

Clase 23

Viernes 27/11/2020

La clase pasada vimos:

- i) Diagramas conexos y desconexos
- ii) Degeneraciones de diagramas
- iii) Autoenergía, autoenergía propia
- iv) Ecuación de Dyson para la función de Green

Hoy vemos:

- i) Ejemplo: diagramas de primer orden de G
- ii) Análisis diagramático de $G(\mathbf{k}, \omega)$

Chapter 23

Diagrammatic calculation of the single-particle Green's function: Dyson's equation

Chapter 24

Diagrammatic analysis of the Green's function $G(\mathbf{k}, \omega)$

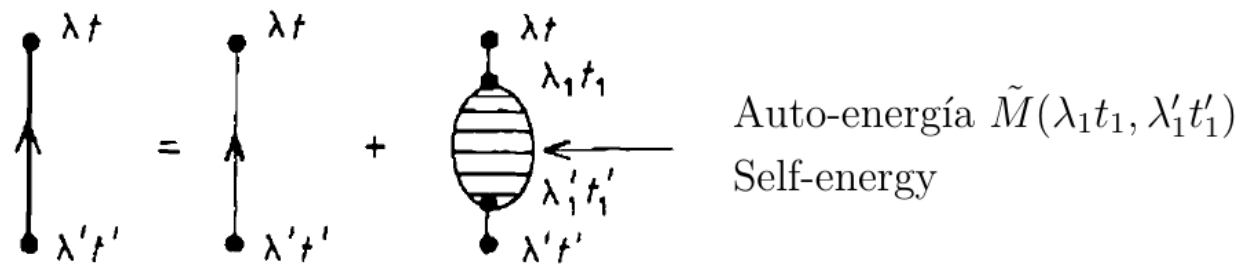


Dos resultados sobre diagramas conexos y desconexos:

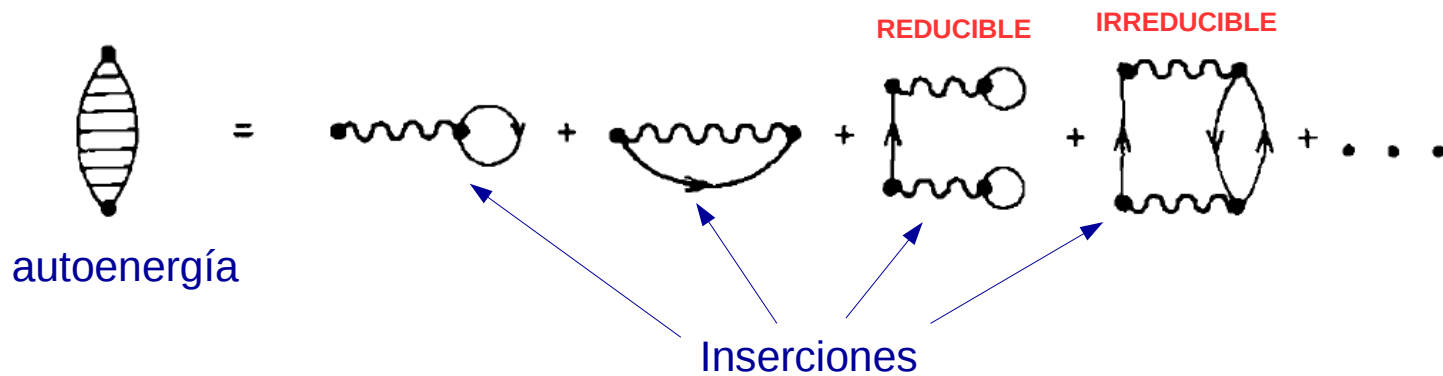
- * Los diagramas desconexos del numerador de la función de Green se cancelan con el denominador (amplitud de vacío).
- * La amplitud de vacío se puede calcular considerando solo los diagramas conexos – Teorema de Linked-clusters de Goldstone (1957).

Teniendo en cuenta las degeneraciones de los diagramas se obtiene:

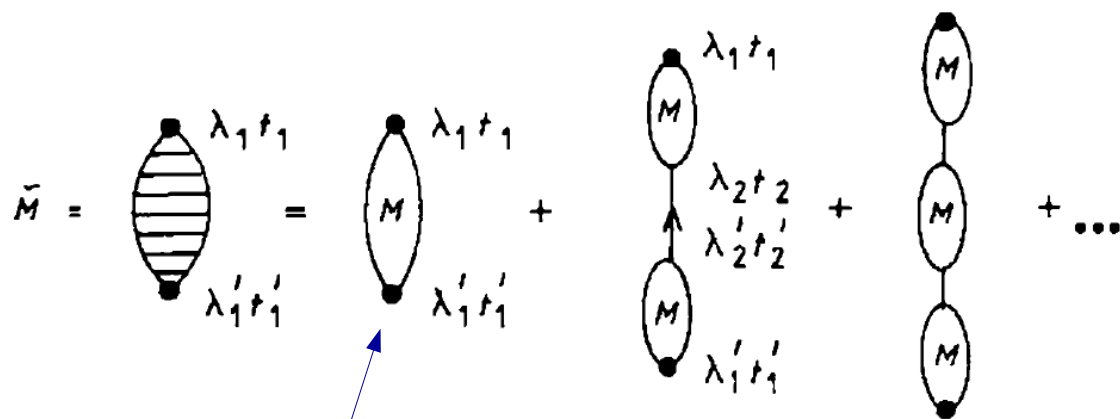
(7) El prefactor de los diagramas **no-indexados** y **conectados** de la función de Green es : i^n



REPASO

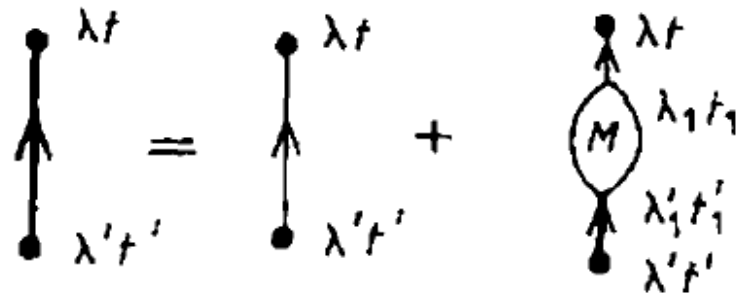
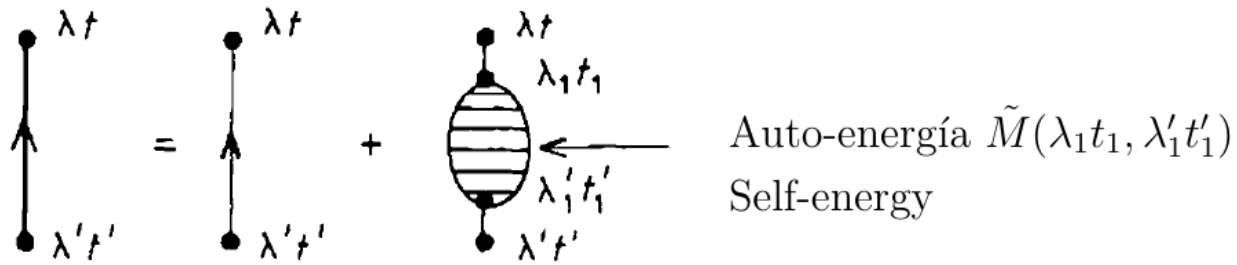


(son diagramas conexos con una entrada y una salida vacantes)



Autoenergía propia (contiene todos los diagramas irreducibles)

REPASO

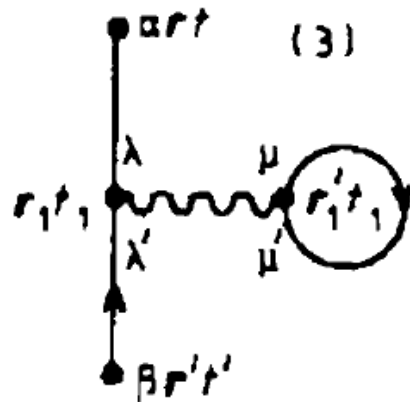


Ecuación de Dyson

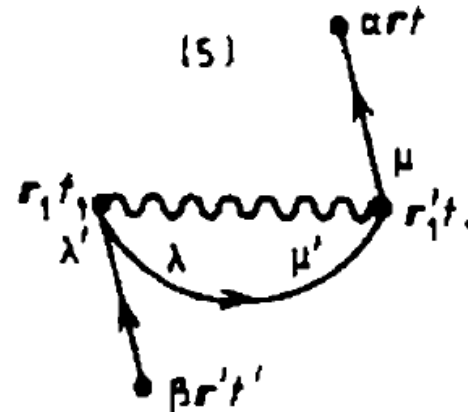
$$G(\lambda t \lambda' t') = G^{(0)}(\lambda t, \lambda' t') + \sum_{\lambda_1} \sum_{\lambda'_1} \int dt_1 \int dt'_1 G^{(0)}(\lambda t, \lambda_1 t_1) M(\lambda_1 t_1, \lambda'_1 t'_1) G(\lambda'_1 t'_1, \lambda' t')$$

Contribuciones de primer orden de G

Se dibujan todos los diagramas topológicamente distintos:



tadpole
directo



open oyster
intercambio

$$G_{\alpha,\beta}^{(1)}(\mathbf{r}t, \mathbf{r}'t') = \frac{i}{\hbar} \sum_{\lambda\lambda', \mu\mu'} \int_{-\infty}^{\infty} dt_1 \int d\mathbf{r}_1 \int d\mathbf{r}'_1 e^{-\epsilon|t_1|} v(\mathbf{r}_1, \mathbf{r}'_1)_{\lambda\lambda', \mu\mu'} \times$$

$$\left[-G_{\alpha\lambda}^0(\mathbf{r}t, \mathbf{r}_1t_1) G_{\mu'\mu}^0(\mathbf{r}'_1t_1, \mathbf{r}'_1t_1) G_{\lambda'\beta}^0(\mathbf{r}_1t_1, \mathbf{r}'t') \right. \quad \text{tadpole}$$

$$\left. + G_{\alpha\mu}^0(\mathbf{r}t, \mathbf{r}'_1t_1) G_{\mu'\lambda}^0(\mathbf{r}'_1t_1, \mathbf{r}_1t_1) G_{\lambda'\beta}^0(\mathbf{r}_1t_1, \mathbf{r}'t') \right] \quad \text{open oyster}$$

Análisis diagramático de $G(\mathbf{k}, \omega)$

Pasamos a la notación espacio-tiempo 4-dimensional:

$$\mathbf{x} \equiv (\mathbf{r}, t) \quad d^4x = d^3r dt$$

Introducimos: $\mathbf{k} \equiv (\mathbf{k}, \omega) \quad d^4k = d^3k d\omega$

Y el producto escalar de Minkowski: $\mathbf{k} \cdot \mathbf{x} = \mathbf{k} \cdot \mathbf{r} - \omega t$

Esta es una notación útil para una formulación covariante de sistemas relativistas.

En los diagramas de Feynman cada línea de interacción daba una contribución:

$$\int dt e^{-c|t|} \int d^3r \int d^3r' v(\mathbf{r}, \mathbf{r}')$$



Ahora pasamos a:



$$u(x, x') \equiv v(\mathbf{r}, \mathbf{r}') \delta(t - t')$$

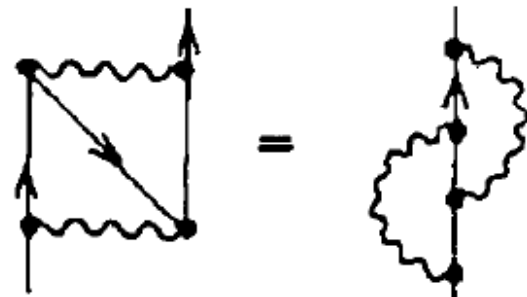
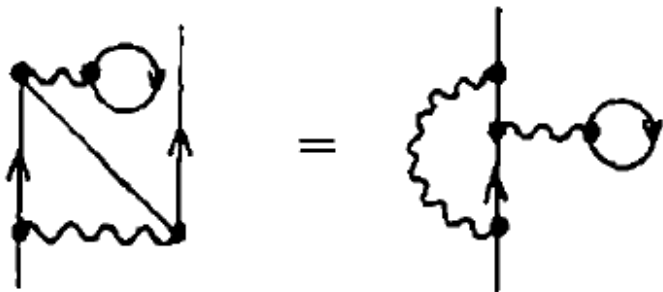
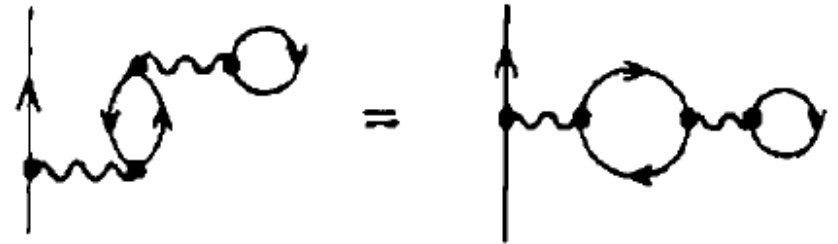
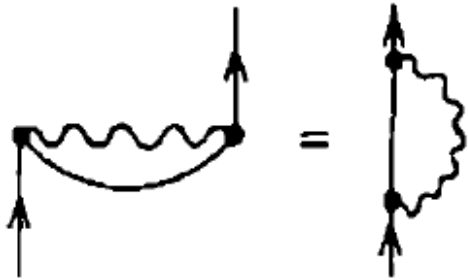
Se integra sobre los índices de los vértices

$$\begin{aligned} \longrightarrow \int dx \int dx' u(x, x') &= \int dt \int d^3r \int dt' \int d^3r' v(\mathbf{r}, \mathbf{r}') \delta(t - t') \\ &= \int dt \int d^3r \int d^3r' v(\mathbf{r}, \mathbf{r}'). \end{aligned}$$

Y hay que agregar el factor de encendido adiabático, pero podemos agregarlo al final.

Análisis diagramático de $G(\mathbf{k}, \omega)$

Si la interacción dejara de ser instantánea la primera regla de Feynman dejaría de aplicarse y podríamos modificar los diagramas:



Análisis diagramático de $G(\mathbf{k}, \omega)$

Supongo invariancia traslacional y temporal:

$$G_{\alpha\beta}(x, y) = \frac{1}{(2\pi)^4} \int d^4k e^{i(x-y)k} G_{\alpha\beta}(k)$$

Con inversa: $G_{\alpha\beta}(k) = \int d^4(x-y) e^{-i(x-y)k} G_{\alpha\beta}(x, y)$

Hacemos lo mismo con el potencial de interacción:

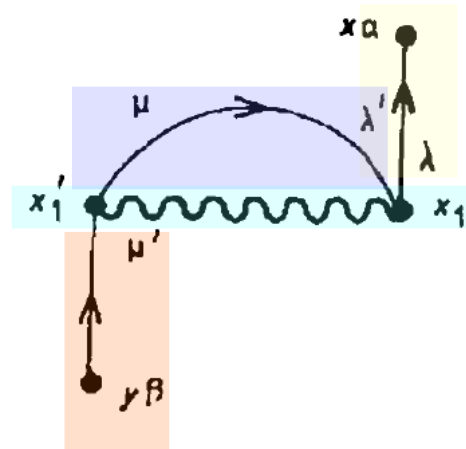
$$u(x-x') = \frac{1}{(2\pi)^4} \int d^4k e^{+ik(x-x')} u(k) \quad u(k) = \int d^4(x-x') e^{-ik(x-x')} u(x-x')$$

Que para sistemas con interacción instantánea se transforma en:

$$u(k) = \int d^3(\mathbf{r}-\mathbf{r}') e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} v(\mathbf{r}-\mathbf{r}') \underbrace{\int d(t-t') \delta(t-t')}_{=1} = v(\mathbf{k})$$

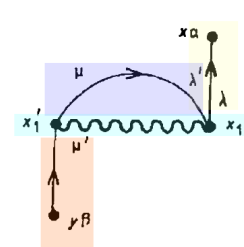
Análisis diagramático de $G(\mathbf{k}, \omega)$

Hacemos transformada de Fourier en el diagrama de primer orden que vimos antes:



$$\begin{aligned}
 & \frac{i^1}{1!2^1} \sum_{\substack{\lambda\lambda' \\ \mu\mu'}} \int d^4x_1 \int d^4x_1' G_{\alpha\lambda}^{(0)}(x, x_1) u(x_1, x_1')_{\lambda\lambda'}_{\mu\mu'} \left[\lim_{t_1' \rightarrow t_1^+} G_{\lambda'\mu}^{(0)}(x_1, x_1') \right] G_{\mu'\beta}^{(0)}(x_1', y) \\
 &= \frac{i}{2} \sum_{\substack{\lambda\lambda' \\ \mu\mu'}} \int d^4x_1 \int d^4x_1' \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x_1)} G_{\alpha\lambda}^{(0)}(k) \\
 & \quad \times \int \frac{d^4q}{(2\pi)^4} e^{iq(x_1-x_1')} u(q)_{\lambda\lambda'}_{\mu\mu'} \int \frac{d^4p}{(2\pi)^4} G_{\lambda'\mu}^{(0)}(p) e^{ip \cdot (x_1-x_1')} \\
 & \quad \times \left[\lim_{t_1' \rightarrow t_1^+} e^{-i\omega_p(t_1-t_1')} \right] \int \frac{d^4p'}{(2\pi)^4} e^{ip'(x_1'-y)} G_{\mu'\beta}^{(0)}(p').
 \end{aligned}$$

Análisis diagramático de $G(\mathbf{k}, \omega)$



Trabajando un poco el tema de los tiempos iguales y reagrupando las integrales se obtiene:

$$\begin{aligned}
 & \lim_{\eta \rightarrow 0^+} \frac{i}{2} \frac{1}{(2\pi)^{16}} \sum_{\substack{\lambda\lambda' \\ \mu\mu'}} \int d^4k \int d^4q \int d^4p \int d^4p' e^{i(kx - p'y)} G_{\alpha\beta}^{(0)}(k) \\
 & \times u(q)_{\lambda\lambda'} \left[e^{i\omega_p \eta} G_{\lambda'\mu}^{(0)}(p, \omega_p) \right] G_{\mu'\beta}^{(0)}(p') \\
 & \times \underbrace{\int d^4x_1 e^{i(q-k+p)x_1}}_{(2\pi)^4 \delta^{(4)}(q-k+p)} \underbrace{\int d^4x_1' e^{i(-q-p+p')x_1'}}_{(2\pi)^4 \delta^{(4)}(p'-q-p)}.
 \end{aligned}$$

Dos deltas de Dirac en los momentos

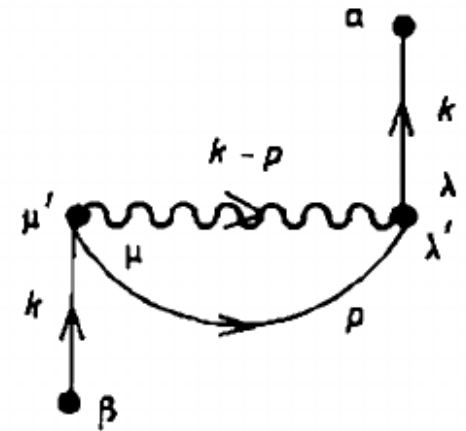
Análisis diagramático de $G(\mathbf{k}, \omega)$

Usando las deltas nos quedan dos integrales en momento:

$$\lim_{\eta \rightarrow 0^+} \frac{i}{2} \sum_{\substack{\lambda\lambda' \\ \mu\mu'}} \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 p}{(2\pi)^4} e^{ik \cdot (x-y)} G_{\alpha\lambda}^{(0)}(k) u(k-p)_{\lambda\lambda'} \left[e^{i\eta\omega p} G_{\lambda'\mu}^{(0)}(\mathbf{p}, \omega_p) \right] G_{\mu'\beta}^{(0)}(k)$$

De acá podemos identificar la transformada de Fourier de la G:

$$G_{\alpha\beta}(x, y) = \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot (x-y)} G_{\alpha\beta}(k)$$

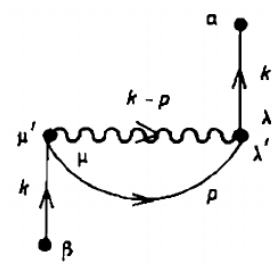


Y nos queda:

$$G_{\alpha\beta}(k) = \lim_{\eta \rightarrow 0^+} \frac{i}{2} \sum_{\substack{\lambda\lambda' \\ \mu\mu'}} \int \frac{d^4 p}{(2\pi)^4} G_{\alpha\lambda}^{(0)}(k) u(k-p)_{\lambda\lambda'} \left[e^{i\eta\omega p} G_{\lambda'\mu}^{(0)}(\mathbf{p}, \omega_p) \right] G_{\mu'\beta}^{(0)}(k)$$

Análisis diagramático de $G(\mathbf{k}, \omega)$

Feynman rules for $G(\mathbf{k}, \omega)$:

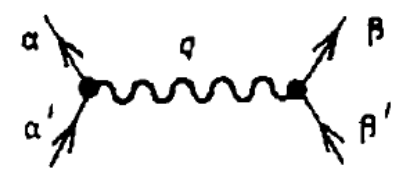


- (1) The Green's function $G_{\alpha\beta}^{(0)}(k)$ is represented by a line labeled by the momentum k and with the spin-indices α and β labeling the end-points (figure 24.7). Each such line gives a factor

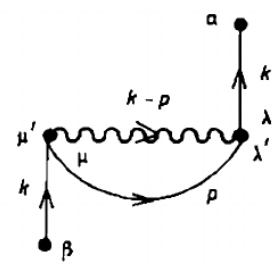
$$G_{\alpha\beta}^{(0)}(k) = \lim_{\xi \rightarrow 0^+} \delta_{\alpha\beta} \left[\frac{\theta(|\mathbf{k}| - k_F)}{\omega - \epsilon_k^{(0)} + i\xi} + \frac{\theta(k_F - |\mathbf{k}|)}{\omega - \epsilon_k^{(0)} - i\xi} \right]$$



- (2) An interaction line is labeled by the momentum q transferred by the interaction, and the vertices are labeled by spin-indices α, α' (first vertex) and β, β' (second vertex) (see figure 24.8). Each interaction line contributes a factor $u(q)_{\alpha\alpha' \beta\beta'}$.



Análisis diagramático de $G(\mathbf{k}, \omega)$

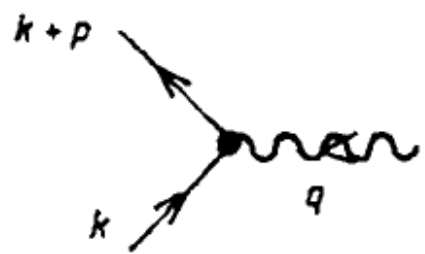


Feynman rules for $G(\mathbf{k}, \omega)$:

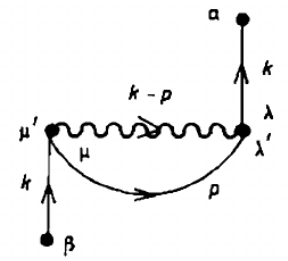
(3) $G^{(0)}$ -lines which begin and end on the same interaction line are interpreted as $e^{i\omega\eta}G^{(0)}(\mathbf{k}, \omega)$, where the limit $\eta \rightarrow 0^+$ is taken at the end of the calculation (figure 24.9).



(4) Momentum is conserved at each vertex. This is illustrated in figure 24.10.



Análisis diagramático de $G(\mathbf{k}, \omega)$



Feynman rules for $G(\mathbf{k}, \omega)$:

- (5) All four-momenta that appear (after satisfying momentum conservation) are integrated over. When the integrals are performed, each is multiplied by a phase-space factor:

$$(2\pi)^{-4} \int d^4p.$$

- (6) The sign of the diagram is given by $(-)^{\ell}$, where ℓ is the number of closed $G^{(0)}$ -loops.

- (7) The prefactor of a diagram of order n is

$$\begin{cases} \frac{(i)^n}{2^n n!} & \text{for an indexed diagram of order } n \\ (i)^n & \text{for an unindexed diagram of order } n \end{cases}$$

Comentario sobre el ejercicio de la guía 4:

3. Hallar la función de Green ordenada temporalmente para el gas de electrones libres y no-interactuantes en el dominio espacio-tiempo, $G(\mathbf{r}t, \mathbf{r}'t')$.

De Bruus y Flensberg:

The corresponding \mathbf{r} -dependent propagator, which expresses propagation of a particle in real space is given by

$$\begin{aligned} \frac{G_0^>(\mathbf{r} - \mathbf{r}', \omega)}{-2\pi i} &= \int \frac{d\mathbf{k}}{(2\pi)^3} (1 - n_F(\xi_{\mathbf{k}})) e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \delta(\xi_{\mathbf{k}} - \omega) \\ &= d(\omega) (1 - n_F(\omega)) \frac{\sin(k_\omega \rho)}{k_\omega \rho}, \quad \frac{k_\omega^2}{2m} = \omega, \quad \rho = |\mathbf{r} - \mathbf{r}'|, \end{aligned} \quad (8.42)$$

where $d(\varepsilon) = m^{3/2} \sqrt{\varepsilon/2} / \pi^2$ is the density of states per spin in three dimensions, see also Eq. (2.31). The propagation from point \mathbf{r}' to \mathbf{r} of a particle with energy ω is thus determined by the density of states, d , the availability of an empty state $(1 - n_F)$, the interference function $\sin(x)/x$ that gives the amplitude of a spherical wave spreading out from the point \mathbf{r}' . See also Exercise 7.3.

De Bruus y Flensberg:

Exercise 7.3

Find the greater propagator, $G^>(\mathbf{r}, \mathbf{r}'; \omega)$ similar to Eq. (8.42), but now in one- and two dimensions. Can you suggest an experiment (at least in principle) that measures this propagator.

Del libro de Wen:

Solving the Schrödinger equation, we find the free-particle propagator for $V(x) = 0$

$$G(x_b, x_a, t) = (-i) \left(\frac{m}{2\pi i t} \right)^{1/2} \exp \left[\frac{i m (x_b - x_a)^2}{2t} \right] \quad (2.1.2)$$

where $t = t_b - t_a$.

$$iG(x_b, t_b, x_a, t_a) \equiv \langle x_b | U(t_b, t_a) | x_a \rangle$$

Esto es para un solo fermión.

FIN de la clase 23

Temas de la clase 23

- i) Diagramas de Feynman de orden 1 de la función Green en coordenadas
- ii) Introducción de notación relativista-compatible
- iii) Suposición de invariancia espacial y temporal: Transformada de Fourier de la función de Green
- iv) Análisis de la transformada de Fourier de un diagrama de orden 1
- v) Reformulación de las Reglas de Feynman
- vi) Comentario sobre la función de Green libre en coordenadas

