_F, \quad \epsilon_j < \epsilon_F$$

Notar que siempre da: $\langle\Phi_0|\mathrm{N}\left[\cdots\right]|\Phi_0\rangle = 0$

REPASO

Pairing: $\underbracket{AB} \equiv AB - N(AB)$

$$\begin{cases} \underbracket{a_i(t)a_j^\dagger(t')} = e^{i\epsilon_j(t'-t)}\delta_{ij} \\ \underbracket{b_i(t)b_j^\dagger(t')} = e^{i\epsilon_j(t-t')}\delta_{ij} \\ \{A, B\} = 0 \quad \Rightarrow \quad \underbracket{AB} = 0 \end{cases}$$

Notar que: $\underbracket{a_i^\dagger(t)a_j}(t') = a_i^\dagger(t)a_j(t') - N[a_i^\dagger(t)a_j(t')] = 0$

Valor de expectación en el vacío:

$$\langle\Phi_0 \mid AB \mid \Phi_0\rangle = \langle\Phi_0 \mid N(AB) + \underbracket{AB} \mid \Phi_0\rangle = \langle\Phi_0 \mid \underbracket{AB} \mid \Phi_0\rangle = \underbracket{AB}$$

REPASO

El orden normal puede incluir pairings:

$$N(ABCDE\ldots XYZ) \equiv (-)^q\, AD\ CY\ N(BE\ldots XZ)$$

Teorema de Wick (I)

$$
\begin{aligned}
A_1 A_2 \ldots A_n \quad = \quad & N(A_1 \ldots A_n) \\
& + N(A_1 A_2\, A_3 \ldots A_n) + N(A_1 A_2 A_3 \ldots) \\
& + \ldots \text{ all other terms with one pairing} \\
& + N(A_1 A_2 \ldots A_n) \\
& + \ldots \text{ all other terms with two pairings} \\
& + \\
& \vdots \\
& + \text{ all completely paired terms} \\
& \quad \text{(they appear only for even } n\text{).}
\end{aligned}
$$

Due to the linearity of normal-ordered products (and of the pairings), Wick's theorem also holds for linear combinations of particle and hole operators, and consequently also for the field operators themselves.

Físicos teóricos suizos:

Gregory Wannier

Physicist

Gregory Hugh Wannier was a Swiss physicist. Wikipedia

Born: December 30, 1911, Basel, Switzerland

Died: October 21, 1983, Portland, Oregon, United States

Books: Statistical physics

Education: University of Basel / Kollegienhaus, Princeton University

Academic advisor: Ernst Stueckelberg

Notable student: Douglas Hofstadter

Ernst Stueckelberg

Swiss mathematician

Ernst Carl Gerlach Stueckelberg was a Swiss mathematician and physicist, regarded as one of the most eminent physicists of the 20th century. Wikipedia

Born: February 1, 1905, Basel, Switzerland

Died: September 4, 1984, Geneva, Switzerland

Education: University of Geneva, University of Basel / Kollegienhaus

Awards: Max Planck Medal

Academic advisors: August Hagenbach, Arnold Sommerfeld

Notable students: André Petermann, Gregory Wannier, Marcel Guénin

Definición: **Contracción**

$$\overbrace{A(t)B}(t') \equiv T\left(A(t)B(t')\right) - N\left(A(t)B(t')\right)$$

Relación entre **Contracción** y **Pairing** $\underbrace{AB} \equiv AB - N(AB)$

$$\overbrace{A(t)B}(t') \quad = \quad T(A(t)B(t')) - N(A(t)B(t'))$$

$$\begin{matrix} t > t' \\ \\ t' > t \end{matrix} \qquad = \begin{cases} A(t)B(t') - N(A(t)B(t')) \\ \\ -B(t')A(t) - N(A(t)B(t')) \;=\; -B(t')A(t) + N(B(t')A(t)) \end{cases}$$

$$\longrightarrow \quad \boxed{\overbrace{A(t)B}(t') = \begin{cases} \underbrace{A(t)B(t')} & t > t' \\ -\underbrace{B(t')A(t)} & t' > t \end{cases}}$$

Finalmente obtenemos para el valor medio de un producto ordenado temporalmente:

$$\langle \Phi_0 \mid T(A(t)B(t')) \mid \Phi_0\rangle = \langle \Phi_0 \mid \contraction{}{A}{(t)}{B}A(t)B(t') + N(A(t)B(t')) \mid \Phi_0\rangle$$

$$= \langle \Phi_0 \mid \contraction{}{A}{(t)}{B}A(t)B(t') \mid \Phi_0\rangle + \langle \Phi_0 \mid N(A(t)B(t')) \mid \Phi_0\rangle$$

$$= \contraction{}{A}{(t)}{B}A(t)B(t')$$

Como vimos para el pairing, se puede incluir contracciones dentro de un producto con orden normal:

$$N(\contraction[2ex]{}{A}{BC}{D}\contraction{A}{B}{CDE\ldots X}{Y}A\,B\,C\,D\,E\ldots X\,Y\,Z) = (-1)^q\,\contraction{}{A}{}{D}A\,D\;\contraction{}{B}{}{Y}B\,Y\;N(C\,E\ldots X\,Z)$$

Teorema de Wick para productos ordenados temporalmente

$$
\begin{aligned}
T(A_1 A_2 \ldots A_n) \quad = \quad & N(A_1 A_2 \ldots A_n) \\
& + N(\overline{A_1 A_2} \ldots A_n) + \\
& + \ldots \text{ all other terms with one contraction} \\
& + N(\overline{A_1 A_2}\, \overline{\ldots}\, A_n) + \ldots \\
& + \ldots \text{ all other terms with two contractions} \\
& + \\
& \vdots \\
& + \text{ all completely contracted terms} \\
& \ \ (\text{they appear only for even } n)
\end{aligned}
$$

Comparar con la primera versión del teorema de Wick que vimos antes:

Wick's theorem for normal products states that

$$
\begin{aligned}
A_1 A_2 \ldots A_n \quad = \quad & N(A_1 \ldots A_n) \\
& + N(\underbrace{A_1 A_2}\, A_3 \ldots A_n) + N(\underbrace{A_1 A_2 A_3} \ldots) \\
& + \ldots \text{ all other terms with one pairing} \\
& + N(\underbrace{A_1 A_2}\, \underbrace{\ldots}\, A_n) \\
& + \ldots \text{ all other terms with two pairings} \\
& + \\
& \vdots \\
& + \text{ all completely paired terms} \\
& \quad \text{(they appear only for even } n\text{).}
\end{aligned}
$$

Veamos contracciones de los operadores originales:

$$\overbrace{c_j(t)c_k^\dagger(t')} = \begin{cases} \text{for } \epsilon_j, \epsilon_k > \epsilon_F : & \overbrace{a_j(t)a_k^\dagger} = \begin{cases} \underbrace{a_j(t)a_k^\dagger(t')} & \text{for } t > t' \\ -\underbrace{a_k^\dagger(t')a_j(t)} = 0 & \text{for } t' > t \end{cases} \\ \text{for } \epsilon_j, \epsilon_k \le \epsilon_F : & \overbrace{b_j^\dagger(t)b_k(t')} = \begin{cases} \underbrace{b_j^\dagger(t)b_k(t)} = 0 & \text{for } t > t' \\ -\underbrace{b_k(t')b_j^\dagger(t)} & \text{for } t' > t \end{cases} \\ \text{otherwise :} & 0 \end{cases}$$

$$= \begin{cases} \delta_{jk} e^{-i\epsilon_j(t-t')} & \epsilon_j > \epsilon_F,\ t > t' \\ -\delta_{jk} e^{-i\epsilon_j(t-t')} & \epsilon_j \le \epsilon_F,\ t' > t \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta_{jk} e^{-i\epsilon_j(t-t')} \left[\theta(t-t')\theta(\epsilon_j - \epsilon_F) - \theta(t'-t)\theta(\epsilon_F - \epsilon_j)\right]$$

$$\overbrace{c_j(t)c_k^\dagger(t')} = \delta_{jk}e^{-i\epsilon_j(t-t')}\left[\theta(t-t')\theta(\epsilon_j-\epsilon_F)-\theta(t'-t)\theta(\epsilon_F-\epsilon_j)\right]$$

Esta expresión es similar la función de Green para fermiones no interactuantes:

$$iG^{(0)}_{\alpha\beta}(\mathbf{k}t,\mathbf{k}'t') = \delta_{\alpha\beta}\delta_{\mathbf{k}\mathbf{k}'}e^{-i\epsilon_k(t-t')}\left[\theta(t-t')\theta(k-k_F)-\theta(t'-t)\theta(k_F-k)\right]$$

Este resultado para la contracción es válido en general:

$$iG^{(0)}(jt,kt') = \langle\Phi_0 \mid T\left[c_j(t)_H c_k^\dagger(t')_H\right] \mid \Phi_0\rangle$$

$$H = H_0$$

$$iG^{(0)}(jt,kt') = \langle\Phi_0 \mid T\left[c_j(t)_I c_k^\dagger(t')_I\right] \mid \Phi_0\rangle = \overbrace{c_j(t)_I c_k^\dagger(t')_I}$$

Se puede ver que:

$$\overbrace{c_k^\dagger(t')c_j(t)} = -\overbrace{c_j(t)c_k^\dagger(t')}$$

$$\overbrace{c_j(t)c_k(t')} = 0 = \overbrace{c_j^\dagger(t)c_k^\dagger(t')}$$

$$\begin{aligned} \overbrace{\psi(xt)\psi^\dagger(yt')} &= -\overbrace{\psi^\dagger(yt')\psi(xt)} \\ &= \langle \Phi_0 \mid T\left(\psi(xt)\psi^\dagger(yt')\right) \mid \Phi_0 \rangle \\ &= iG^{(0)}(xt, yt') \end{aligned}$$

$$\overbrace{\psi(xt)\psi(yt')} = 0 = \overbrace{\psi^\dagger(xt)\psi^\dagger(yt')}$$

CONCLUSION: Las contracciones de operadores de creación y destrucción del H_0 son iguales a las funciones de Green del H_0, o nulas.